4.3 Radiation of caustic beams from a collapsing bullet

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Solitons while perturbed are able to emit small-amplitude radiation with frequency detuned far from the soliton frequency [1–3]. This ability, generally recognized as striking evidence of soliton’s wave nature [4], has gained dramatically in importance in recent years with the discovery of supercontinuum generation in photonic crystal fibres whose applications for frequency comb generation in metrology, spectroscopy and imaging are more than just impressive (see e.g. [5]).

One has to note that small-amplitude radiation from solitons is possible, provided the dispersion relation for the medium deviates from the parabolic law \( \omega(k) = \omega_0 + v(k - k_0) + D/2(k - k_0)^2 \) built-in to the standard (i.e. parabolic) Cubic Nonlinear Schrödinger Equation (CNSE):

\[
\frac{\partial a}{\partial t} + i\omega_0 a + v \frac{\partial a}{\partial z} - \frac{iD}{2} \frac{\partial^2 a}{\partial z^2} + iN |a|^2 a = 0 \tag{1}
\]

Here \( a \) is the envelope function for the propagating wave packet, \( z \) is the direction of propagation, and \( N \) is the nonlinear coefficient (the term \( iN |a|^2 a \) is called “cubic nonlinearity” and gives rise to the equation name). The dispersion law is expressed as a series expansion of the frequency \( \omega \) in terms of the wave number \( k \) of plane linear waves supported by the medium. The sign of the second-order dispersion coefficient \( D = \partial^2 \omega(k)/\partial k^2 \) is of paramount importance for soliton formation: solitons are formed provided \( DN < 0 \).

Karpman [6] theoretically predicted that in the same CNSE framework the fourth-order two-dimensional (2D) dispersion \( \omega(k_y, k_z) \) with a set of coefficients of proper signs should lead to wave irradiation from \((2+1)\)-dimensional wave packets \( a(x, y, t) \) as well. He considered an axially symmetric wave packet in an isotropic medium. In his work and in subsequent numerical simulations [7] it was shown that an axially symmetric wave packet in an isotropic medium is destroyed by radiating energy into a non-trapped circular wave.

In contrast to the isotropic case considered by Karpman [6] here we demonstrate that in a medium with uniaxial anisotropy this radiation takes the form of two narrow-aperture beams. This phenomenon is found for a spin-wave bullet propagating in a thin magnetic film.

Stable \((2+1)\)D localized nonlinear spin-wave excitations, termed spin-wave bullets, have been previously observed in thin ferrimagnetic films of yttrium iron-garnet (YIG) magnetized along the propagation direction [8–11]. The waves propagating along the field are called backward volume magnetostatic spin waves. They are waves of backward nature, which means that their group velocity and wave vector point in opposite directions.

Two-dimensional pulses are intrinsically unstable and undergo nonlinear narrowing leading to collapse as nonlinearity overcompensates linear broadening due to 2D parabolic dispersion. Weak magnetic losses in a real magnetic film may balance the nonlinear narrowing. This results in a quasi-2D spatially localized bell-shaped waveform called a bullet and ensures its stability for some distance of propagation [8]. This regime is well described by the \((2+1)\)D CNSE.

Stable bullets are observed in a certain range of initial powers for the wave packets. For larger input powers waveform collapse is unavoidable [8]. The advanced stage of the collapse is beyond

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In this work we experimentally study the behavior of spin-wave bullets beyond the power range of their stability. The experiment is carried out using a longitudinally magnetized yttrium iron garnet (YIG) film stripe which is \(2.5\) mm wide \((w = 2.5\) mm\) and \(5\,\mu \text{m}\) thick. The magnetizing field \(H\) is \(1831\) Oe. The spin waves are excited by a rf magnetic field created with a \(25\,\mu \text{m}\) wide microstrip antenna placed across the stripe and driven by \(20\) ns long microwave current pulses at a carrier frequency of \(7.125\) GHz. The spatio-temporal behavior of the traveling spin-wave packets is probed by means of space- and time-resolved Brillouin light scattering spectroscopy [12]. For relatively small input powers we observe formation of quasi-stable \((2+1)\)D wave packets - guided spin-wave bullets (Fig. 1, upper row) - reported previously for the same geometry [11]. If we increase the power beyond the range of bullet stability the wave packet collapses (Fig. 1, lower row). The most prominent feature of the collapse is the pair of rays irradiated from the packet in the backward direction (second panel in the lower row of Fig. 1). One clearly sees that the rays have narrow apertures and are directed at well defined angles to the longitudinal axis of the ferromagnetic stripe: the value of the angle between the rays is \(64^\circ\).

In order to understand the particular value of the angle between the two irradiated beams it is appropriate to consider the phenomenon of caustics which will play a central role in the following.

In an anisotropic medium with a dispersion law \(\omega(k)\), the direction of the wave group velocity \(v_g = \frac{\partial \omega(k)}{\partial k}\) indicating the direction of energy flow does not, in general, coincide with the direction of the wave vector \(k\). When the medium anisotropy is sufficiently strong, the direction of the group velocity of the waves in the vicinity of a certain wave vector \(k_c\) may become practically independent of their wave vectors. In this case the energy of the waves is channeled along this specific direction, which is called caustic direction, and forms a so-called caustic beam.
In a YIG film magnetocrystalline anisotropy is negligible; application of an external static field \( \mathbf{H} \) in the film plane orients the static magnetization along the field direction and thus imposes a uniaxial symmetry necessary for the formation of caustics. Thus, two pairs of spin-wave caustic beams are directed at specific angles with \( \mathbf{H} \). The angles depend on the magnitude of the applied field, film saturation magnetization, film thickness, and spin wave frequency \([8, 13–17]\). For an immobile excitation source and for the conditions of our current experiment the angle \( \phi \) between the beams in each pair should amount to \( 84^\circ \) as our calculation based on the theory in \([15]\) shows. Note that this value is significantly different from the angle seen in Fig. 1.

To reveal the origin of the irradiation we carry out numerical simulations. An original reciprocal-space approach is used which effectively accounts for all dispersion orders of the real spin-wave dispersion \([11, 18]\). In the reciprocal space CNSE takes the form of a system of coupled equations for the amplitudes of the wave packet’s spatial Fourier components:

\[
i \frac{\partial F_{n,k}(a)}{\partial t} + (\omega_{n,k} + i\Gamma - \omega_I)F_{n,k}(a) + NF_{n,k}(|a|^2a) = f_{n,k}(t)
\]

where \( a = a(y,z,t) \) is the spin-wave scalar amplitude and

\[
F_{n,k}(a) = \frac{1}{2\pi W} \int_{-W/2}^{W/2} \sin \left( \frac{n\pi y}{W} \right) dy \int_{-\infty}^{\infty} ae^{-ikzd} dz
\]

denotes a 2D Fourier transform which is continuous along the stripe \( k_z \) takes continuous values in the model) and discrete in the transverse direction \( k_y = n\pi/W, n = 1, 2, 3, \ldots \). This term is responsible for nonlinear coupling of Fourier components with different values of \( n \) and \( k_z \). \( \Gamma \) denotes the coefficient of spin-wave linear damping for the medium and \( f_{n,k}(t) \) is the Fourier transform of the time envelope of the driving microwave field which excites the spin-wave packet at the entrance into the waveguide medium. Similar to the experiment, \( f_{n,k}(t) \) is taken in the form of a 20 ns-long rectangular pulse. The carrier frequency of this driving pulse is \( \omega_I/2\pi \) (see Fig. 2).

The term \( \omega_{n,k} = \omega(k_y,k_z) \) is the 2D spectrum of spin waves, shown in Fig. 2 as \( k_z \)-dependence for a family of different discrete values of \( n \) which we take here equal to \( n\pi/W \). The calculated dispersion corresponds to the conditions of our experiment. Physically this \((n,k_z)\)-representation describes a family of guided modes for a film waveguide. The transverse profiles of the width modes are given by \( \sin(n\pi y/W) \) with the integer number \( n \) indicating each particular mode. One sees that the dispersion slope \( \partial \omega_{n,k}/\partial k_z \) is negative for all the modes. This reflects the fact that the modes are of backward type. Consequently, a localized wave packet which propagates in the positive direction of the axis \( z \) has a carrier wave number \( k_z < 0 \).

As previously shown for both 1D \([2]\) and isotropic 2D media \([6]\), to ensure radiation from nonlinear waveforms it is important to have a dispersion law for a medium which contains terms higher than parabolic. More precisely, the curvature \( \partial^2 \omega/\partial k^2 \) should change sign along the dispersion curve. From Fig. 2 one sees that, on one hand, for each particular width mode the sign of \( \partial^2 \omega_{n,k}/\partial k_z^2 \) changes from negative to positive in the vicinity of \( k_z = 0 \). On the other hand, the spectrum in Fig. 2 is quite dense. Considering this spectrum as quasi-continuous one retrieves the 2D continuous dispersion of plane spin waves in ferromagnetic films \([19]\) which is characterized by the variation of the dispersion sign \( \partial^2 \omega/\partial k^2 \) as a function of the propagation angle with respect to the direction of the applied field \( \mathbf{H} \) \( (k^2 = k_y^2 + k_z^2, \phi = \arctan(k_y/k_z)) \). Waves excited in this medium by an immobile point source are prone to forming caustics \([13]\). All this suggests that irradiation from a bullet may indeed be possible and may take the form of narrow beams at particular angles to the bullet propagation direction.
Fig. 2: Spectrum of the guided modes for the longitudinally magnetized film waveguide. The calculation assumes pinned spins at the stripe edges. The family of symmetric modes $n = 1, 3, ... , 65$ is shown. The mode which is the lowest in frequency is the fundamental width mode for the waveguide $n = 1$. Inset: a section of the spectrum close to the frequency $\omega_B/2\pi$. The horizontal dashed lines in the main field of the figure and in the inset show the carrier frequency $\omega_0/2\pi$ of the input microwave pulse. The oblique lines show the Doppler-shifted frequencies of an excitation source which moves in the positive direction. The bold sections of these lines show the modes responsible for formation of the respective caustics.

Our simulations confirm the validity of Karpman’s idea that the sign of dispersion should vary along the dispersion curve in order to generate a wave which is not trapped by the original nonlinear wave packet. The result is shown in Fig. 3a (same parameters as for Fig. 2 and standard for YIG films have been used in this calculation.) One sees good agreement with the experiment. In particular, we obtain the same radiation angle of 64° as in the experiment. Important here is that in order to obtain the correct collapse scenario we have to include all the modes shown in Fig. 2 into the simulation. Since in our experiment the carrier frequency for the excitation pulse is located about 45 MHz below the frequency of the in-plane ferromagnetic resonance $\omega_B/2\pi$ (Fig. 2), it would be natural to exclude the modes located above the gap $\omega = \omega_0$ from the calculation. However, if we limit the wave packet spectrum in this way the simulation delivers a completely different collapse scenario. In particular, in this case the velocity of the collapsing bullet is zero and beams are irradiated at quite different angles. This fundamental difference in simulated scenarios evidences the crucial importance of the specific form of the 2D dispersion relation for the collapse dynamics.

Now let us understand the specific angle for the radiation indicated both in Fig. 1 and in Fig. 3a. In the following we will show that the respective directions are the caustic angle directions which have been rotated because the excitation source is moving. First one notices that a collapsing spin-wave bullet in a magnetically saturated ferromagnetic film meets the size criterion $d < 2\pi/k\zeta$ for a quasi-point source of linear excitation of caustic waves. The source moves with the velocity of the bullet $v = 2.5$ cm/µs. A point source located on the axis of the waveguide will excite all waveguide modes which have an anti-node on the axis. These are the symmetric modes $n = 1, 3, 5, ...$. If the source were immobile, the frequency of the excited waves would be equal to the frequency of the source. This condition is indicated by the dashed horizontal line in Fig. 2: all the symmetric modes whose dispersion lines cross the dashed line will be excited with the frequency given by the
Fig. 3: Numerical simulation of irradiation of caustic beams from the collapsing bullet a), slowness curves for the immobile and the mobile excitation sources b) and the respective angles c). In b): bold solid lines are the slowness curves for the frequency given by the horizontal dashed line in Fig. 2. Thin solid lines are the slowness curves for the Doppler-shifted frequencies, as explained in the text. The bold dashed line connects zero-curvature points of the thin solid lines. Dark arrows normal to the thick solid line show the caustic directions for excitation by an immobile source. Grey arrows normal to the family of the thin solid lines show the directions of irradiation by the moving source. The arrows are located at the points of the slowness curves, where their curvature vanishes. In c): the arrow indicates the direction of source motion. Thin solid lines: caustic directions for the immobile source. Dotted lines: directions of energy radiation by the moving source. Dashed lines: instant directions along the irradiated beams.

ordinate of the point of the cross-section. It is a short exercise [14] to find which of the modes are responsible for formation of the standard caustics: those modes are shown by the bold part of the dashed horizontal line.

The moving source is described in Fig. 2 by the solid oblique line $\omega = \omega_0 + \nu k_z$ which can be considered as the time-space Fourier transform of the moving point source (see e.g. Eq. (14) in [20]). The term $\nu k_z$ is obviously the Doppler frequency shift for the excited modes. One sees that now each mode is excited with its own frequency. Furthermore, one sees that crossings are possible only for positive $k_z$ values. Importantly, our calculations show that, similar to the previous case of the immobile source, the modes which satisfy the Doppler shift condition $\omega_n(k_z) = \omega_0 + \nu k_z$ are also able to form caustic beams. The family of modes which are found to be responsible for formation of the modified caustics is shown by the bold section of the oblique line.

The modified caustic angles are obtained by considering the family of slowness curves for the Doppler-shifted frequencies. A slowness curve is a constant-frequency line in the $(k_y, k_z)$ plane (Fig. 3b) calculated for a film which is continuous in both in-plane directions [15]. The group velocities of all plane waves which exist at the respective frequency are directed perpendicular to this curve. For this reason the caustic direction is given by the normal to the slowness curve at the point where its curvature is zero. A range of frequencies for which the slowness directions are close to each other will contribute to formation of the modified caustics. The thin solid lines in Fig. 3b are the slowness directions for the frequencies which correspond to the crossing points of the oblique solid line with the dispersion lines in Fig. 2. The bold dashed line runs across zeros of curvature for these curves. One sees that normals to the thin solid lines at these points point in the same direction which suggests that this direction is the modified caustic direction. By calculating the direction of the maximum energy flow [15] for this set of slowness curves one finds a rigorous value for the angles between the caustic beams and the bullet velocity: $\pm 136.5^\circ$ (Fig. 3c).
First one sees that the angles exceed 90° which reflects the fact that the carrier waves are the waves of backward nature. (Note that the Doppler frequency shift for backward waves is anomalous \([21,22]\).) Second one notices that the angle between the two caustics \(\phi_M = 87^\circ\) is increased by just 3° with respect to \(\phi_I = 84^\circ\) for the excitation by an immobile source (Fig. 3c). However, this is not the full story yet, since \(\phi_M\) is the angle at which the energy is irradiated by the moving source, but not the instant direction along the irradiated beam. The angle correction due to this effect depends on the source velocity \(v\) solely, and is the same for waves of backward and forward types. Upon introducing this correction one finds a net angle between the two backwards-irradiated beams. The net angle is 64° which is in the excellent agreement with the experimental data and results of numerical simulation shown in figures 1 and 3a, respectively.

In conclusion, we experimentally studied the collapse scenario for an intense two-dimensional wave packet in a ferromagnetic medium with cubic nonlinearity and induced uniaxial anisotropy. We showed that before being self-destroyed the wave packet irradiates narrow-aperture beams of continuous waves at very specific angles to its propagation direction. We believe that this effect is of fundamental importance for physics of nonlinear waves and may exist for other types of wave excitations in other materials. Nonlinear 2D-media from a broad class which is characterized by cubic nonlinearity and 2D-dispersion of an order higher than parabolic, provided uniaxial anisotropy of dispersion is available or induced in the medium should be suitable candidates to observe the effect.

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References