Theory of a frequency-shifted feedback laser

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Abstract

We present a new approach to the description of the output from a frequency-shifted feedback (FSF) laser seeded by a phase-fluctuating but stationary continuous-wave (CW) laser. We illustrate the new analysis by showing how short frequency-chirped pulses arise for appropriate operating conditions. We show the equivalence of two common viewpoints of the FSF laser output as either a moving comb of equidistant frequencies or as a fixed set of discrete frequencies. We also consider operation of a FSF laser when there is no external seeding laser, and instead the cavity radiation originates with spontaneous emission.

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1. Introduction

Conventional lasers rely on multiple passes through a gain medium to reinforce a preselected frequency, thereby obtaining near-monochromatic output. For many purposes, such as excitation of Doppler-broadened atomic transitions, it is desirable to have light spread over a broader range of frequencies. One means of fulfilling that objective is to introduce a frequency shifting element into the laser feedback loop such that successive passes of a wavepacket take place with different carrier frequencies. The operation of such a frequency shifting feedback (FSF) laser has been considered by several research groups [1–46] and it has been used for a number of practical applications [12,21,36,41,47,48]. Although much is known about the behavior of FSF lasers and some of their surprising properties (see e.g. [20,24]), there remain some fundamental questions. Even the precise nature of the field emerging from such a laser remains a matter for discussion and confusion [33].

Although all authors agree on the elements that comprise a FSF device, there are conflicting approaches concerning detailed description of the electric field $E(t)$ that emerges from the device. In part this is because a FSF laser can operate in a variety of regimes: the output may range from
extreme irregularity to very regular pulse trains (cf. [24,46]), depending on such controllable properties as the amount of gain (controlled by the pump power) and the relative values of various time constants or frequency bandwidths of the device.

In this paper we reexamine some of the properties of this device, as deduced from equations that describe the time dependence of the (complex-valued) electric field amplitude. We also provide further clarification of the relationship between two apparently conflicting views of the output electric field. We are here concerned with steady-state operation, and will not consider the very interesting possibilities of irregular behavior.

The terms “steady state” and “stationary” deserve some comment when applied to pulsed lasers. As an example, one can consider the operation of a conventional mode-locked laser whose output consists of a train of short pulses. On a short time scale (that of one pulse) the intensity, and even the frequency, depends on time and thus on this time scale the output appears nonstationary. Nevertheless, on a time scale that encompasses many pulses the properties of the output is unchanged, and one can consider this as a stationary process.

We begin, in Section 2, with a general description of a prototype FSF layout. In Section 3 we derive equations of motion describing the two coupled systems that comprise the laser, namely the electric field (including both a seeding field and spontaneous emission as well as the stimulated emission) and the atomic inversion. Our field equations deal with complex-valued amplitudes, not just with photon numbers, as in rate equation treatments (e.g. [24]). Thus our formulation of the FSF laser equations includes phases. These are important for the injection seeded FSF: all the pulse chirping effects considered in Sections 4–6 rely on phase relationships. For the FSF laser seeded by spontaneous emission phase is not important.

In Section 4 we present solutions to these equations, allowing for seeding by a noisy but otherwise continuous-wave signal. We show here some special cases and some specific examples, and we reconcile (see Section 6.2) two common but apparently conflicting views of the FSF laser (as discrete frequencies, Section 5 or as a moving comb of frequencies, Section 6). Section 7 briefly discusses operation of a FSF laser whose field originates entirely from spontaneous emission, and Section 8 connects the present formalism to the earlier rate-equation model.

2. The FSF laser layout

We consider a simple idealized model of a frequency-shifted feedback laser: a ring cavity of perimeter $L$ in which there is a section having gain, some elements that provide spectral filtering, and, most importantly, an element that induces a frequency shift (of angular frequency $\Delta$) on each wavepacket that passes through it. The round trip time is $\tau = L/c$. For definiteness we assume that the frequency shift is provided by an acousto-optic modulator (AOM) and that this acts as a grating, to deflect light (into first order) that has been frequency shifted from $\omega$ to $\omega + \Delta$. The light that remains in the undeflected beam (zero-order diffraction) leaves the cavity as output.

We follow common practice and idealize the cavity field as having spatial variation only along the longitudinal coordinate $z$ (i.e. we unwind the mirror action). We take the entrance face of the AOM to be $z = 0$ and regard the various optical elements, such as the gain medium and the AOM, as having negligible length.

Fig. 1 shows a prototype schematic layout for a seeded FSF cavity. The input is through the AOM and the output is light that is undeflected by the AOM. With each pass around the cavity some light is lost. The most important loss sources are

![Fig. 1. Schematic diagram of a ring cavity showing mirrors (M), frequency shifting AOM, amplifier (gain), as well as input (seed) and output beams.](image)
the mirror transmittance and the loss to the output beam (through the AOM), both of which are frequency dependent. Often the cavity will include a filter that introduces controllable frequency-dependent loss. To simplify the presentation and to avoid unnecessary model-dependent details, we will consider only the case of a cavity whose loss varies negligibly within the frequency interval $\Delta$. In this case the ordering of the elements (AOM, mirrors, broadband gain and filter) are not important, and we can lump all losses into one element, which we call the filter. We consider this to be infinitesimally thin, time independent, and to decrease the field amplitude entering it with frequency $\omega$ by a factor $f(\omega) = \exp\left(-\frac{\omega - \omega_f}{\Gamma_f}\right)$. We assume that the filter function $f(\omega)$ varies slowly with frequency. Such broadband operation is found in almost all experimentally realized FSF lasers, and it is a prerequisite for use in high-resolution interferometry. In keeping with this requirement, we assume that the effect of the losses can be expressed in quadratic form

$$ f(\omega) = f_m + \frac{1}{2} \left( \frac{\omega - \omega_f}{\Gamma_f} \right)^2. $$

(1)

The filter function $f(\omega)$ has its minimum value, $f_m$, when $\omega = \omega_f$, and it increases from that value quadratically with the frequency difference $\omega - \omega_f$. The parameter $\Gamma_f$ characterizes the bandwidth of the filter while the central frequency $\omega_f$ provides a convenient fiducial for defining frequency detunings.

In order to sustain operation and produce a steady-state output, the cavity must include a section having gain, where amplification replaces the light lost to the output beam. As with the loss, we consider the gain to be localized in an infinitesimally thin element, having the property that a field entering it with frequency $\omega$ will be increased in amplitude by a factor $\exp[\mathcal{J}(\omega)]$. We assume that the gain $g(\omega)$, like the loss $f(\omega)$, varies slowly and smoothly with frequency.

### 3. The coupled field-atom equations

As in all dynamical descriptions of a laser, we require an equation relating the time evolution of the electric field amplitude in the presence of polarizable atoms (some form of the Maxwell equations), and equations describing the time evolution of atomic properties in the presence of a given field (some form of a density-matrix equation). We here derive the needed coupled equations by considering the passage of a pulse through the region of gain (the amplifier), and then considering the effects of the filter and the frequency shift.

Our model of the FSF laser differs in essential respects from some previous models, and readers familiar with those may question how one can consider the output of the FSF laser to be stationary. Where we to consider (as we do not) just a single pulse in the FSF laser, then this pulse will evolve with each round trip, changing frequency repetitively and undergoing growth through gain. The output in such a scenario is clearly not stationary, for each output pulse differs in amplitude and frequency from all others.

Although it is useful to describe the frequency shifting properties of a FSF laser by following the succession of frequency shifts occurring to a specific pulse, our mathematical treatment does not follow such a single pulse. Instead we consider a system in which the FSF laser is continuously seeded, either by spontaneous emission or by continuous injection of a seeding laser. We then examine the characteristics of the FSF output after startup transients have ceased. Because the seeding source is assumed to be stationary the FSF output, after some initial transient time development, is also stationary; operation of the FSF laser reaches a steady state in which the statistical properties of the output signal do not change.

We assume that the properties of the filters, though frequency dependent, are independent of time and of the intracavity field. However, we allow the gain to be influenced by both the pump power and by the cavity field; time variations of the cavity field will then induce time-dependent changes in the gain.

We assume that the relaxation time of the gain-medium polarization is much shorter than all other relaxation times under consideration, so that we can use standard techniques of adiabatic elimination and neglect the time derivative of the medium polarization. With this approximation we deal
only with rate equations for number densities or probabilities (populations), rather than with the Bloch-type equation for a density matrix. We assume that the population is spatially uniform within the (infinitesimally thin) gain medium, and that during one round trip within the cavity there is at most only a small change in the population (and so the gain can change only slowly with time).

In keeping with our interest in Ti:Sapphire as the gain medium we idealize the amplifier as a collection of identical four-level atoms (cf. [49], whose notation we follow), with the lasing transition between levels 1 and 2, while the pump radiation moves population between levels 0 and 3. Rapid decay from level 3 fills the upper lasing level 2, and rapid decay from level 1 to the ground state 0 ensures that the lower-level probability \( \rho_1(t) \) is negligible, and the population inversion \( w(t) \equiv \rho_2(t) - \rho_1(t) \) can be replaced by the population \( \rho_2(t) \).

The FSF laser is a system where the Fourier transform in its normal form does not appear particularly useful. As one learns in Fourier analysis, sampling a signal for a time interval \( \tau \) gives a valid representation of the spectral properties of the signal if \( \tau \) is long enough so that the effect of the discontinuities at the beginning and end of the segment are insignificant. Then the Fourier transform, from which one evaluates the spectrum, will be independent of any change in \( \tau \).

Our goal is different. We want to obtain equations which describe all regimes of FSF laser operation – steady state, transient, or pulse evolution.

### 3.1. The field amplitude equation

We take the center frequency of the effective filter, \( \omega_c \), as a carrier frequency and express the time dependence of the complex-valued electric field \( E(t) \) at position \( z = 0 \), taken to be just prior to the AOM in a path around the cavity, by the complex envelope \( \delta(t) \)

\[
E(t) = \delta(t) \exp(-i\omega_c t).
\]

The FSF laser is one of many examples where the frequency content of a signal changes with time and conventional Fourier transform methods are not sufficient. This problem, of explicitly time-dependent spectra, has been discussed by Eberly and Wodkiewicz [51]. We overcome the difficulties by using a sliding-window Fourier transform (SWFT). For any fixed fiducial reference time \( T \) we introduce a Fourier decomposition of the envelope \( \delta(t) \), valid in the time window \( T - \tau < t < T \),

\[
\delta(t) = \int_{-\infty}^{\infty} d\omega \delta(\omega, T) \exp(-i\omega t)
\]

for \( T - \tau < t < T \).

Note that the Fourier frequency \( \omega \) is not the optical frequency; it is a detuning from the carrier frequency \( \omega_c \). The windowed Fourier components are obtained from the integral

\[
\delta(\omega, T) = \frac{1}{2\pi} \int_{T-\tau}^{T} dt \delta(t) \exp(+i\omega t).
\]

It should be understood that our choice of the window duration as the round trip time is not related to any requirements of Fourier transforms, nor is it to impose any phase closure condition, for example, a requirement that accumulated phase during a round trip must be an integral multiple of \( 2\pi \).

Because we deal with a SWFT rather than a conventional FT, the value of the frequency shift relative to the fiducial time \( T \) is not important. We obtain a set of finite-difference equations which can be used to describe the FSF laser. The smallness of the frequency shift can subsequently be used to simplify these finite-difference equations as approximate differential equations.

The intensity \( I(t) \) at \( z = 0 \) during the time interval \( T - \tau < t < T \) is expressible either as the absolute square of the time-dependent electric field envelope or as a double Fourier integral

\[
I(t) = \frac{c}{8\pi} |\delta(t)|^2
\]

\[
= \frac{c}{8\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega_1 d\omega_2 \delta(\omega_1, T)\delta(\omega_2, T)^* \times \exp[-i(\omega_1 - \omega_2)t].
\]

The windowed Fourier components \( \delta(\omega, T) \), based on windowed Fourier analysis, serve as the basic quantities for theory; we derive the desired equations by considering the connection between \( \delta(\omega, T) \) and \( \delta(\omega, T - \tau) \). Prior to arriving at the reference position, this Fourier component \( \delta(\omega, T) \)
has made a circuit in which it was filtered, underwent a frequency shift, and underwent gain. It began that circuit with frequency \( \sigma - \Delta \) at time \( T - \tau_r \), and after immediately shifting frequency it acquired a phase increment \( (\omega_t + \sigma) \tau_r \) from spatial propagation. We express this behavior by writing
\[
\epsilon'(\sigma, T) = \delta(\sigma - \Delta, T - \tau_r) \exp\left[i(\omega_t + \sigma) \tau_r + g(\omega_t + \sigma, T) - f(\omega_t + \sigma)\right].
\]  
We assume that there is little change in the gain during one round trip and we assume the filter has only a slight frequency dependence
\[
g(\omega_t + \sigma, T - \tau_r) \simeq g(\omega_t + \sigma, T),
\]
\[
f(\omega_t + \sigma + \Delta) \simeq f(\omega_t + \sigma).
\]
To emphasize the discreteness of the periodic frequency shift it proves useful to consider a two-dimensional time-frequency space
\[
\vec{X} = (\sigma, T),
\]
in which successive circulations involve a change by the vector \( \vec{X}_0 = (\Delta, \tau_r) \). Then the basic equation (6) for the field change in one round trip can be written as
\[
\epsilon'(\vec{X}) = \delta(\vec{X} - \vec{X}_0) \exp\left[G(\vec{X})\right] + \epsilon(\vec{X}) + \xi(\vec{X}),
\]
where the effect of gain (and loss) is present through the function
\[
G(\vec{X}) = i(\omega_t + \sigma) \tau_r + g(\omega_t + \sigma, T) - f(\omega_t + \sigma)
\]
and the two additional terms represent various sources of radiation (in Fourier space):
- \( \epsilon(\vec{X}) \) is the seeding field source (if present),
- \( \xi(\vec{X}) \) is the source of the spontaneous emission field during a time interval \( \tau_r \).

Eqs. (8) and (9) are our basic equations for the field amplitude emerging from the FSF cavity. Missing from our treatment is any explicit non-linearity, such as four-wave mixing due to Kerr effect or gain saturation. Because we neglect such non-linearities our resulting theory cannot explain results such as those of Ito and coworkers [29,32] where the chirped comb (though a very small component of the FSF laser output) was observed in an unseeded FSF laser.

The source terms have the following properties.

### 3.1.1. The seed laser source

The seeding laser produces the source term \( \epsilon(\vec{X}) \). We take this to be a source of continuous radiation at the frequency
\[
\omega_s = \omega_t + \sigma_s
\]
(that is, \( \sigma_s \) is a frequency offset from the central filter frequency \( \omega_t \) of the carrier field). We allow a finite bandwidth to this laser by taking the windowed Fourier transform to have the form
\[
\epsilon(\vec{X}) = \epsilon_c \frac{1}{2\pi} \int_{T - \tau_r}^{T} dt \exp[-i\sigma_r t - i\varphi_s(t) + i\sigma t].
\]

The time varying phase \( \varphi_s(t) \) appearing here represents the phase of the seed laser. To permit treatment of the unavoidable finite bandwidth of any seed laser we treat this phase as a stochastic process. The quantity \( \epsilon_c \) is the amplitude of the seeding laser field within the cavity. It parametrizes the seeded intensity \( I_c \) within the cavity,
\[
I_c = \frac{c}{8\pi} |\epsilon_c|^2.
\]

### 3.1.2. Spontaneous emission source

Spontaneous emission (within the gain medium) is described by the term \( \xi(\vec{X}) \). This is similar to a Langevin force [50]. This frequency domain source term relates to the time domain source via Fourier integral within one round trip time
\[
\xi(\vec{X}) = \xi(\sigma, T) = \frac{1}{2\pi} \int_{T - \tau_r}^{T} dt \xi_{sp}(t) \exp(i\sigma t).
\]

The time-varying emission source \( \xi_{sp}(t) \) describes the field created from spontaneous emission events. These occur at random times, and successive emissions are completely uncorrelated. This property will be quantified in Section 7.

### 3.2. The population inversion equation

To complete the description of the atom-field system we require an equation for the time dependence of the gain coefficient \( g(\omega, T) \). This is
directly proportional to the population inversion $w(t) = P_2(t) - P_1(t) \approx P_2(t)$ of the lasing transition (assuming the two levels have equal degeneracies). We express this by the equation

$$g(\omega, T) = g_0(\omega)w(T).$$  \hspace{1cm} (14)

We take the frequency dependence of $g_0(\omega)$ to be of Lorentzian form

$$g_0(\omega) = \frac{g_m}{1 + (\omega - \omega_g)^2/I_g^2},$$  \hspace{1cm} (15)

where $\omega_g$ is the frequency of maximum gain and $1/I_g$ is the relaxation time for the off-diagonal terms of the density matrix.

To avoid unnecessarily complicated formulas we assume that the width of the frequency-dependent loss, as incorporated into the filter $f(\omega)$, is much smaller than the gain width: $I_g \gg I_\epsilon$. With this assumption we can regard the gain as being independent of frequency, and set $g_0(\omega) = g_m$. This simplification is not essential, and the results we present are readily adapted to the case where only the gain, but not the filter, has significant frequency dependence. For simplification we have taken $g_0(\omega)$ and $f(\omega)$ to be purely real functions of frequency. It can be shown that, in the quadratic approximation, the inclusion of imaginary parts (dispersion) would lead only to a redefinition of the round trip time.

Within the approximations discussed here, the inversion $w(T)$ changes from its value at the earlier time $T - \tau_r$ due to the following effects, each of which acts for time $\tau_r$:

- growth by pumping (into level 3 with rapid decay to 2) at rate $R$;
- loss by spontaneous emission at rate $\gamma_s w(T - \tau_r)$;
- loss by stimulated emission at rate $\gamma_s f(t)$.

Taking these effects into account we write the following expression for the inversion after one round-trip time:

$$w(T) = w(T - \tau_r) \left[1 - \gamma_s \int_{T - \tau_r}^{T} \frac{I(t)}{I_{sat}} \, dt - \gamma_s \tau_r \right] + R \tau_r.$$  \hspace{1cm} (16)

In the absence of lasing, $I(t) = 0$, this equation gives the stationary inversion $w(T) = w(T - \tau_r) = w_m = R/\gamma_s$. Then the inversion equation reads

$$\frac{w(T) - w(T - \tau_r)}{\tau_r} = -\gamma_s \left[w(T - \tau_r) - w_m + w(T - \tau_r) \int_{T - \tau_r}^{T} \frac{I(t)}{I_{sat}} \, dt \right].$$  \hspace{1cm} (17)

In traditional treatments of population inversion one considers the limiting case of an infinitesimal time increment (here the fixed increment $\tau_r$)

$$\frac{w(T) - w(T - \tau_r)}{\tau_r} \rightarrow \frac{dw}{dT}$$

and thereby obtains an equation for the time derivative of the inversion. However, in the present work the discreteness of the round trip time is an essential part of the physics of the field, and so we here retain also a similar discrete form for the evolution equation for the inversion.

The time integral over intensity can be taken as the definition of the average intensity during a round trip

$$I(T) = \frac{1}{\tau_r} \int_{T - \tau_r}^{T} \, dt I(t) \equiv \int_{-\infty}^{\infty} \, d\sigma I(\sigma, T),$$  \hspace{1cm} (18)

where, in the frequency domain,

$$I(\sigma, T) = \frac{c}{8\pi} \int_{-\infty}^{\infty} \, d\sigma' \delta(\sigma, T) \delta(\sigma + \sigma', T)^* \times \frac{1 - \exp[-i\sigma'\tau_r]}{i\sigma'\tau_r} \exp[i\sigma' T].$$  \hspace{1cm} (19)

With this definition we can write the equation for the inversion as

$$w(T) = w(T - \tau_r) \left[1 - \gamma_s \frac{I(T)}{I_{sat}} - \gamma_s \tau_r \right] + R \tau_r.$$  \hspace{1cm} (20)

This equation completes the necessary set of equations for the FSF laser.

### 3.3. Steady-state operation

We are interested in steady-state operation of the FSF laser, as contrasted with irregular or transient behavior. We therefore carry out a stochastic average $\langle \cdots \rangle$ of the inversion and the average intensity.
\[ \bar{w} \equiv \langle w(T) \rangle = \langle w(T - \tau_i) \rangle, \quad \bar{I} \equiv \langle I(T) \rangle. \quad (21) \]

This step rules out possible relaxation oscillations of the inversion, but it does not require that the phase be constant. With this approximation we deal with the average gain coefficient \( g_m \bar{w} \).

In this work we consider a laser whose spectrum has a width much larger than the free spectral range \( \Delta \omega_{\text{axial}} = 2\pi/\tau_r \). Under this restriction we can replace \( w(T) \) and \( w(T + \tau_r) \) with \( \bar{w} \) and \( \bar{I} \) in Eq. (20). Then we find that the stationary average inversion \( \bar{w} \) and the average intensity \( \bar{I} \) are related by the equation

\[ \bar{w} = \frac{w_m}{1 + I/I_{\text{sat}}}. \quad (22) \]

### 3.4. Solving the coupled equations

The dynamics of a laser with frequency-shifted feedback can be described by two stochastic finite-difference equations: Eq. (8) for the time-dependent spectral component \( \mathcal{E}(\tilde{X}) \) and Eq. (20) for the inversion \( w(T) \). The procedure for solving the equations is as follows.

1. We find the solution \( \mathcal{E}(\tilde{X}) \) of Eq. (8) with \( \bar{w} \) treated as an unknown parameter.

2. Using \( \mathcal{E}(\tilde{X}) \) in Eq. (19) we obtain \( I(\sigma, T) \).

3. We do the stochastic averaging of \( I(T) \) over realizations of the spontaneous emission source \( \zeta(\tilde{X}) \) and (or) realizations of the stochastic phase \( \varphi_s(t) \) of the seeding laser, and from these we obtain \( \bar{I} \).

4. We solve Eq. (22) for \( \bar{w} \).

5. Using the expression obtained for \( \bar{w} \) we obtain \( \mathcal{E}(\tilde{X}) \) and the intensity \( I(T) \) of items 1 and 2. We also obtain the spectral density of the intensity, \( \Gamma(\sigma, T) \).

The following sections discuss implementations of this program with various approximations and simplifications for the following important cases:

- A laser with injection seeding, but without spontaneous emission, \( \varepsilon(\tilde{X}) \neq 0, \zeta(\tilde{X}) = 0 \).
- A laser with spontaneous emission, but without seeding, \( \varepsilon(\tilde{X}) = 0, \zeta(\tilde{X}) \neq 0 \).

### 4. The FSF laser with (noisy) seeding

An important simple case of the FSF laser occurs when a seeding laser provides the only source term. We take this to be a CW laser, of frequency \( \omega_s \), but we allow finite bandwidth by including phase noise \( \varphi_s(t) \).

#### 4.1. The electric field

We here neglect the spontaneous emission source \( \zeta(\tilde{X}) \), so that Fourier components of the laser field are determined by the equation

\[ \mathcal{E}(\tilde{X}) = \mathcal{E}(\tilde{X} - \tilde{X}_0) \exp \left( \Gamma(\tilde{X}) \right) + \varepsilon(\tilde{X}). \quad (23) \]

Here we use the function \( G(\tilde{X}) \), defined by Eq. (9), with stationary and frequency-independent gain coefficient \( g_m \bar{w} \), so that it becomes

\[ G(\tilde{X}) = i(\omega_s + \omega_r) \tau_r + g_m \bar{w} - f(\omega_s + \omega_r). \quad (24) \]

As in Eq. (11) we take \( \varepsilon(\tilde{X}) \) to have only phase variation (i.e. constant amplitude).

The solution to the difference equation (23) is the infinite sum

\[ \mathcal{E}(\tilde{X}) = \mathcal{E}(\tilde{X}) + \sum_{n=1}^{\infty} \exp \left( \sum_{l=0}^{n-1} G(\tilde{X} - l\tilde{X}_0) \right) \times \varepsilon(\tilde{X} - n\tilde{X}_0), \quad (25) \]

where

\[ \varepsilon(\tilde{X} - n\tilde{X}_0) = \frac{v_c}{2\pi} \int_{T-n\tau_r}^{T-(n+1)\tau_r} \text{d}t' \exp \left[ -i\sigma t' - i\varphi_s(t') \right] + i(\sigma - n\Delta t'). \quad (26) \]

From the windowed Fourier transform of expression (25), over the time interval \( T - \tau_r < t < T \), one finds (after straightforward but lengthy algebra) the time-dependent envelope of the field to be

\[ \mathcal{E}(t) = v_c \exp[-i\sigma t - i\varphi_s(t)] + v_c \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \text{d}\sigma \exp(-i\sigma t) \times r_n(\sigma) \times q_n(\sigma), \quad (27) \]

where \( r_n(\sigma) \) incorporates all consequences of gain and loss.
The function \( q_n(\sigma) \) as defined earlier,

\[
\begin{align*}
r_n(\sigma) & \equiv \exp \left[ \ln(\omega_i + \sigma_k + [n + 1]A/2) \tau_f \\ & + \sum_{l=0}^{n-1} \left[ g_m \tilde{w} - f(\omega_i + \sigma - lA) \right] \right], \quad (28)
\end{align*}
\]

and \( q_n(\sigma) \) expresses all the effects of phase fluctuations (seed-laser bandwidth)

\[
q_n(\sigma) \equiv \frac{1}{2\pi} \int_{T-x} \exp[-\text{i}\phi_s(t - n\tau_f) + \text{i}(\sigma - \sigma_k - nA)t'] dt'. \quad (29)
\]

The function \( q_n(\sigma) \) is maximum near \( \sigma = \sigma_k + nA \). The width of this maximum is roughly \( 1/\tau_r + \delta\omega_s \), where \( 2\pi/\tau_r \) is the free spectral range and \( \delta\omega_s \) is the bandwidth of the seeding laser. We assume that the width of the filter \( \Gamma_f \) is much larger than these

\[
\Gamma_f \gg 1/\tau_r + \delta\omega_s. \quad (30)
\]

This equation says that the function \( r_n(\sigma) \) is a slowly varying function of the detuning \( \sigma \) compared with the narrow function \( q_n(\sigma) \). Thus one can replace \( r_n(\sigma) \) by its value at the point \( \sigma = \sigma_k + nA \). After integration over \( \sigma \) and using the expression (1) one can write the electric field envelope \( \delta(t) \) as a sum over terms with constant real-valued amplitudes \( a_n \)

\[
\delta(t) = \tilde{\nu}_c \sum_{n=0}^{\infty} a_n \exp[-\text{i}\Phi_n - \text{i}\Phi_s(t - n\tau_f) \\ - \text{i} (\sigma_k + nA)t] \quad (31)
\]

Section 4.2 provides a simple interpretation of this expression, and of the indexing integer \( n \). The constant real-valued amplitudes are expressible as \( a_n = \exp(S_n) \) with

\[
S_n = n \left[ g_m \tilde{w} - f_m \\ - \frac{6\sigma_k^2 + 6\sigma_k nA + 6\sigma_i A + 2n^2 A^2 + 3A^2 n + A^2}{12\tau_f^2} \right]. \quad (32)
\]

As defined earlier, \( f_m \) is the filter loss at maximum transmission (\( \omega = \omega_i \)) and \( \sigma_k \) is the detuning of the seed laser frequency from \( \omega_i \). The phase of the \( n \)th component has a stochastic portion \( \phi_s(t) \) and a time-independent part that is quadratic in \( n \)

\[
\Phi_n = -\tau_f n[(\omega_k + (n + 1)A/2)]. \quad (33)
\]

The steady population inversion \( \bar{\nu} \) affects only the amplitudes \( a_n \) of Eq. (31), not the phases of the Fourier expansion (31). The inversion depends upon the steady mean intensity \( \bar{I} \) through the condition of steady-state operation, Eq. (22). In turn, the mean intensity \( \bar{I} \)

\[
\bar{I} = \frac{c}{8\pi} |\delta(t)|^2 = I_c \sum_{n=0}^{\infty} |a_n|^2, \quad (34)
\]

involves \( \bar{w} \) through the dependence of \( a_n \) shown in Eq. (32). Eqs. (22) and (34) combine to give a single transcendental equation for either \( \bar{w} \) or \( \bar{I} \).

4.2. Simplified interpretation

For the specific case of a seeded FSF laser the expression (31) for the envelope \( \delta(t) \) of the intra-cavity field \( E(t) = \delta(t) \exp(-\text{i}\omega_i t) \) can be interpreted in the following way. At any time \( t \) the field \( E(t) \) inside the cavity at \( z = 0 \) is the superposition of fields entering the cavity at earlier times that underwent at least one round trip. For simplicity we assume the incident field to be perfectly monochromatic at frequency \( \omega_s \), meaning that there is no random phase, \( \phi_s(t) = 0 \). Then the incident seed field (prior to passing through the input mirror) can be written as

\[
E_{\text{in}}(t) = \delta_s \exp(-\text{i}\omega_s t). \quad (35)
\]

In this case the amplitude \( \tilde{\nu}_c \) of the seeding source is \( \tilde{\nu}_c = \delta_s \mathcal{T}_m \), where \( \mathcal{T}_m \) is the transmittance of the input mirror for the incoming light.

The cavity field entering the AOM at time \( t \) is expressible as the sum

\[
E(t) = \sum_{n=0}^{\infty} E_n(t), \quad (36)
\]

where component \( E_n(t) \) at time \( t \) has, by definition, made \( n \) round trips and has undergone \( n \) frequency shifts. The field emerging at time \( t \) as the output beam is evaluated by considering the transmittance of each beam, \( \mathcal{F}_n \equiv \mathcal{F}(\omega_k + nA) \), through the AOM without frequency shifting

\[
E_{\text{out}}(t) = \sum_{n=0}^{\infty} \mathcal{F}_n E_n(t). \quad (37)
\]
The amplitude of an individual component is determined by the effective gain $Q(\omega)$,

$$Q(\omega) = g_m w - f_m - \frac{\sigma^2}{2I_t} = Q_m \left(1 - \frac{\sigma^2}{\sigma_0^2}\right),$$  \hspace{1cm} (38)

defined as the difference between saturated gain and filter losses. Here $Q_m$ is the maximum of the effective gain and $\sigma_0$ is the larger of the two frequency detunings for which the effective gain is equal zero

$$Q_m \equiv g_m w - f_m, \quad \sigma_0 \equiv \sqrt{2Q_m I_t}. \hspace{1cm} (39)$$

It is well known that for a conventional laser, in CW operation, the saturated gain is equal to the loss. However for a FSF laser the gain is not saturated completely, even when the laser operates in the stationary regime. The field components can be evaluated from the following consideration. The component $E_n(t)$ entered the cavity at time $t - n\tau_t$, with frequency $\omega_n$. It decreased in amplitude by the factor $T_{in}$, shifted frequency by $\Delta n$, and underwent attenuation or gain [depending on the sign of the effective gain $Q(\omega_n + \Delta n)$] by a factor $R_1 = \exp(Q(\omega_n + 1.\Delta n)).$ On the next round trip it underwent frequency shift from $\omega_n + \Delta n$ to $\omega_n + 2\Delta n$ and again underwent attenuation or gain, this time by the factor $R_2 = \exp(Q(\omega_n + 2\Delta n)).$ It makes $n$ such round trips, on each of which it changes amplitude by a factor $R_n(\omega)$ and acquires a phase shift $\omega_n \tau_t$ because it propagates the distance $\tau_t / c$. It follows that the various components are

$$E_0(t) = \delta_s T_{in} \exp[-i\omega_0 t],$$  \hspace{1cm} (40)

$$E_1(t) = \delta_s T_{in} R_1 \exp[-i(\omega_n + \Delta) t] \exp[i(\omega_n + \Delta) \tau_t],$$  \hspace{1cm} (41)

$$E_2(t) = \delta_s T_{in} R_1 R_2 \exp[-i(\omega_n + 2\Delta) t] \exp[i(\omega_n + 2\Delta) \tau_t],$$  \hspace{1cm} (42)

$$\vdots$$

$$E_n(t) = \delta_s T_{in} R_1 R_2 \cdots R_n \exp[-i(\omega_n + n\Delta) t] \exp[i \sum_{m=1}^{n} (\omega_n + m\Delta) \tau_t],$$  \hspace{1cm} (43)

where the reflectivity $R_n$ is

$$R_n = \exp\left[Q_n - \frac{(\omega_n + n\Delta)^2}{2T_t}\right].$$  \hspace{1cm} (45)

The first exponential factor describes the time variation appropriate to frequency $\omega_n = \omega_n + n\Delta$. The last exponential factor describes the spatial phase $\sum_n (\omega_n/c)L$ acquired by propagating $n$ full round trips through the cavity, each of length $L = \tau_t / c$ but at a different frequency $\omega_n$.

Note that in the absence of any frequency shift, in the absence of the gain, and with $R(\omega_n) = R < 1$, the individual components have the form $E_n(t) = \delta_s T_{in} \exp[-i\omega_n t + i\omega_n \tau_t]$ and the field has the construction

$$E(t) = \delta_s T_{in} \exp[-i\omega_n t] \frac{1 - R \exp[-i\omega_n \tau_t]}{1 - R \exp[i\omega_n \tau_t].}$$  \hspace{1cm} (46)

The maximum of this field occurs when $\tau_t \omega_n = 2\pi q$, for any integer $q$. This is the standard Fabry–Perot cavity condition for axial modes.

5. The discrete frequency model

Comparison of the heuristically derived Eqs. (35) and (44) with the field presented in Eq. (31) is done by introducing an envelope $\delta_{DF}(t)$ as in Eq. (2),

$$E(t) = \exp[-i\omega_0 t] \delta_{DF}(t).$$  \hspace{1cm} (47)

Using Eq. (10) and the definitions (32) and (33) of $S_n$ and $\Phi_n$ we find

$$\delta_{DF}(t) = \sum_{n=0}^{\infty} \delta_s T_{in} \exp[S_n - i\Phi_n]$$

$$\times \exp[-i(\omega_n + n\Delta)t]$$

$$= \exp[-i\omega_0 t] \sum_{n=0}^{\infty} A_n \exp[-in\Delta t].$$  \hspace{1cm} (48)

This is just the amplitude of Eq. (31) with $\varphi_0 = 0$ (i.e. a monochromatic seed) and $\varepsilon_s = \delta_s T_{in}$.

The amplitude $\delta_{DF}(t)$ defined by the summation (48) is periodic, with the period $T_p = 2\pi / \Delta$, and hence the expression is in the form of a Floquet series: a time-dependent prefactor $\exp[-i\omega_0 t]$ multiplying a periodic function, expressed as a Fourier series with constant complex-valued
Fourier coefficients $A_n = e_n a_n \exp[-i\phi_n]$. Therefore the field in the seeded FSF cavity is a comb of discrete frequency components, separated by the increment $\Delta$, and starting from the seed frequency $\omega_s$. The phase $\phi_n$ varies quadratically with $n$, whereas the exponent of $S_n$, defined in Eq. (32), varies cubically. However, in practice the variation of $S_n$ is significant only near its maximum $S_{\text{max}} = S_{\text{max}}$ where it also is quadratic.

The magnitude of $A_n$ is governed by the real-valued exponent $S_n$ the sign of which is determined critically by the saturated inversion $\bar{w}$. In turn, $\bar{w}$ depends on the total intensity and hence on the seed detuning $\sigma_0$ and on other parameters of the laser. We will show in Section 5.1 that, for wide range of parameters, $\bar{w}$ is determined mainly by the filter width $F$ and the frequency shift $\Delta$. Using the definitions of Eq. (39) for the peak effective gain $Q_m$ and $\sigma_0$ we rewrite the real-valued exponent $S_n$ of Eq. (32) as

$$S_n = Q_m n \left(1 - \frac{6\sigma_0^2 + 6\sigma_0 n \Delta + 6\sigma_0 \Delta + 2n^2 \Delta^2 + 3\Delta^2 n + \Delta^2}{6\sigma_0^2}\right).$$

If the seeding laser is tuned near the lower frequency where the effective gain is zero, so $\sigma_s \gg -\sigma_0$, then starting from $n = 0$, the value of $S_n$ grows steadily with $n$ as long as the gain exceeds the loss, i.e. $Q(\sigma_0 + n\Delta) > 0$. Eventually, for $n > (\sigma_0 + |\sigma_s|)/\Delta$, the loss dominates the gain, and the exponent $S_n$ falls with further increase of $n$. Thus there is a maximum of $S_n$, determined from the solution of the equation $(d/dn)S_n = 0$. This value, $S_{\text{max}}$, occurs for the index $n = [n_{\text{max}}]$, where $[\ldots]$ means the integer part. For large $\sigma_0 \gg \Delta$ the peak occurs at

$$n_{\text{max}} \approx (\sigma_0 + |\sigma_s|)/\Delta.$$

This is a value of $n$ for which $Q(\sigma_0 + n\Delta) = 0$. The variation of the magnitude $a_n = \exp[S_n]$ of the frequency component with index $n$ is therefore very nearly of Gaussian shape centered around $n = n_{\text{max}}$; the exponential argument $S_n$ departs from quadratic form in $n$ only when $n$ is large or small compared with $n_{\text{max}}$.

Fig. 2(b) illustrates the variation of the components $a_n$ with $n$, normalized to unit value at the maximum. The peak of $a_n$ is very close to the location of the value $Q(\sigma_0 + n\Delta) = 0$. For reference, Fig. 2(a) shows the effective gain $Q(\sigma)$ associated with these components. Fig. 5 of [8] shows an experimentally observed spectrum which looks exactly like our Fig. 2(a).

5.1. Gaussian approximation

The $S_n$ of Eqs. (32) or (49) follows a quadratic dependence on $n$ for appropriate conditions. This gives the amplitude $a_n$ a Gaussian form. To identify those conditions, we consider the field given by Eq. (31) when the seeding laser is detuned far away from the central frequency of the laser spectrum. In this case the first frequency component is very
weak and the shape of the laser output spectrum is close to Gaussian. To quantify this relationship we express \( S_n \) as a Taylor series around \( n_{\text{max}} \), where \( dS_n/\text{d}n = 0 \), retaining only the quadratic term. The magnitudes \( a_n \) can then be written as

\[
a_n \simeq \mathcal{A} \exp \left[ - \left( \frac{n - n_{\text{max}}}{n_w} \right)^2 \right],
\]

where the width parameter \( n_w \) obeys the relationship

\[
(n_w)^2 = \frac{2(\Gamma_f)^2}{\sigma_0 \Delta}
\]

and \( n_{\text{max}} \) is given by Eq. (50). The common amplitude \( \mathcal{A} \) is expressible as

\[
\mathcal{A} = \exp \left[ \frac{2\sigma_0^3 + \sigma_3^3 - 3\sigma_3 \sigma_0^2}{6\gamma^3} \right],
\]

where the frequency parameter \( \gamma \) is defined as

\[
\gamma \equiv (\Delta \Gamma_f^2)^{1/3}.
\]

The Gaussian approximation of Eq. (51) is valid if \( n_w \) is large enough, meaning

\[
n_w \gg \Gamma_f^2 / \sigma_0^2.
\]

This condition depends, through \( \sigma_0 \) and Eq. (39), on the still-unknown saturated inversion \( \tilde{w} \),

\[
\tilde{w} = \frac{f_m}{g_m} + \left( \frac{\sigma_0^2}{2g_m(\Gamma_f)^2} \right).
\]

To find \( \tilde{w} \) we assume that the Gaussian approximation is valid, so that the mean number \( n_w \) of discrete components is large, \( n_w \gg 1 \). Then we replace the summation over \( n \) in Eq. (34) by an integration

\[
\tilde{I} = I_c \sigma f^2 \int_{-\infty}^{\infty} \text{d}x \exp \left( - \frac{2x^2}{n_w^2} \right) = I_c \sqrt{\frac{\pi}{2}} \sigma f^2 n_w.
\]

Taking into account the condition \( |\Delta| \ll \Gamma_f \), we have from Eq. (22) the result

\[
\exp \left[ \frac{2\sigma_0^3 + \sigma_3^3 - 3\sigma_3 \sigma_0^2}{3\gamma^3} \right] \sqrt{\frac{2\gamma}{\sigma_0}} = \frac{1}{\beta}.
\]

The small parameter \( \beta \ll 1 \) introduced here is

\[
\beta \equiv \sqrt{\frac{\pi}{2}} \frac{I_c}{2I_{\text{sat}}(\eta - 1)} \left( \frac{\Gamma_f}{\Delta} \right)^{2/3}
\]

with the threshold parameter \( \eta \) being defined as

\[
\eta = \frac{g_m}{f_m}.
\]

Eq. (58) gives an implicit relationship between the value of \( \sigma_0 \) (and hence of \( \tilde{w} \)) and the parameters \( \gamma \), \( \beta \) and \( \sigma_3 \). Fig. 3 shows the numerical solution of this equation for different values of detunings \( \sigma_3 \) of the seeding laser as a function of the parameter \( \beta \). The dependence on \( \beta \) is very weak.

A simple approximation, based on inspection of this figure, is to take the parameters of the Gaussian profile to be

\[
\sigma_0 \approx 3\gamma, \quad \tilde{I} \approx \sqrt{\frac{2}{3}\gamma}, \quad n_w \approx \sqrt{\frac{2}{3} \Delta}.
\]

With this approximation the amplitudes of the Fourier components depend only on the detuning \( \Delta \) and the filter bandwidth \( \Gamma_f \), through the combination \( \gamma \).

More detailed calculations, taking into account the parameters \( I_c \), \( I_s \) and \( \eta \), make only a small change. When \( \beta \) is extremely small (so \( 1/\ln(1/\beta) \) is also small) the transcendental equation (58) can be solved approximately for the specific choice of detuning \( \sigma_3 = 0 \). In this case one obtains the approximation

\[
\sigma_0 \approx \left[ \frac{3}{2} \ln \left( \frac{1}{\beta} \right) \right]^{1/3} \gamma.
\]

![Fig. 3. The shift \( \sigma_0 \) of the position of the spectrum maximum (solid lines) and the width \( \tilde{I} \) of the spectrum (dashed lines) in units of \( \gamma \) as a function of the parameter \( \beta \). The detuning of the seeding laser is: (1) \( \sigma_3 = -8\gamma \); (2) \( \sigma_3 = 0 \); (3) \( \sigma_3 = -2\gamma \).](image)
The maximum of the spectrum is located at
\[ \omega_{max} = \omega_c + \sigma_0. \]  
(63)

The mean number of components (the “width parameter” \( n_w \)) is given by the formula
\[ n_w = \frac{\sqrt{2}}{[(3/2) \ln(1/\beta)]^{1/6}} \Delta \]  
(64)
and the mean intensity \( I \) is
\[ I = I_{sat}(\eta - 1). \]  
(65)

The spectral width, \( \Delta \), and the mean intensity are both nearly independent of the seeding laser intensity.

We have derived here, from first principles, formulas for the width and the peak of the laser intensity. The right-hand side of Eq. (70) is a Floquet series representation of the FSF laser can be fully described.

5.2. Explicit Fourier formula

When the Gaussian approximation is justified for the amplitude of Eq. (48), the construction for the electric field \( E(t) \) reads
\[ E(t) = \exp[-i\omega_c \tau_c \omega_A] \times \sum_{n=0}^{\infty} \exp \left[ -\frac{(n - n_{max})^2}{n_w^2} \right] \times \exp[-in\Delta t]. \]  
(66)

Because the frequency width of the spectrum \( \Gamma \) is much smaller than the filter width \( \Gamma_c \) one can neglect the frequency dependence of the AOM efficiency near \( \omega_{max} \). Then the output field \( E_{out}(t) \) is directly proportional to the intracavity field \( E(t) \).

Because we assume \( n_w \) to be very large we can approximate \( n_{max} \) by the integer \( n_{max} \approx \lceil \sqrt{n_{max}} \rceil \). For purposes of display, the largest-magnitude Fourier component, which occurs for \( n = n_{max} \), serves as a useful reference point for expressing the others. We rewrite the argument of the exponential,
\[ S_n - i\Phi_n = -\frac{(n - n_{max})^2}{n_w^2} + i\tau_s [\omega_s + (n + 1)\Delta/2], \]
in terms of the difference between \( n \) and \( n_{max} \), using the variable \( m \)
\[ S_{n_{max} + m} = -(m/n_w)^2, \]  
(67)
\[ \Phi_{n_{max} + m} = \Phi_{n_{max}} - m[\omega_s \tau_t + (n_{max} + 1)\Delta \tau_t/2] - m^2(\Delta \tau_t/2). \]  
(68)
We eliminate the occurrence \( n_{max} \) from this formula by shifting the time \( t \rightarrow t - t_0 \) with an appropriately chosen initial time \( t_0 \) appropriately. The choice
\[ t_0 = \tau_t \left[(n_{max} + 1)/2 + \omega_s / \Delta \right] \]  
(69)
available when \( \Delta \neq 0 \), eliminates the linear variation with \( m \) from the phase, and eliminates also the dependence on the seed frequency \( \omega_s \). This choice leads to the expression
\[ E'(t) = \exp[-i(\omega_s + n_{max} \Delta)(t - t_0)] \times \sum_{m=-\infty}^{\infty} A'_m \exp[-im\Delta(t - t_0)] \]  
(70)
with
\[ A'_m = A_0 \exp[-(m/n_w)^2 - \text{Im}^2(\Delta \tau_t/2)], \]  
(71)
where the new normalization constant \( A_0 \) incorporates all constant phases and the factor \( \omega_c \). With this choice the seed frequency \( \omega_s \) appears only in the common time-dependent prefactor \( \exp[-i\omega_s(t - t_0)] \), but not in the summation. The prefactor does not affect the intensity of the output radiation, \( |E'(t)|^2 \), and it gives only a constant contribution to the instantaneous frequency \( \Omega(t) \), defined by
\[ \Omega(t) = \frac{d}{dt} \arctan \left( \text{Re}[E'(t)] / \text{Im}[E'(t)] \right). \]  
(72)
The right-hand side of Eq. (70) is a Floquet series in which the coefficients are Gaussians with a phase that varies quadratically with the index \( m \). Apart from normalization, the amplitudes depend on only two parameters: the phase change in one
round trip, $\Delta \tau_r$, and the width of the effective gain distribution, $n_w$.

### 5.3. Examples of pulsing output

It is instructive to examine illustrative examples of the output, Eq. (70). For that purpose we introduce the dimensionless time variable $x = t/T_p$, thereby expressing time in units of the basic repetition period $T_p = 2\pi/\Delta$. We then deal with the function

$$F(x) = F_0 \sum_{m=-M}^{+M} \exp[-im2\pi(x - x_{\text{ref}}) - (m/n_w)^2]
- im^2(\pi\tau_r/T_p)].$$  \hfill (73)

The function $F(x)$ depends on three parameter combinations: $n_w$, $x_{\text{ref}} = t_0/T_p$ and $\tau_r/T_p$. The first of these determines the width of the Gaussian distribution of Fourier amplitudes, and thereby controls the number of components that can add coherently. The relative time $x_{\text{ref}}$ merely adjusts the position of the pulse peaks along the $x$ axis. The most important parameter combination is $\tau_r/T_p = 2\pi\Delta \tau_r$. Whenever the parameter $\tau_r/T_p$ is an integer, the field is a sequence of short pulses, one per period. Such pulses are described by the function

$$F(x) = F_0 \sum_{m=-M}^{+M} \exp[-im2\pi(x - x_{\text{ref}}) - (m/n_w)^2].$$  \hfill (74)

The duration of an individual pulse is inversely proportional to the number of Fourier components that contribute significantly to the sum. The present work parameterizes this number by $n_w$, the width of the Gaussian distribution of Fourier components. Thus as the $n_w$ increases, the pulse durations can become shorter. The time-dependent frequency of such pulses, defined by Eq. (72), remains constant during the presence of a pulse.

Fig. 4 shows examples of these short pulses, for several values of $n_w$. Earlier such pulses were reported in [20]. For uniformity of plots we normalize the field by taking the quantity $1/F_0$ to be real and equal to the sum (we take the summation limits to be $M = 4n_w$)

![Fig. 4. The relative intensity $I(t)$ of the output field vs. time $t$, in units of the period $T_p = 2\pi/\Delta$. Plots are for $n_w = 10, 20, 30$ and $40$.](image)

$$1/F_0 = \sum_{m=-M}^{+M} \exp[-(m/n_w)^2].$$  \hfill (75)

More generally, trains of short pulses occur whenever $\tau_r/T_p$ is the ratio of two integers. These become less intense as they become more numerous in one period. Fig. 5 presents examples of such

![Fig. 5. Intensity $I$ vs. time $t$, in units of $T_p$, for values of $\tau_r/T_p = 1, 3/2, 4/3$, and $5/4$.](image)
pulses, for several choices of $s_r = T_p$. As the difference between $s_r = T_p$ and an integer increases, the pulses become longer, and the instantaneous frequency develops a chirp. The chirp rate increases until adjacent pulses begin to overlap. Fig. 6 illustrates the chirp. With further increase of $|s_r = T_p|$ the overlapping pulses interfere, and a more complex structure develops.

The predicted chirp rate for the frequency chirp shown in Fig. 6 can be estimated from the following argument. The spectral width of the short pulses does not depend on the difference between round trip time $s_r$ and the time interval $T_p = \frac{2\pi}{\lambda}$. Thus, for $s_r = T_p$ we have short pulses with duration $T_p/n_{\omega}$ and with the spectral width $n_{\omega}A$. If $s_r$ is not equal to $T_p$ (as it is in Fig. 6) the duration of a pulse is about $s_r$, but the spectral width is the same, $n_{\omega}A$. This width is due entirely to chirp, and so one can estimate the chirp rate as $n_{\omega}A/s_r$.

6. The moving comb model

The preceding section presented a description of the FSF laser as a static comb of frequency components. An alternative viewpoint of the FSF laser output is that in stationary operation the output electric field $E_{MC}(t)$ consists of a moving comb of chirped frequencies [14,18,29,36,39]. There are an infinite number of frequency components (“teeth”) in this comb, each with a time varying amplitude $B_n(t)$

$$E_{MC}(t) = \varepsilon_0 \sum_{n=-\infty}^{+\infty} B_n(t) \exp[-ip_n(t)].$$

(76)

Associated with the real-valued amplitude $B_n(t)$ is a phase $p_n(t)$, whose time derivative is the instantaneous frequency of the $n$th component

$$\omega_n(t) = \frac{\partial}{\partial t} p_n(t).$$

(77)

The moving comb model assumes that each frequency $\omega_n(t)$ increases linearly with time; for the $n$th component one can write the time-dependent frequency as

$$\omega_n(t) = \omega_{\text{max}} + \frac{\Lambda}{s_r} (t - n s_r).$$

(78)

That is, starting from a very large negative value in the remote past (as $t \rightarrow -\infty$), this chirped frequency increases steadily, exceeding the value $\omega_{\text{max}}$ when $t = n s_r$. The frequency continues to increase indefinitely thereafter, while the amplitude $B_n(t)$ is vanishingly small in the remote past, when its instantaneous frequency lies outside the range for which the gain exceeds the loss. As time progresses the chirp brings the frequency into the frequency domain of net amplification, and the amplitude grows. The amplitude reaches a maximum, by definition, when $t = n s_r$; the frequency $\omega_n(t)$ then is $\omega_{\text{max}}$. As time increases the instantaneous frequency grows beyond the region of net gain, and losses cause the amplitude to diminish. In the remote future, when the instantaneous frequency has become very large, the amplitude again becomes negligible.

It is generally assumed, e.g. [32], that the amplitude $B_n(t)$ has a Gaussian form

$$B_n(t) = \exp \left[ -\frac{(t - n s_r)^2}{(n_0 s_r)^2} \right].$$

(79)
peaking at $t = n\tau_s$ and with a Gaussian width parameter of $n_0\tau_r$. We take the phase $p_n(t)$ to be

$$p_n(t) = \frac{A}{2\tau_r} (t - n\tau_s)^2 + \omega_{\text{max}}(t - n\tau_s). \quad (80)$$

From Eq. (80) we determine the instantaneous frequency to be equal to the value given by Eq. (78). Combining the amplitude with the phase, and summing over all possible comb components, we obtain the field

$$\mathcal{E}_{\text{MC}}(t) = \mathcal{E}_0 \sum_{n=-\infty}^{+\infty} \exp \left[ -\frac{(t - n\tau_s)^2}{(n_0\tau_r)^2} - i \frac{A}{2\tau_r} (t - n\tau_s)^2 - i\omega_{\text{max}}(t - n\tau_s) \right]. \quad (81)$$

This is the fundamental expression for the field in the moving comb picture, when individual amplitudes are idealized as Gaussians.

### 6.1. Interpretation

It is customary to depict this moving-comb field by presenting a density plot of the (real) amplitude $B_n(t)$ as a function of time $t$ (along the horizontal axis) and of instantaneous frequency $\omega$ (along the vertical axis). Such a plot appears as a set of slanted ridges, each corresponding to one of the frequency components. Fig. 7 illustrates a portion of such a comb. The intensity distribution shown in the picture has the following properties.

- For fixed frequency (a horizontal line), one finds a set of peaks, spaced evenly with time interval $\tau_s = 2\pi/\Delta$, the repetition period of the field.
- At a fixed time (a vertical line), one finds a set of ridges separated in frequency by the mode spacing, $2\pi/\tau_r$. This set of ridges forms a comb in frequency space. As time increases, the peaks shift toward higher frequency.
- The $n$th ridge peaks at time $t = n\tau_s$, at which moment the instantaneous frequency is $\omega_{\text{max}}$. The ridges all become negligibly small as the instantaneous frequency becomes very different from $\omega_{\text{max}}$.
- Each ridge has the same slope, $(d/dt)\omega_n(t)$, equal to the frequency-chirp rate $\gamma_c = \Delta/\tau_r$.

This model, of a moving comb of frequencies, explains a number of observed properties of FSF lasers. However, it has been questioned by some authors. In particular, it seems puzzling that, although the frequency shift occurs only within a single short segment of the ring cavity (namely the length of the AOM), the output field shows only a smoothly increasing instantaneous frequency [33].

The model has been used to describe the output of an unseeded FSF laser [37], that is, one in which the internal field originates with amplified spontaneous emission. Presumably a set of discrete-frequency components such as shown by a fixed-time slice of Fig. 7 derive from a single spontaneous emission event. These are thereafter phase coherent with this initial phase. But many spontaneous emission events occur, each at random times, and so it is not obvious, for example, what sets the time $t_0$ at which any component has a specified reference frequency. It is likely that frequency feedback may offer a similar means of stabilizing a comb pattern, by selecting one discrete comb out of a continuum of combs that differ only in the time $t_0$. Any irregularities or noise will introduce a departure from the regular pattern depicted in Fig. 7. A seeded FSF laser deals with this ambiguity by providing a definite phase and frequency of the input signal.
6.2. Unification of models

In the model presented in Section 5 there is no continuously varying frequency, no comb apparently steadily advancing with time. Instead, the frequency takes a succession of constant values, being shifted by \( \Delta \) with each round trip. The output field \( E_{\text{DF}}(t) \), where DF refers to the discrete-frequency model, is the superposition of the original seed field and fields that have made all possible round trips within the cavity.

A glance at Eq. (81) shows that the field associated with the moving-comb model is unchanged when \( t \) is replaced by \( t + mt_c \), where \( m \) is any integer. This field therefore shares with the field of the discrete-frequency model a periodicity, of period \( t_c = 2\pi/\Delta \). We therefore know that this field can be represented as a Fourier series. We write this series representation of the moving-comb model in the form

\[
E_{\text{MC}}(t) = \delta_0 \sum_m C_m \exp[-im\Delta(t - t_{\text{MC}})],
\]

where \( t_{\text{MC}} \) is a reference time. The Fourier coefficients \( C_m \) are determined by integrating over one period

\[
\delta_0 C_m = \frac{\Delta}{2\pi} \int_{t_0}^{t_0 + \tau_c} dt E_{\text{MC}}(t).
\]

For the moving-comb model defined by Eq. (81) the Fourier components can be evaluated in closed form. They are

\[
C_m = C_0 \exp \left[ - \left( m - \frac{\omega_{\text{max}}}{n_0} \right)^2 \right] \times \exp[+im\Delta t_{\text{MC}}].
\]  

We are interested in the case where there are many Fourier components, which occurs when \( n_0 \gg 1 \). We also assume that the frequency shift \( \Delta \) is comparable to the cavity mode separation, meaning that \( \Delta \tau_c / 2\pi \) is comparable with unity. In this case, with the choice of reference time as \( t_{\text{MC}} = \omega_{\text{max}} \tau_c / \Delta \), the Fourier components can be written as

\[
C_m = C_0 \exp \left[ - \left( m - \frac{\omega_{\text{max}}}{n_0} \right)^2 \right] \times \exp[+im\Delta t_{\text{MC}}].
\]  

Here the factor \( C_0 \) incorporates the common phase, the first exponential describes a Gaussian distribution of amplitudes, and the final exponential describes the (quadratic) dependence of phase upon \( m \).

Although the discrete frequency model of the previous section deals with a seeded laser, there is no such reference frequency for the moving-comb model. Nevertheless, the two pictures have the same Fourier transforms, and hence they describe the same output field, if only one assumes that the frequency of the seed laser of the discrete-frequency model is some large integer multiple of \( \Delta \). The connection between the two models is completed with the identification of the Gaussian widths, \( n_0 = n_w \).

6.3. Comments

Our analytical expressions of the earlier sections provide a first-principles derivation of the width and the peak of the discrete-frequency model and, from the proven equivalence of the two models, the moving-comb model as well. The quadratic \( m \)-dependence of the phases of the Fourier components of the moving-comb model, \( C_m \), given in Eq. (85) are identical with the phases of the amplitudes \( A_m \) of the discrete-frequency model. With simple assumptions about the gain and loss properties of the device, one obtains in both cases Gaussian distributions of amplitudes. Thus the two models are identical when they both describe a FSF laser seeded by a CW field. Then Eq. (85) is exactly the field of a seeded FSF laser whose seed frequency is equal to some (very large) integer multiple of \( \Delta \).

The equivalence of the two descriptions means, in particular, that even though the moving comb model appears to have a continuously varying linearly chirped frequency, in fact the field contains only discrete frequencies when it is seeded.

The moving comb model has been applied not only to a seeded system but to a system that grows
from spontaneous emission. In this case there is no identifiable seed carrier frequency \( \omega_s \). Nevertheless, the system retains the same periodicity established by the presence of a fixed frequency shift \( \Delta \) at regular intervals.

In all cases only these discrete frequencies \( n\Delta \) are present in the field. This means that unless the field passes through some nonlinear element, then it cannot acquire any additional frequencies. This implies that for use in interferometry that depends on measuring a beat frequency whose value is proportional to delay time between two paths [28,29,36] such a field can not be used directly.

7. The FSF laser with spontaneous emission

The procedure used to solve Eq. (23) is also applicable to the treatment of a FSF laser based on the growth of spontaneous emission – the laser acts as a regenerative amplifier of spontaneous emission. The relevant equation is

\[
\delta(\vec{x}) = \delta(\vec{x} - \vec{x}_0) \exp \left[ G(\vec{x}) \right] + \zeta(\vec{x}). \tag{86}
\]

The solution to Eq. (86) is

\[
\delta(\vec{x}) = \xi(\vec{x}) + \sum_{n=1}^{\infty} \exp \left[ \sum_{l=0}^{n-1} G(\vec{x} - l\vec{x}_0) \right] \times \zeta(\vec{x} - n\vec{x}_0). \tag{87}
\]

We now use the properties of our assumed noise model to evaluate stochastic averages.

7.1. Stochastic averages

We are interested in the intensity \( I(\omega, T) \) of Eqs. (18) and (19) averaged over stochastic realizations of the Langevin force \( \zeta(\vec{x}) \)

\[
J(\sigma, T) \equiv \langle I(\sigma, T) \rangle. \tag{88}
\]

For the steady-state cases considered in most of this paper, this average is independent of \( T \); in Section 8 we consider more general situations. To carry out the stochastic average \( \langle \cdots \rangle \) and obtain the spectrum \( J(\sigma, T) \), we assume that the spontaneous emission process \( \zeta_{sp}(t) \) is a delta-correlated (Wiener–Levy) stochastic process

\[
\langle \zeta_{sp}(t_1) \zeta_{sp}^*(t_2) \rangle = \frac{\gamma_s^2}{\gamma_0} \delta(t_1 - t_2). \tag{89}
\]

The parameter \( \gamma_s^2 \) that characterizes the spontaneous emission intensity depends on the inversion \( n \), the spontaneous emission rate \( \gamma_s \), and the geometry of the laser cavity. Because the width of the filter is much larger than the free spectral range \( \Gamma_f \gg 1/\tau_s \), the result of this averaging is the spectrum

\[
J(\sigma, T) = \frac{c \tau_s \gamma_s^2}{16 \pi^2} \sum_{n=1}^{\infty} \exp \left[ 2 \sum_{l=0}^{n-1} \left[ g_m \bar{w} - f_m - \frac{1}{2} \left( \frac{\sigma - l\Delta}{\Gamma_f} \right)^2 \right] \right]. \tag{90}
\]

After integrating Eq. (90) over \( \sigma \) we obtain the averaged intensity

\[
\bar{I} = \int_{-\infty}^{+\infty} d\sigma J(\sigma, T)
\]

\[
= \frac{c \tau_s \Gamma_f \gamma_s^2}{16 \pi^{3/2}} \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \exp \left[ n \left( 2 \left( g_m \bar{w} - f_m \right) - \frac{A^2}{12 \Gamma_f^2} (n^2 - 1) \right) \right]. \tag{91}
\]

7.2. Simplification

We assume, as before, that \( \Delta \ll \Gamma_f \). In this case we can replace the summation over \( n \) in (91) by integration and write

\[
\bar{I} = \int_{-\infty}^{+\infty} d\sigma J(\sigma, T),
\]

\[
X = \frac{\Delta}{\sqrt{12 \Gamma_f^3}} \left[ 2(g_m \bar{w} - f_m) + A^2/(12 \Gamma_f^2) \right]^{-3/2}, \tag{92}
\]

where the function \( F(x) \) is defined as

\[
F(x) = x^{1/3} \int_{0}^{\infty} dt \exp \left[ t^2 (1 - x^2 t^2) \right]. \tag{93}
\]

and the magnitude of the spontaneous emission source is characterized by the parameter

\[
L_{sp} = \frac{c \tau_s \gamma_s^2}{\sqrt{\pi^3}} \left( \frac{\sqrt{3} \Gamma_f^4}{2 \gamma_0^2 A} \right)^{1/3}. \tag{94}
\]

By taking into account that \( g_m \bar{w} \approx f_m \) one can transform Eq. (22) for \( \bar{w} \) into
\[ \beta_{sp} F(X) = 1, \]  
where we have introduced the (small) parameter \[ \beta_{sp} = \frac{I_{sp}}{I_{sat}(\eta - 1)} \ll 1. \]  
Under typical operating conditions the parameter \( \beta_{sp} \) is very small: \( \beta_{sp} \approx 10^{-18} \). Then we can use the asymptotic form of the function \( F(x) \) for large \( x \)  
\[ F(x) \approx \frac{\pi^{1/2}}{2} x^{-1/3} \exp \left( \frac{2}{3^{2/3} x} \right). \]  
In this case the initial inversion \( w_0 \) can be found from the condition  
\[ \sqrt{\frac{\pi}{3^{1/2} 2^{2/3}}} \beta_{sp} \exp(Y) = Y^{1/3}, \]  
where  
\[ Y = \frac{2}{3^{2/3}} \frac{4 I \Gamma}{3 \Delta} \left[ 2(g_m \bar{w} - f_m) + \Delta^2 / (12 I \Gamma^3) \right]^{3/2}. \]  
The solution to Eq. (98) for \( \beta_{sp} \ll 1 \), and thus for \( \ln(1/\beta_{sp}) \gg 1 \), is  
\[ Y \approx \ln \frac{1}{\beta_{sp}}. \]  
Then the requirement for inversion is  
\[ 2(g_m \bar{w} - f_m) = -\frac{\Delta^2}{12 I \Gamma} + \left[ \frac{3 \Delta}{4 I \Gamma} \ln \frac{1}{\beta_{sp}} \right]^{2/3}. \]  
Taking into account the condition \( \Delta \ll I \Gamma \) we find that \( g_m \bar{w} \approx f_m \), as is expected. The spectrum \( J(\sigma, T) \) can be written now as  
\[ J(\sigma, T) = I_{sat}(\eta - 1) \frac{1}{\sqrt{\pi} \tilde{\Gamma}} \exp \left[ -\frac{(\sigma - \delta \omega)^2}{\tilde{\Gamma}^2} \right], \]  
whose width \( \tilde{\Gamma} \) is  
\[ \tilde{\Gamma} = \gamma / \sqrt{\sigma} \]  
and whose peak is offset from \( \sigma = 0 \) by \( \delta \omega = \sigma \gamma \)  
with  
\[ \sigma = \left( \frac{3}{4} \ln \frac{1}{\beta_{sp}} \right)^{1/3}. \]  
Recall that \( \sigma \) is the frequency offset from the filter center \( \omega \), meaning that the spectral maximum occurs at \( \omega_{\text{max}} = \omega \gamma + \sigma \gamma \). This result for the continuous spectrum is to be compared with that of Section 5.1 for the distribution of discrete frequency components within a Gaussian envelope.

As in the case of the seeded laser we have here a first-principles derivation of the width and the peak of the spectrum. Again there is only a weak dependence on \( \beta_{sp} \) and the only important parameter is \( \gamma = [4 I \Gamma]^{1/3} \). It is noteworthy that parameter \( \sigma \) can be easily determined from the ratio \( \delta \omega / \tilde{\Gamma} = \sigma \gamma / \tilde{\Gamma} \). In principle these are known for a particular laser, and so the operation of the FSF laser can be determined from these known quantities [24].

8. Rate equations

A previous paper [24] has used a rate-equation model to describe the operation of the FSF laser. This model is based on two sets of coupled equation, one for the population inversion and another for the spectral density of photons. It is interesting to compare our approach based on the Fourier expansion of the electric field with the rate-equation model.

Considering the FSF laser with spontaneous emission and using Eq. (86) for \( \delta' (\lambda) \) in the model of spontaneous emission, Eq. (89), we obtain an equation for the averaged spectral density \( J(\sigma, T) \):
\[ J(\sigma, T) = J(\sigma - \Delta, T - \tau_r) \]
\[ \times \exp(2[g_0(\sigma)w(T) - f(\sigma)]) + \frac{e}{8\pi} \frac{\gamma^2}{(2\pi)}. \]

This equation was derived earlier using the condition of broad loss \( f_t \gg 1/\tau_r \). Here we do not assume steady-state operation and we allow the averaged spectral density \( J(\sigma, T) \) to be time dependent (on the time scale of inversion relaxation). In traditional treatments of laser dynamics one assumes changes of \( J(\sigma, T) \) are small during one round trip, \( J(\sigma, T) \approx J(\sigma - \Delta, T - \tau_r) \). In this limiting case Eq. (106) can be written as
\[ \left[ \frac{\partial}{\partial T} + \frac{\Delta}{\tau_r} \frac{\partial}{\partial \sigma} \right] J(\sigma, T) \]
\[ = J(\sigma, T)(2[g_0(\sigma)w(T) - f(\sigma)]) + \frac{e}{8\pi} \frac{\gamma^2}{(2\pi)}. \]

With the same assumptions the differential equation for the inversion reads
\[ \frac{\partial}{\partial T} w(T) = -w(T) \left[ \frac{\gamma_s}{\tau_r} \int \frac{d\sigma J(\sigma, T)}{I_{\text{sat}}} + \gamma_s \right] + \gamma_s. \]

Here \( \gamma_s \), as defined in Section 3.2, is the spontaneous emission rate and \( \gamma^2 \), defined in Eq. (89), parameterizes the spontaneous emission intensity. Apart from notational changes, these equations differ from the earlier rate equations of [24] only in the presence of the partial frequency derivative in Eq. (107) for the spectral density. In [24] the authors discretized the frequency space. For a smooth spectrum, which we assumed here, this discretization replaces Eqs. (107) and (108) with partial difference equations, leading to the approximate solutions discussed in [24].

9. Summary and conclusions

We have presented a formalism for treating the coupled equations linking electric field output and population inversion for a frequency-shifted feedback laser. The equations take full account of the phase of the field, and include a possible seed laser, which may have arbitrary phase modulation or fluctuation.

Using these equations we have derived simple expressions for the steady-state output electric field. These have a simple interpretation in terms of multiple passes of a wavepacket through the system, gaining or losing energy while undergoing a periodic frequency shift \( \Delta \).

We show that the distribution of spectral components follows a Gaussian form, as is often assumed. The width of this distribution determines the minimum duration of pulses. The characteristic parameters of the Gaussian amplitudes are almost entirely determined by the frequency shift \( \Delta \) and the filter bandwidth \( f_t \), through the combination \( \gamma = (\Delta f_t)^{1/3} \).

The form of the output varies with the relative phases of these spectral components, and depends critically on the ratio of the cavity round-trip time \( \tau_r = L/c \) to the repetition period \( T_p = 2\pi/\Delta \). We have presented illustrative examples of the steady-state output in which there occur various forms of short pulses. Under appropriate conditions these pulses exhibit a linear frequency chirped.

We have also shown the equivalence of two popular models for the output field of the FSF laser, one in which there is a moving comb of frequencies and the other in which there is a fixed discrete set. Our analytical expressions provide a derivation from first principles of the width and the peak of the spectrum of the moving comb model. We also show that the rate equations used previously in modeling the FSF laser can be recovered from the equations presented in this paper.

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References
