Ranging and interferometry with a frequency shifted feedback laser

L.P. Yatsenko a, B.W. Shore b,*, K. Bergmann b

a Institute of Physics, Ukrainian Academy of Sciences, Prospect Nauki 46, Kiev-39, 03650, Ukraine
b Universität Kaiserslautern, 67653 Kaiserslautern, Germany

Received 15 May 2004; accepted 30 August 2004

Abstract

The potential advantages of chirped pulses for very precise measurement of distance, through frequency-domain ranging, has prompted consideration of frequency shifted feedback (FSF) lasers as sources of interferometer light. We here derive theoretical limitations to the spatial accuracy one can expect in such applications, by considering analytical expressions for the electric field emerging from a frequency shifted feedback (FSF) laser seeded by a CW laser whose finite bandwidth originates in phase fluctuations. We also consider consequences of fluctuations in cavity size. We show that, for surfaces flat within the laser footprint, such a system can provide the subwavelength accuracy of conventional interferometry but without dependence on material-dependent phase shifts. Although noise has been important for previous uses of FSF lasers in optical ranging and interferometry, we here show that a frequency modulated seeding laser can be used to better advantage than noise.

© 2004 Elsevier B.V. All rights reserved.

PACS: 42.55.-f; 42.60.Da; 42.55.Ah

Keywords: Optics; Lasers; Frequency shifted feedback; Interferometry; Profilometry; Frequency-domain ranging

1. Introduction

Rapid and nonintrusive measurements of distances from millimeter to kilometer with accuracies of microns or less are now feasible using techniques of optical frequency domain ranging (OFDR) wherein a measurement of distance is obtained from a measurement of frequency differences [1–5]. The technique can be regarded either as a form of chirped optical radar [6] or as a form of interferometry using chirped laser input. As has been demonstrated, [7,8] the needed interferometer seed laser can use the technique of frequency-
shifted feedback (FSF) [9–17]. In such a device feedback introduces a fixed frequency shift \( \Delta \) with each passage of a wave packet around the cavity (during the round-trip time \( \tau_r = L/c \) in a cavity of perimeter \( L \)). By interpreting the output as a moving comb of frequencies [18,19,15,20–22] Nakamura et al. [7,8] analyzed the use of FSF for distance measurements. A key parameter of their approach is the chirp rate \( \gamma_c = \Delta/\tau_r \); the difference in lengths of the two interferometer arms \( h \) can be found from the measured beat frequency \( \omega_B \) as \( h = c\omega_B/2\gamma_c \).

An alternative view is also possible. Under proper conditions the output of a seeded FSF laser can be a regular train of pulses [9,10,18,23,13,24] and the individual pulses can exhibit a frequency chirp [26]. The apparent similarity with the longer-wavelength pulses used in chirped radar for accurate ranging suggests that operation at optical wavelengths could offer accuracy adequate for nanoscale structures. However, this simplistic view of FSF laser output is misleading. The chirp of individual pulses from a FSF laser is not connected with the average chirp rate \( \gamma_c \) but, as was shown in [26], is instead determined by the deviation of the frequency increment \( \Delta \) from \( c/L \).

Although the literature on FSF lasers is extensive, there remain some fundamental questions [25]. Some of these concern the ultimate limits on accuracy attainable when such a laser is used in interferometry. Earlier we have developed a first-principles treatment of the FSF laser [26]. In the present paper, we reexamine some of the properties of this device, with particular emphasis on interferometric applications. Our results provide useful estimates of the limiting accuracy one can achieve using a FSF laser for distance measurement. We clarify the importance of noise in providing the beat frequency that is used in the frequency-domain ranging, and we suggest an alternative approach, using controlled frequency modulation rather than uncontrolled noise, to produce the desired signal.

The present paper is organized as follows. In Sections 1.1 and 1.3, we present a simple discussion of relevant aspects of interferometry and of frequency-domain ranging based on chirped pulses. In Section 2, we discuss the properties of the FSF laser, and the form of the output field. The analytic expressions given here provide the basis for our discussion of FSF interferometry that follows.

In Section 3, we consider interferometry using the field presented in Section 2. We point out the importance of phase fluctuations in providing the frequencies needed for range measurement.

In Section 4, we consider the FSF laser seeded by a phase-noisy cw input field. We derive expressions for the beat-frequency spectrum. The width of this spectrum determines the accuracy with which the peak position (the beat frequency) can be measured, and hence the accuracy with which one can measure the distance.

In Section 5, we consider the effects of unavoidable fluctuations in the cavity length, and the effects these have on the accuracy of beat-frequency measurement.

In Section 6, we consider using a frequency-modulated seed to the FSF laser. We show that this has advantages for improved accuracy. We consider both phase sensitive and phase insensitive detection of the interference intensity.

Section 7 discusses the accuracy obtainable, using either a noisy seed or deliberate frequency modulation to provide the needed frequencies.

Section 8 summarizes our results.

An appendix provide details of our noise model.

1.1. Interferometry

Interferometry offers a means for obtaining extremely accurate distance measurements, such as are needed for determining the heights of nanometer-scale features on lithographed surfaces [27]. In a typical Michelson implementation, a laser beam passes through a beam splitter from which one beam travels a fixed distance in a fixed reference arm. The other beam travels to a sample, where it is reflected. The field of the returning beam is combined with that of the reference beam and the resulting intensity pattern is recorded; changes of intensity provide a measure of phase difference between the two beams. Apart from a fixed (but important) phase increment upon reflection, the phase is proportional to the difference in
lengths of the two paths. Thus a measurement of intensity, converted to a measurement of phase, gives a measurement of distance.

One standing difficulty with interferometric techniques applied to surfaces comprising several materials, often of unknown composition, is the presence of a material-dependent phase shift of the reflected field. For reliable measurements of surface heights to tolerances of a few nanometers, it is essential that this material phase shift be accurately known [28]. This shift is particularly difficult to determine when the surface comprises granules embedded in a substrate, unless the granules are either much smaller than a wavelength, so that the surface can be characterized by a complex index of refraction for an effective medium, or are much larger than a wavelength, so that they can be resolved readily. Even then it is difficult to evaluate the phase shift. A recent paper offers one means of overcoming this limitation [27]. As we will show, chirped pulse interferometry also offers the potential to overcome these handicaps.

1.2. Ranging

Conventional interferometry with a single wavelength $\lambda$ provides subwavelength accuracy of path differences, but because phase has an intrinsic periodicity, it can only provide this accuracy modulo $\lambda$, i.e., it cannot distinguish between distance $h$ and $h + \lambda$. Various techniques are used to overcome this limitation, such as the use of two different wavelengths.

For measurements of distances of meters or miles, radar techniques are often used. Typically these transmit a train of pulses and measure the time delay before a return pulse is observed. The requirement for very precise measurement of time intervals makes it difficult to obtain positions of even stationary objects with micron accuracy. The technique of chirped radar [29] replaces time measurements with beat-frequency measurements, but has much in common with interferometry. The technique described here can be used for measurements of large distances, as is done with optical ranging, as well as for small distances; all distances are measured with equal accuracy.

1.3. Chirped-pulse interferometry

It is easy to show that if one has a chirped frequency as input to a Michelson interferometer, see Fig. 1, then the Fourier transform (i.e., the spectrum) of the beat intensity includes a frequency equal to the product of the chirp rate and the delay time between the two arms. Thus one can anticipate construction of an interferometer in which a measurement of frequency gives a measurement of distance. To show this connection consider a simple model of a field whose carrier frequency varies linearly with time (a chirped frequency) at rate $\gamma_c$ during the time interval $0 < t < T_0$. Assume a constant amplitude $\delta$. Then the complex-valued field at a fixed position is

$$E(t) = \delta \exp[-i(\omega_0 + \gamma_c t/2)t].$$

We use this field as input to a Michelson interferometer in which one arm has length $z$ and the other has length $z + h$, where $h$ is to be determined. Fig. 1 illustrates the general layout of the interferometer. Our interest here is in sub-micron metrology of surfaces, such as occur with lithography, wherein the reference beam can be reflected from the substrate that holds the sample; in other work the arms are often of very different lengths [7,8].

Upon reflection the electric field of beam $j$ acquires a phase shift $\phi_j$, which depends on the properties of the surface. The difference in arm lengths gives a relative time delay

$$T = 2h/c.$$
For simplicity, we assume perfect reflection at the reference and target surfaces and \( T < T_0 \). Then the interferometer output intensity is
\[
I(t, T) = |E(t) + \exp(i\Delta \phi)E(t + T)|^2 \\
= |E(t)|^2 + |E(t + T)|^2 \\
+ 2\text{Re}\{\exp(i\Delta \phi)E^*E(t + T)\}.
\] (3)
where \( \Delta \phi \) is the difference between the material-dependent phase shifts \( \phi_j \) in the two arms. We are interested in the interference-term signal
\[
S(t) = 2\text{Re}\{\exp(i\Delta \phi)E^*E(t + T)\}
= 2|\delta|^2 \cos(2h\gamma_c/c + \Delta \phi + \Phi(T)).
\] (4)
This expression shows that a measurement of the beat frequency [the oscillation frequency of \( S(t) \)]
\[
\omega_B \equiv 2\pi\nu_B \equiv \gamma_c T = (\gamma_c/c)2h
\] (5)
converts, using the chirp rate \( \gamma_c \), directly into a measure of \( T \) and thence to a measurement of distance \( h \). The material-dependent phase difference \( \Delta \phi \) adds a constant to the geometrical phase increment
\[
\Phi(T) \equiv \omega_0 T - \gamma_c T^2/2
\] (6)
to produce the interferometer phase \( \Phi(t) + \Delta \phi \). Thus any determination of \( T \) from the phase of the interference term must rely on values of the material phase shift \( \Delta \phi \). By contrast, measurements based on beat frequency determination are insensitive to this phase.

As will be explained, interferometry with the FSF laser also offers a means of avoiding dependence of distance measurements upon the material phase shift \( \Delta \phi \), and hence it offers the potential for accurate surface topography measurements without knowledge of surface characteristics. Furthermore, by giving a measurement of heights of both reflecting and partially reflecting surfaces, it can reveal structure underlying a partially transparent layer with known refractive index but of arbitrary thickness.

The FSF laser is particularly suitable for optical ranging: it provides a strictly linear chirp that can have a very large chirp rate, say 100 MHz in 5 ns, or \( 5 \times 10^{13} \) s\(^{-2} \). The FSF laser also functions without an external seed, driven by spontaneous emission in the gain medium. The work of Nakamura et al. [7,8] is done with such a laser. Our work demonstrates that, when the properties of the laser are dominated by an external seed, then much improved accuracy of distance measurements can be achieved. The bandwidth of the seeding laser determines the accuracy, but the origin of the bandwidth is also significant. We show that bandwidth from controlled phase modulation is particularly useful.

1.4. Accuracy and resolution

When the laser beam covers a surface pattern having several heights, then a histogram of distance values displays several peaks. To resolve these using frequency chirping the difference in heights must be greater than half a wavelength. This limitation can be understood in the following way.

From the relationship (5) between the measured feature height \( h \) and beat frequency \( \omega_B \) it follows that the height error \( \delta h \) is proportional to the beat-frequency error, \( \delta h = (c/2\gamma_c)\delta \omega_B \). This error is approximated as \( \delta \omega_B \approx 2\pi/T \), where \( T \) is the duration of a pulse. During a pulse the frequency changes by \( \gamma_c T \). This change cannot be larger than the optical frequency \( \omega = 2\pi c/\lambda \), and hence the frequency error cannot be less than \( \delta \omega \approx (\gamma_c c/\lambda) \). From this limit it follows that the resolution limit is set by the optical wavelength \( \lambda \),
\[
\delta h > \lambda/2.
\] (7)

When the laser beam covers a flat surface then, as we will note, the error in a measurement of surface height relative to a reference surface can be significantly less than a wavelength. This error, the accuracy of interferometer measurement, is determined by the accuracy with which one can measure the location of the peak value of a spectral profile, i.e., by the error \( \delta \omega_B \) in the beat frequency \( \omega_B \). In turn, this is set by the resonance width \( \Gamma_{\text{res}} \) of the spectral profile. As we will note (see section 7), accuracy of 100 nm appears possible.

2. The FSF laser

In brief, a FSF laser contains a feedback cavity (which is typically either a closed loop, of perime-
ter \( L \), or a Fabry–Perot cavity of length \( L/2 \) in which there is a section having gain, one or more spatial or frequency filters and, most importantly, an element that induces a frequency shift (of angular frequency \( \Delta \)) on each wavepacket that passes through it. Typically the frequency shift is provided by an acousto-optic modulator (AOM), acting as a grating that feeds the first order diffraction back into the cavity, while the zero-order light emerges as the output of the ring cavity. Fig. 2 illustrates an example of such a ring cavity [25].

The properties of such a laser have been extensively reported, e.g., [30–37,13,24,14,38]. The important parameters of the FSF cavity itself, apart from gain and loss, are as follows:

\[
\begin{align*}
\Delta &= \text{the frequency shift (rad/s) per round trip; } \\
\tau_r &= L/c = \text{the round trip time in the cavity; the free spectral range is } 2\pi/\tau_r, \\
\tau_s &= 2\pi/\Delta = \text{the repetition period of the field; } \\
\gamma_c &= \Delta/\tau_r = \text{the average chirp rate of the instantaneous frequency of the cavity field.}
\end{align*}
\]

Another set of parameters describe the net gain within the cavity and the loss due to reflection and filters. These are distributed over some finite range of frequencies. We treat them as localized in an infinitesimally thin segment of the path. Fig. 3 shows the distribution of gain and loss as a function of frequency (discretized by \( n \)).

A key parameter controlling the beat spectrum, from which follows the achievable accuracy of beat frequency measurement, is

\[
n_w = \text{the effective number of discrete frequency components within the output spectrum from the seeded FSF laser. (The output bandwidth divided by the the frequency shift).}
\]

\[
\Gamma_f = \text{the bandwidth of the frequency limiting elements of the cavity (a filter).} \\
\gamma = (\Delta^2 \Gamma_f)^{1/3} = \text{a frequency parameter largely determining the properties of the FSF laser output.}
\]

A FSF laser can operate in a variety of regimes: the output may range from extreme irregularity to very regular pulse trains or even cw, depending on such controllable properties as the amount of gain and the relative values of various time constants or frequency bandwidths of the device. We are here concerned only with steady-state operation (i.e., infinite pulse trains or cw).

### 2.1. The moving-comb model of FSF output

One viewpoint of the FSF laser output, used extensively by Nakamura et al. [15,16], is that in
stationary operation the output electric field \( E_{MC}(t) \) consists of a moving comb of chirped frequencies. There are an infinite number of frequency “teeth” in this comb, each with a time varying amplitude \( B_n(t) \):

\[
E_{MC}(t) = \delta_0 \sum_{n=-\infty}^{\infty} B_n(t) \exp[-i\phi_n(t)].
\] (8)

Associated with the real-valued amplitude \( B_n(t) \) of tooth \( n \) is a phase \( \phi_n(t) \) whose time derivative, the instantaneous frequency \( \omega_n(t) \), varies linearly with time (a frequency chirp)

\[
\omega_n(t) \equiv \frac{d}{dt} \phi_n(t) = \omega_{\text{max}} + \frac{A}{\tau_t} (t - n\tau_s).
\] (9)

Typically the amplitude \( B_n(t) \) is taken to be of Gaussian form,

\[
B_n(t) = \exp \left[ -\frac{(t - n\tau_s)^2}{(n_w\tau_t)^2} \right],
\] (10)

peaking at \( t = n\tau_s \) and with a Gaussian width parameter \( n_w\tau_t \).

Because the output of the FSF laser can be viewed as a comb of chirped frequencies, it is natural to consider using it as a source of radiation for chirped-pulse interferometry. However, this use is not as direct as one might think. In reference [26] we showed that this output of the FSF laser can also be written in the form of a Floquet series

\[
E_{MC}(t) = \delta_0 \sum_{n} C_m \exp[-i\alpha(t - t_{MC})],
\] (11)

where \( t_{MC} \) is a reference time. We take \( t_{MC} = \omega_{\text{max}}\tau_t/A \) and \( n_w \gg 1 \) and thereby write the Fourier coefficients as

\[
C_m = C_0 \exp \left[ -\frac{(m - (\omega_{\text{max}}/A))^2}{(n_w^2)} \right] \exp \left[ +im^2 \Delta\tau_t \frac{\tau_t}{2} \right].
\] (12)

Here the factor \( C_0 \) incorporates the common phase, the first exponential describes a Gaussian distribution of amplitudes with width \( n_w \), and the final exponential describes the (quadratic) dependence of phase upon \( m \). This decomposition shows that the field comprises a discrete set of time-independent frequencies, separated by the frequency shift \( \Delta \). By contrast, the field needed for interferometry should comprise a continuum of frequencies if it is to include the beat frequency \( 2\gamma_c T \) for arbitrary delay time \( T \).

In [26] we have shown that, because of the discreteness of the frequencies actually present in the moving comb model, it can be viewed as a discrete-frequency model (our term) seeded by a cw laser whose frequency is some integer multiple of the frequency shift \( \Delta \).

### 2.2. FSF cavity seeded with variable phase

The frequencies required for frequency-domain interferometry can be supplied in several ways. To observe a beat signal at some frequency it is essential that the seed laser spectrum contain this frequency. Noise fluctuations provide one potential source of the desired bandwidth. Previous authors [15,16], have relied on random amplitude fluctuations to provide seed bandwidth. Here we discuss the use of a modulated seed pulse, of carrier frequency \( \omega_s \). When we allow a variable phase \( \varphi_s(t) \) on this input to the FSF cavity, the output field has the form

\[
E(t) = \sum_{n=0}^{\infty} E_n(t) \exp \left[ -i(\omega_s + n\Delta)t \right],
\] (13)

where the complex-valued time-dependent amplitude is

\[
E_n(t) = \hat{\varphi}_s(t - n\tau_s) \exp \left[ -i\Phi_n - i\varphi_s(t - n\tau_s) \right].
\] (14)

The factor \( \hat{\varphi}_s \) provides a normalization: it is related to the seeding-laser intensity inside the cavity \( I_e \) through the formula

\[
I_e = \frac{c}{8\pi} |\hat{\varphi}_s|^2.
\] (15)

The dimensionless amplitude \( \hat{\varphi}_s \) is, in the limit of interest here (large \( n_w \), see [26]),

\[
\hat{\varphi}_s \simeq \mathcal{A} \exp \left[ -\left( \frac{n - n_m}{n_w} \right)^2 \right]
\] (16)

where amplitude \( \mathcal{A} \), width \( n_w \) and central position \( n_m \) are determined by laser parameters (see [26]). Here we assume that \( n_w \gg 1 \). The phase has a time-independent contribution that is quadratic in \( n \).
\[ \Phi_n = -\tau_n \omega_n + (n + 1) \Delta / 2. \]  

The field of Eq. (13) also receives a contribution from time varying phases \( \varphi_s(t) \), either random or deterministic.

3. Interferometry with a FSF laser

We are interested in the homodyne beat signal created by the interference between two electric field amplitudes with relative time delay \( T = 2\hbar / c \) and with a phase difference \( \Delta \phi \) originating in the difference between the material dependent phase shifts of reflected light in the two arms. This signal is proportional to the real part of the complex-valued product \( \exp(\imath \Delta \phi) E(t) E^\ast(t + T) \):

\[
S(t) = \exp(\imath \Delta \phi) E(t) E^\ast(t + T) + \text{c.c.}
\]

\[
= \sum_{n,l=0}^{\infty} E_n(t) E^\ast_l(t + T) \exp \left[ \imath \Delta \phi - \imath (\omega_n + n \Delta) t + \imath (\omega_l + l \Delta) (t + T) \right] + \text{c.c.}
\]

\[
= \exp[\imath \Delta \phi + \imath \omega_l T] \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} E_n(t) E^\ast_{n-m}(t + T)
\]

\[
\times \exp [\imath (n - m) \Delta T - \imath m \Delta t] + \text{c.c.} \quad (18)
\]

(Note that we need not consider the spectrum of \( E(t) E^\ast(t) \) or \( E(t + T) E^\ast(t + T) \) because these contain no information about the distance.) This signal contains many harmonics \( m \), originating with combinations of the various frequencies in the sum. We will assume that the frequency \( T \gamma_c = T \Delta / \tau_c \) produced during the time \( T \) is much less than the frequency shift \( \Delta \). This implies that \( T \ll \tau_c \) and, in turn, that the possible beating frequency is small. For application to surface profilometry, where one is interested in measuring small variations in height, it is always possible, in principle, to adjust the reference arm relative to the sample so that \( T \ll \tau_c \). For use with large distances, in optical ranging, this is not always possible and it is necessary to resolve the phase ambiguity of successive chirped pulses. That is, the time \( T \) is determined by interferometry only within some multiple of the repetition time \( \tau_c \). To resolve this ambiguity one can use a second AOM frequency to determine the order \( m \) to be used with Eq. (18), as was done by Nakamura et al. [8].

Without loss of generality we will here consider only the terms in (18) for which \( l = n \) or \( m = 0 \). The signal is therefore

\[
S(t) = \sum_{n=0}^{\infty} E_n(t) E^\ast_n(t + T) \times \exp [\imath \Delta \phi + \imath (\omega_n + n \Delta) T] + \text{c.c.} \quad (19)
\]

The only time dependence of this signal originates in phase variation of the seeding laser, as described by \( \varphi_s(t) \) in \( E_n(t) \).

Note that in the absence of phase variation, \( \varphi_s(t) = 0 \), only discrete frequencies contribute to the electric field: these differ from the seed frequency by \( n \) increments of the AOM frequency shift \( \Delta \). In particular, the interferometer generates no new frequencies. The time shift \( T \) leads only to a phase shift of beating at the frequency \( \Delta \) and at the higher harmonics.

We emphasize that when there is no variation of the laser phase (e.g., no noise and no deliberate modulation), then there is no beat signal. Experimental observations [14] have confirmed that, when feedback in the FSF laser causes mode locking (and a consequent removal of noisy phase fluctuations), then any interferometric beat signal disappears. Thus to understand the characteristics of the beat signal, and the accuracy attainable in interferometry, we must consider the effect of noise of the seed laser. As we will show, it is possible to produce a beat signal whose power spectrum has a very sharply defined maximum at the frequency \( \omega_B = T \gamma_c \). We also consider the possibility of introducing frequency modulation of the seeding laser, rather than relying on noise. We will show that this technique offers promise of high accuracy.

Our task now is to compute the power spectrum \( J_{s^2}(\omega) \) of the beat signal \( S(t) \). This is determined as the Fourier transform

\[
J_{s^2}(\omega) = 2 \int_0^\infty \, d\tau \, G(\tau) \cos \omega \tau = \int_{-\infty}^{\infty} \, d\tau \, G(\tau) \exp(\imath \omega \tau) \quad (20)
\]

of the autocorrelation function \( G(\tau) \).
\[ G(\tau) \equiv \langle S(t)S(t+\tau) \rangle. \]  

Here and elsewhere \( \langle \cdot \rangle \) denotes a stochastic average. Because we deal with steady state, the autocorrelation function is independent of time \( t \).

4. FSF interferometry with a noisy seed

In the presence of phase variation of the seed laser the autocorrelation function \( G(\tau) \) can be written as

\[
G(\tau) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} |c_n|^4 |a_n|^2 a_l^2 F_+ (\tau - l\tau_r + n\tau_r, T) \times \exp [i2\Delta\phi + i(2\omega_n + n\Delta + l\Delta)] + \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} |c_n|^4 |a_n|^2 a_l^2 F_- (\tau - l\tau_r + n\tau_r, T) \times \exp [-i(n - l)\Delta T] + \text{c.c.} \tag{22}
\]

where we define

\[
F_\pm (t_1, t_2) \equiv \langle \exp [-i\varphi_s(t) + i\varphi_s(t + t_1) \pm i\varphi_s(t + t_2 + t_1)] \rangle. \tag{23}
\]

Note that the only dependence upon the material phase shift \( \Delta\phi \) occurs with the first summation, in the exponential associated with \( F_+ \).

We now consider the Fourier transform of this correlation function. From the above expressions we obtain the formula

\[
J_s(\omega) = F_+ (\omega, T) \mathcal{L}(\omega + \gamma_c T) \mathcal{L}'(-\omega + \gamma_c T) \times \exp [+i\Delta\phi + i2\omega_c T] + F_+ (\omega, T) \times \mathcal{L}'(-\omega - \gamma_c T) + 2F_- (\omega, T) \times \mathcal{L}'(-\omega - \gamma_c T) + \mathcal{L}'(\omega - \gamma_c T) \tag{24}
\]

where all of the dependence upon phase fluctuation occurs in the factors

\[
F_\pm (\omega, T) = \int_{-\infty}^{\infty} dt \ F_\pm (\tau, T) \exp [i\omega\tau], \tag{25}
\]

while the remaining factors

\[
\mathcal{L}(\omega) \equiv \sum_{n=0}^{\infty} |a_n|^2 \exp [+i\omega\tau, n] \tag{26}
\]

serve as profile functions independent of any phase fluctuation.

4.1. The profile function

Using formula (16) we write the profile function \( \mathcal{L}(\omega) \) as

\[
\mathcal{L}'(\omega) \simeq A^2 \sum_{n=0}^{\infty} \exp [+i\omega\tau, n] \exp \left[ -2 \left( \frac{n - n_m}{n_w} \right)^2 \right]. \tag{27}
\]

When the number of frequency components becomes large this is a very narrow function of \( \omega \). In this situation, with \( n_w \gg 1 \) and \( n_m \gg 1 \) the summation in this expression can be replaced by integration from \(-\infty\) to \(+\infty\), with the result

\[
\mathcal{L}'(\omega) \simeq n_w A^2 \sqrt{\frac{\pi}{2}} \exp [+i\omega\tau, n_m] \exp \left[ - \frac{(\omega - \gamma_c T)^2}{8} \right]. \tag{28}
\]

In this case, we can extract a phase and an amplitude from the profile function \( \mathcal{L}'(\omega) \) and introduce a normalized Gaussian profile function

\[
\mathcal{L}_g(\omega) \equiv \exp \left[ - \frac{(\omega - \gamma_c T)^2}{8} \right]. \tag{29}
\]

Then the spectrum takes the form

\[
J_s(\omega) = \pi(n_w A^2)^2 |c_c|^4 \left[ F_+ (\omega, T) \cos [2\Delta_0 T + 2\Delta\phi] \times \mathcal{L}_g (\omega + \gamma_c T) \mathcal{L}_g (\omega - \gamma_c T) + 2F_- (\omega, T) \times \mathcal{L}_g (\omega - \gamma_c T) \right]. \tag{30}
\]

In this formula, like the general expression of Eq. (24), the contributions from \( F_+ \) involve the product of profile functions \( \mathcal{L}_g(\omega) \) centered about the values \( +\gamma_c T \) and \(-\gamma_c T \). The width of \( \mathcal{L}_g(\omega) \) is \( \sqrt{8/n_w} \). When this width is much less than \( \gamma_c T \), as it is for large \( n_w \), then the overlap of these functions will be negligible, and the only contribution to the spectrum \( J_s(\omega) \) will come from the last term. Under these conditions, we obtain the formula

\[
J_s(\omega) = \pi(n_w A^2)^2 |c_c|^4 F_- (\omega, T) \mathcal{L}_g (\omega - \gamma_c T). \tag{31}
\]

Because the spectral width of \( F_- (\omega, T) \) is usually quite large compared with the width of \( \mathcal{L}_g(\omega) \), Eq. (31) expresses a very narrow resonance, in
which position and width depend on laser parameters (through $\mathcal{L}$) and the magnitude is determined (through $F_-$) by the noise characteristics.

4.2. Modeling seed-laser noise

The function $F_-(\omega, T)$ depends on the fluctuation spectrum. To evaluate this dependence we assume that the time derivative of the seed-laser phase $\varphi_s(t)$ is a zero-mean stochastic process $\xi(t)$ [see Appendix, Eq. (A.4)],

$$\dot{\varphi}(t) = \xi(t).$$

In Appendix A, we discuss using an exponentially correlated (Ornstein–Uhlenbeck) process, with zero mean and exponential correlation

$$\langle \xi \rangle = 0, \quad \langle \xi(t)\xi(t') \rangle = GD\exp(-|G|t-t')$$

(33)

to carry out the stochastic averaging in expression (23). We use here the results for a limiting case, a delta-correlated process $G \to \infty$, for which simple analytic expressions can be obtained.

As shown in Appendix A, the width of the function $F_-(\omega, T)$ is larger than 1/T. The spectrum given by Eq. (31) is appreciable only within a narrow interval, of width $\delta\omega \approx \sqrt{8/(n_w\tau_c)}$ near the resonance value $\omega_B = \gamma_c T$. Because the delay is small, $T \ll \tau_c$, the width obeys (for any $n_w$) the inequality $\delta\omega \ll 1/T$, and we can take $F_-(\omega, T)$ to be $F_-(\gamma_c T, T)$. Moreover, from the conditions $\Delta\tau_r \approx 1$ and $T \ll \tau_c$ we have $\gamma_c T \ll 1/T$, and so we can take $F_-(\omega, T)=F_-(\gamma_c T)=F_-(0, T)$. Using expression (A.13) we obtain the result

$$F_-(0, T) = (1/D)[1 - (1 + 2DT)\exp(-2DT)].$$

(34)

Thus the expression (31) can be written as

$$J_S(\omega) = J_{res} \exp \left[ -(\omega - \gamma_c T)^2 (\tau_n T)^2 \right],$$

(35)

where

$$J_{res} = \pi(n_w A^2 \gamma_c)^4 (1/D)[1 - (1 + 2DT)\exp(-2DT)].$$

(36)

These expressions describe a narrow resonance in the spectrum of the beat signal. The maximum of this resonance coincides with the frequency $\omega_B = \gamma_c T$ and the width of this resonance is equal to

$$\Gamma_{res} = 2/\langle \tau_n T \rangle.$$  

(37)

Neither of these parameters depends on the fluctuation properties. The only dependence on the fluctuations appears in the amplitude of the signal, which depends on the intensity of the noise as parameterized by $D$. For small $DT \ll 1$ the magnitude $J_{res}$ is proportional to $DT^2$. There is an optimal value of $DT = 0.89664$ for which the amplitude is maximum, with value $J_{res} = 0.59697T\pi[n_w A^2 \gamma_c]^4$.

5. Fluctuations of the optical length

Whereas uncontrolled fluctuations of the seed-laser frequency can be useful, any fluctuations of the optical path will degrade the accuracy. In this section we estimate the effect of such fluctuations.

Consider the influence on the resonance profile (31) caused by fluctuations in the optical length of the cavity. We write the optical length as $L(t) = L_0 + \delta L(t)$, where $\delta L(t)$ is a zero-mean fluctuation of the optical length. Taking into account these fluctuations it is easy to show that Eqs. (13), (14) for the field can be rewritten as

$$E(t) = \delta_c \sum_{n=0}^{\infty} a_n \exp[-i\Phi_n - i\Phi_s(t - n\tau_c) - i(\omega_s + n\Delta)\tau_c] \exp[i\delta\Phi_n(t)],$$

(38)

where

$$\delta\Phi_n(t) = -\delta L(t - n\tau_c) \approx -\frac{\omega_0}{c} \sum_{l=0}^{n-1} \delta L(t - l\tau_c).$$

(39)

For the conditions of the observations of the narrow resonance (31) the correlation function (21) can be written as

$$G(\tau) = |\xi_c|^4 \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} |a_n|^2 |a_l|^2 \exp[+i(n - l)\Delta T] \times F_-(\tau - l\tau_c + n\tau_c, T)F_3(T, \tau, l, n),$$

(40)
where

\[ F_3(T, \tau, l, n) \equiv \langle \exp \left[ -i \delta \Phi_3(t) + i \delta \Phi_3(t + T) \right. \\
- i \delta \Phi_3(t + \tau + T) + i \delta \Phi_3(t + \tau) \rangle \rangle. \]

(41)

To evaluate the functions \( F_3(T, \tau, l, n) \) we assume that the fluctuation \( \delta L(t) \) of the optical length is an Ornstein–Uhlenbeck process \([39,40]\) whose parameters we denote by \( G_L \) and \( D_L \).

\[ \langle \delta L \rangle = 0, \]

\[ \left( \frac{\omega_0}{L_0} \right)^2 \langle \delta L(t) \delta L(t') \rangle = G_L D_L \exp(-G_L |t - t'|). \]

(42)

We use formula (A.5) to evaluate stochastic averages of exponentiated noise and obtain for \( F_3(T, \tau, l, n) \) the result

\[ F_3(T, \tau, l, n) = \exp \left\{ -\left| g_m(0) + g_{ll}(0) \right. \right. \\
- g_m(T) - g_{ll}(T) + g_{nl}(\tau + T) \\
+ g_{nl}(\tau - T) - 2g_{nl}(\tau) \right\}. \]

(43)

where

\[ g_{nl}(t) = \sum_{m=1}^{n} \sum_{k=1}^{l} \tau_t^2 G_L D_L \exp[-G_L |t + (m - k)\tau_r|]. \]

(44)

We assume that fluctuations of the optical length are relatively slow, \( G_L \tau_r \ll 1 \). This is always true for fluctuations caused by variation of pump intensity or vibrations. Then we replace the summation over \( m \) and \( k \) in Eq. (44) by an integration, and for the most interesting case \( nG_L \tau_r, lG_L \tau_r \gg 1 \) write

\[ g_{nl}(t) \simeq G_L D_L \int_{t_1}^{\tau_t} dt_1 \int_{t_1}^{\tau_t} dt_2 \exp(-G_L |t + t_1 - t_2|) \]

\[ \simeq 2D_L \tau_r - \frac{D_L}{G_L} [\exp(-G_L |t|) \]

\[ + \exp(-G_L |t + n\tau_r - l\tau_r|)] \]

(45)

where \( t_1 \) is the lesser of \( t_1 = n\tau_r \) and \( t_2 = l\tau_r - t \). Using Eq. (45) for small \( T \ll 1/G_L \) one can obtain the expression

\[ F_3(T, \tau, l, n) = \exp \left\{ -v[2 - \exp(-G_L |\tau|) \right. \\
- \exp(-G_L |\tau + n\tau_r - l\tau_r|)] \right\}, \]

(46)

where

\[ v \equiv T^2 D_L G_L. \]

(47)

Note that we made no assumptions about the intensity \( D_L \) of the fluctuations. Thus the quantity \( v \) can be large, and we cannot generally expand the exponential. For the spectrum \( J_S(\omega) \) we obtain the formula

\[ J_S(\omega) = \sum_{n,l=0}^{\infty} |a_n|^2 |a_l|^2 \int_{-\infty}^{\infty} \mathrm{d} \tau F_-(\tau, T) \]

\[ \times \exp(-v[2 - \exp(-G_L |\tau|) \]

\[ + (l - n)\tau_r]) - \exp(-G_L |\tau|)] \]

\[ \times \exp \left[ i\omega t + i\omega (l\tau_r - n\tau_r) + i(n - l)\Delta T \right]. \]

(48)

As follows from the above consideration of the delta-correlated fluctuations of the seeding laser the function \( F_-(\tau, T) \) can be considered as a very narrow function of \( \tau \). Therefore we use the approximation (16) for the amplitudes \( a_n \) and write the spectrum \( J_S(\omega) \) as

\[ J_S(\omega) = J_{res} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{d} x \exp(-x^2) \]

\[ \times \exp(-v[1 \exp(-G_L |\tau|) \]

\[ \times \exp \left[ -ixn_x(\omega - \gamma T)\tau_r \right] \]

(49)

with \( J_{res} \) given by (36).

This is a general result valid for any choice of the noise parameters. For fluctuations originating in various mechanical or technical variations, \( G_L \) is about \( 10^4 \) to \( 10^5 \) s\(^{-1}\). Then for \( \tau_r = 1 \) ns we have \( n_x G_L \tau_r = n_x (10^{-4} \sim 10^{-5}) \). Thus there are two different possibilities. The first one is a laser with moderate numbers of the frequency components, \( n_x \ll 10^4 \), and the second one is a laser with broadband gain (Ti-Sapphire laser) with \( n_x \gg 10^4 \) to \( 10^5 \). For given laser properties the two possibilities correspond to correlation times \( 1/G_L \) much longer than or much shorter than \( n_x \tau_r \). We consider these two cases separately below.

5.1 Long correlation time

For the first case, when \( G_L n_x \tau_r \ll 1 \), one can expand \( \exp(-G_L |\tau|) \) as

\[ \exp(-G_L |\tau| n_x |x|) = 1 - G_L \tau_r n_x |x| \]

(50)

and obtain the resonance shape as a Voigt profile:
We now consider two limiting cases of the exponential argument $v G_L \tau_r n_w$.

- For $v G_L \tau_r n_w \ll 1$ the fluctuations of the optical length do not influence the resonance shape. As noted earlier, in this case a narrow resonance exists in the spectrum of the beating signal. The maximum of this resonance coincides with the frequency $\omega_B = \gamma_c T$, and the width is $\Gamma_{\text{res}} = 2(\tau_r n_w)$.

- For $v G_L \tau_r n_w \gg 1$ the spectrum $J_S(\omega)$ is Lorentzian,

$$J_S(\omega) = \frac{1}{1 + (\omega - \gamma_c T)^2 / (v G_L)^2}. \quad (52)$$

The maximum of this resonance again coincides with the frequency $\omega_B = \gamma_c T$ but the width $\Gamma_{\text{res}}$ of this resonances is larger than $2/(\tau_r n_w)$, and is equal to $\Gamma_{\text{res}} = v G_L = T^2 D_L G_L^2$. The amplitude $J_{\text{res}}$ of the resonance is equal to $J_{\text{res}} = J_{\text{res}} 2/ (\sqrt{\pi} v G_L \tau_r n_w)$ and is much smaller than $J_{\text{res}}$.

These formulas show that even with slow fluctuations there will be a limit to the accuracy, i.e., a finite bandwidth of the beat spectrum.

### 5.2. Short correlation time

Consider now the second case, when $G_L n_w \tau_r \gg 1$. For large detunings $|\omega - \gamma T| \gg 1/(n_w \tau_r)$ the main contribution in the integral over $x$ in Eq. (49) and so this integral can be written as

$$J_S(\omega) = J_{\text{res}} \frac{\exp(-v)}{\sqrt{\pi} G_L \tau_r n_w} 2 \Re \int_0^\infty dy \{\exp[+v \exp(-y)] - 1\} \times \exp\left[-iy(\omega - \gamma_c T) / G_L\right]. \quad (53)$$

This integral describes a resonance which is close to Lorentzian with width $\Gamma_{\text{res}} = G_L$. The amplitude of this resonance is equal to

$$J_S(0) = J_{\text{res}} \frac{2 \exp(-v)}{\sqrt{\pi} G_L \tau_r n_w} \int_0^\infty dy \{\exp[+v \exp(-y)] - 1\}. \quad (54)$$

This expression has the limiting values

$$J_S(0) = J_{\text{res}} \frac{2}{\sqrt{\pi} G_L \tau_r n_w} \begin{cases} v & \text{for } v \ll 1, \\ 1/v & \text{for } v \ll 1. \end{cases} \quad (55)$$

The maximum value of this function of $v$ is $1.312 J_{\text{res}}/(G_L \tau_r n_w)$, occurring for $v = 1.503$.

For small detunings $|\omega - \gamma T| \leq 1/n_w \tau_r$, the main contribution to the integral over $x$ in Eq. (49) comes from large $|x| \gg 1$. The result can be written as

$$J_S(\omega) \approx \exp(-v) J_{\text{res}} \times \exp\left[-(\omega - \gamma_c T)^2 (\tau_r n_w / 2)^2\right] + J_S(0).$$

Thus for relatively fast fluctuations, meaning $n_w G_L \tau_r \gg 1$, and when there is only partial correlation between changes of the phases of the frequency components of the laser spectrum, the beat-frequency spectrum $J_S(\omega)$ is the superposition of a narrow resonance (56) and a wide resonance (53). The ratio of the amplitudes of the narrow and wide resonances is $\sqrt{\pi} n_w / (2T_2 D_L) \gg 1$ for small $v = T^2 D_L G_L < 1$ and is exponentially small, $\exp(-v) \sqrt{\pi} G_L \tau_r n_w v / 2$ for large $v \gg 1$. (Fig. 4) shows the beat signal for various $v$ and $G_L$.

### 6. FSF interferometry with an FM seed

Earlier discussion emphasized that some modulation of the seed laser is essential if an interferometer beat signal is to be observed. The needed modulation will automatically occur if the seed laser is noisy. But one may also consider introducing structured modulation, i.e., a frequency-modulated seed laser. Here we analyze this possibility.
Let us consider the case when the seeding laser is frequency modulated. We take the phase of \( \varphi_s(t) \) to be, instead of a stochastic quantity, a sinusoid

\[
\varphi_s(t) = \varphi_0 + \beta \sin(\Omega_m t),
\]

where \( \Omega_m \) is the modulation frequency, \( \beta \) is the modulation index and \( \varphi_0 \) is the initial phase. Then the FSF laser field (13), (14) reads

\[
E(t) = \sum_{n=0}^{\infty} E_n(t) \exp[-i(\omega_n + n\Delta)t],
\]

where the amplitude is now

\[
E_n(t) = e_c \sum_n a_n \exp[-i(\Phi_n - i\varphi_0 - \beta \sin(\Omega_m(t - n\tau_r))].
\]

The homodyne beat signal \( S(t) \) given by Eq. (18) can be rewritten as

\[
S(t) = \exp(i\Delta \phi)|\dot{\epsilon}_c|^2 \sum_{n=0}^{\infty} a_n a_n^* \\
\times \exp \{-i(\Phi_n - \Phi_0) - \beta \sin(\Omega_m(t - n\tau_r)) \}
\]

\[
+ \beta \sin(\Omega_m(t + T - \tau_r)) \exp[-i(\omega_x + n\Delta)T + i(\omega_x + n\Delta)(t + T)] + \text{c.c.}
\]

We are interested in the part of the signal \( S(t) \) that oscillates with the seed modulation frequency \( \Omega_m \). More particularly, we consider slow frequency modulation \( \Omega_m = \gamma e T \ll \Delta \). To evaluate such modulation we need only the terms in Eq. (60) for which \( l = n \). Then we have the expression

\[
S(t) = \exp(i\Delta \phi)|\dot{\epsilon}_c|^2 \sum_{n=0}^{\infty} |a_n|^2 \exp[i(\omega_x + n\Delta)T] \\
\times \exp \{+2i\beta \sin(\Omega_m T/2) \}
\]

\[
\times \cos[\Omega_m(t - n\tau_r + T/2)] \} + \text{c.c.}
\]

Using the formula

\[
\exp(iM \cos \Phi) = \sum_{n=-\infty}^{\infty} i^n J_n(M) \exp(in\Phi),
\]

where \( J_n(x) \) is the \( n \)-th order Bessel function, we obtain the result

\[
S(t) = \exp(i\Delta \phi)|\dot{\epsilon}_c|^2 \sum_{n=0}^{\infty} |a_n|^2 \\
\times \exp [i\Omega_m(t - n\tau_r + T/2)] \\
\times \exp [+i(\omega_x + n\Delta)T] \\
\times \sum_{l=\infty}^{-\infty} i^l J_l(2\beta \sin(\Omega_m T/2)) + \text{c.c.}
\]

The processing of this signal can proceed in two ways: either by means of a phase sensitive detector, or else with a detector that is insensitive to phase. As we will show next, there are notable advantages for the latter.

### 6.1. Phase sensitive detectors

Consider phase-sensitive detection of the modulation signal. Typically this produces, as a function of modulation frequency \( \Omega_m \) and synchronous detector phase \( \Phi_s \), a signal
\[ S_{\text{phase}}(\Omega_m, \phi_d) = \frac{Q_m}{2\pi} \int_0^{2\pi/Q_m} dt \, S(t) \sin(\Omega_m t + \phi_d). \]  

(63)

With the same assumption which were made to obtain Eq. (35) the signal of Eq. (62) can be written as

\[ S_{\text{phase}}(\Omega_m, \phi_d) = -n_w A^2 |c_0|^2 \sqrt{\frac{\pi}{2}} J_1(2\beta \sin \frac{\Omega_m T}{2}) \times \cos \left[ (-\Omega_m + \gamma_c T) \tau_r n_m + \Omega_m T/2 - \phi_d + \omega_s T + \Delta \phi \right] \times \exp \left[ -\frac{(\Omega_m - \gamma_c)^2 \tau_s^2 n_m}{8} \right]. \]  

(64)

An examination of the argument of the exponential reveals that the signal \( S_{\text{phase}} \) is a narrow resonance function of the detuning of the modulation frequency \( \Omega_m \) from the frequency \( \gamma_c T \). This signal depends, through the argument of the cosine, on the phase

\[ (-\Omega_m + \gamma_c T) \tau_r n_m + \Omega_m T/2 - \phi_d + \omega_s T + \Delta \phi. \]  

(65)

This dependence can lead to a resonance shape that differs significantly from a Gaussian form.

6.2. Phase insensitive detection

Phase sensitive detection of the interference beat signal introduces a dependence upon the material phase shift \( \Delta \phi \). Thus it will be much more convenient to use amplitude detection. Then the signal is proportional to the square of the amplitude of the variable components,

\[ S_{\text{amp}} = \sqrt{(S_{\phi_{\gamma=0}})^2 + (S_{\phi_{\gamma=\pi/2}})^2} \]

\[ = J_1(2\beta \sin \frac{\Omega_m T}{2}) |c_0|^2 n_w A^2 \sqrt{\frac{\pi}{2}} \times \exp \left[ -\frac{(\Omega_m - \gamma)^2 \tau_s^2 n_m}{8} \right]. \]  

(66)

Here there is no dependence upon the material phase shift \( \Delta \phi \). Note that this signal can easily be obtained using a standard RF-spectrum analyzer. Regarded as a function of modulation frequency the signal appears as a Gaussian profile. The width of this intensity profile is the same as that of the amplitude profile shown in the beat-frequency spectrum without cavity fluctuations of Fig. 4(a).

To make use of the formula given in Eq. (66) for a determination of \( h \), one would vary the FM frequency \( \Omega_m \) to find the value \( \Omega_{\text{max}} \) for which \( S_{\text{amp}} \) is a maximum. The delay time is then

\[ T = \frac{\Omega_m}{\gamma_c}. \]  

(67)

Once an approximate delay time is established, the strength of the peak value can be optimized by adjusting the modulation index \( \gamma_c \). For application to surface profilometry, the reference arm of the interferometer would be adjusted so that the beat frequencies have values within a convenient range, much larger than the width \( \Gamma_{\text{res}} \), and the modulation index \( \beta \) would be adjusted to optimize heights within a range of values.

7. Achievable accuracy

We will show here that the accuracy of interferometry based on the FSF laser can be comparable to the accuracy of conventional interferometry, i.e., a fraction of a wavelength. The analysis starts by expressing the error in step-height measurement \( \delta h \) in terms of beat frequency error \( \delta \omega \) as

\[ \delta h = \frac{c}{2\gamma_c} \delta \omega = \frac{L}{2A} \delta \omega. \]  

(68)

We can make this distance error small, for fixed error in beat frequency, by increasing the chirp rate \( \gamma_c \), by making the cavity circumference \( L \) smaller, or by increasing the frequency shift per round trip \( A \). Ultimately, however, the achievable error is set by the accuracy with which one can measure the beat frequency. The accuracy of this measurement is determined primarily by the width of the beat-frequency spectrum \( \Gamma_{\text{res}} \). We take the error to be some fraction \( \epsilon \) of this width:

\[ \delta \omega = \epsilon \Gamma_{\text{res}} \]  

(69)

and write the height error as proportional to a coherence length \( h_c \).
\[ \delta h = \varepsilon c. \]  

We can then use the result
\[ \Gamma_{\text{res}} = 2nh_0 \tau_t \]  and \( \gamma_c = \Delta / \tau_t \) to write this parameter variously as
\[ h_c = \frac{c\Gamma_{\text{res}}}{2\gamma_c} = \frac{c}{n_w A} = \frac{c}{\delta \omega_L}, \]  

where \( \delta \omega_L = n_w A \) is the width of the FSF spectrum. In reference [26] we give several estimates of this width. We show, for example, that \( \delta \omega_L \) and \( n_w \) are related to \( \Gamma_{\text{f}} \), the bandwidth of the filtering mechanisms within the FSF cavity, through the approximation
\[ \delta \omega_L = n_w A \approx (\Delta \Gamma_{\text{f}}^2)^{1/3}. \]  

Thus we can write
\[ h_c = \frac{c}{(\Delta \Gamma_{\text{f}}^2)^{1/3}}. \]  

For example, with filter bandwidth of 100 nm for light with central wavelength 800 nm (a bandwidth of 47 THz), and a frequency shift per round trip of 200 MHz, this length is 63 \( \mu \)m.

To estimate \( \varepsilon \) we here consider the error present in various realizations of FSF interferometry.

**7.1. Phase noisy seed**

First we consider the FSF laser to be seeded by a cw laser subject to phase noise. When the resonance width \( \Gamma_{\text{res}} \) is sufficiently small, one can consider a direct measurement of the beat frequency using a frequency meter to count cycles within some observation time interval \( T_0 \). This time should be significantly longer than the resolving time for the resonance width, \( T_0 \gg 1/\Gamma_{\text{res}} \). If we consider a measurement time of \( T_0 = 10^{-4} \) s, then it is necessary that the resonance width be larger than 10 kHz, meaning a width of a few tens of kHz, in order that \( T_0 \gg 1/\Gamma_{\text{res}} \). In this case \( \varepsilon \) is roughly
\[ \varepsilon \approx \frac{1}{\sqrt{\Gamma_{\text{res}} T_0}}. \]  

This same accuracy is obtainable with other methods. For example in the frequency domain, using a spectrum analyzer that scans the frequency, one can measure the frequency in the peak in the spectral distribution. The result is again Eq. (74).

**7.2. Frequency modulated seed**

Next we consider using a frequency modulated seed and measuring the modulation frequency \( \Omega_m \) for which the output signal of Eq. (66) maximizes. There are several possible contributions to the error in measuring the frequency of this peak value.

To use the resonance (66) for the measurement of the delay time \( T \) one can additionally slowly modulate the frequency \( \Omega_m \). We assume that
\[ \Omega_m(t) = \Omega_m^{(0)} + \Omega_m \cos(2\pi ft) \]  where \( f \) is a slow modulation frequency and neglect optical length fluctuations. Then for small modulation amplitude \( \delta \Omega_m \) and detunings \( (\delta \Omega_m, [\Omega_m^{(0)} - \gamma_c T] \ll \Gamma_{\text{res}}) \) the magnitude of the detected useful signal varies linearly with modulation detuning \( \Omega_m^{(0)} - \gamma_c T \),
\[
S_{\text{useful}} = J_1(2\beta \sin[\Omega_m T/2])|\bar{\varepsilon}_c|^2(n_w A^2 \sqrt{\pi}) \times \frac{(\Omega_m - \gamma_c T)(\tau_m n_w)^2}{4 \delta \Omega_m}. \]  

The noise signal \( S_{\text{noise}} \) can be estimated using (34) for delta-correlated phase fluctuations and small \( DT \ll 1 \) as
\[
S_{\text{noise}} = \sqrt{J_1(\gamma_c T)\sqrt{\delta \omega_d}} \approx \sqrt{2\pi|\bar{\varepsilon}_c|^2 n_w A^2 \sqrt{T^2D^2}} \sqrt{\delta \omega_d},
\]  

where \( \delta \omega_d = 1/T_0 \) is the filter passband of the detection system. We take the frequency error \( \delta \omega \) to be the value of the detuning \( \Omega_m - \gamma_c T \) where signal and noise are equal, \( S_{\text{noise}} = S_{\text{useful}} \). This leads to the estimate
\[
\delta \omega \approx \frac{2(\Gamma_{\text{res}})^2 \sqrt{T^2} \delta \omega_d}{J_1(2\beta \sin[\Omega_m T/2]) \delta \Omega_m}. \]  

For rough estimation one can take \( \delta \Omega_m \approx \Gamma_{\text{res}} \) and, with appropriate choice of the modulation index \( \beta \), \( J_1(2\beta \sin[\Omega_m T/2]) \approx 0.5 \). Then
\[
\delta \omega \approx 4\Gamma_{\text{res}} \sqrt{T^2D \delta \omega_d},
\]  

which leads to the result
\[
\varepsilon \approx 4T \sqrt{D \delta \omega_d} = \frac{1}{\sqrt{T_0 \Gamma_{\text{res}}}} \times 4T \sqrt{D \Gamma_{\text{res}}}, \]  

where
\[ \delta \omega \approx 2\Gamma_{\text{res}} \sqrt{T^2D \delta \omega_d} \]  

for delta-correlated phase fluctuations and small \( DT \ll 1 \) as
\[ S_{\text{noise}} = \sqrt{J_1(\gamma_c T)\sqrt{\delta \omega_d}} \approx \sqrt{2\pi|\bar{\varepsilon}_c|^2 n_w A^2 \sqrt{T^2D^2}} \sqrt{\delta \omega_d}, \]  

where \( \delta \omega_d = 1/T_0 \) is the filter passband of the detection system. We take the frequency error \( \delta \omega \) to be the value of the detuning \( \Omega_m - \gamma_c T \) where signal and noise are equal, \( S_{\text{noise}} = S_{\text{useful}} \). This leads to the estimate
\[
\delta \omega \approx \frac{2(\Gamma_{\text{res}})^2 \sqrt{T^2} \delta \omega_d}{J_1(2\beta \sin[\Omega_m T/2]) \delta \Omega_m}. \]  

For rough estimation one can take \( \delta \Omega_m \approx \Gamma_{\text{res}} \) and, with appropriate choice of the modulation index \( \beta \), \( J_1(2\beta \sin[\Omega_m T/2]) \approx 0.5 \). Then
\[
\delta \omega \approx 4\Gamma_{\text{res}} \sqrt{T^2D \delta \omega_d},
\]  

which leads to the result
\[
\varepsilon \approx 4T \sqrt{D \delta \omega_d} = \frac{1}{\sqrt{T_0 \Gamma_{\text{res}}}} \times 4T \sqrt{D \Gamma_{\text{res}}},
\]  

where
where \( D \) is the bandwidth of the seed laser. For example, with \( D = 2\pi \times 1 \text{ MHz} \) and \( h = 1 \text{ cm} \), the error of Eq. (79) is \( 2 \times 10^{-4} \) times smaller than Eq. (74).

### 7.3. Quantum noise

Spontaneous emission in the seed laser itself sets a lower bound on the achievable bandwidth \( D \). As discussed in textbooks on laser theory, this Schawlow–Townes limit can be as small as a few Hz. This bandwidth imposes a fundamental limit on the accuracy of the FM method: when all “technical” noise has been eliminated, there remains an irreducible bandwidth attributable to white noise. As an example, for a linewidth of \( D = 2\pi \times 1 \text{ Hz} \), \( T = 0.5 \text{ ns} \) and \( \Gamma = 2\pi \times 10 \text{ kHz} \) the factor \( 4T\sqrt{\Gamma_{\text{res}}} \) is \( 10^{-6} \).

Another ultimate limit on observation comes from shot noise in the detector. We estimate this to be

\[
\epsilon \approx \sqrt{\frac{1}{T_0} \left( \frac{h\omega}{\eta P} \right)} = \frac{1}{\sqrt{\Gamma_{\text{res}} T_0}} \times \sqrt{\frac{\Gamma_{\text{res}}}{R_{\text{phot}}}},
\]

(80)

where \( P \) is the laser power and \( \eta \) is the detector efficiency. \( R_{\text{phot}} \) is the photon counting rate of the detector (laser power times efficiency divided by photon energy \( h\omega \)). With reasonable estimates of these parameters (say \( P = 1 \text{ W}, \eta = 0.5 \)) this value of \( \epsilon \) is \( 2 \times 10^{-7} \) times smaller than Eq. (74), and an order of magnitude smaller than the Schawlow-Townes limit mentioned above.

### 7.4. Examples

Table 1 gives examples of the accuracy possible, using the preceding formulas and the following parameters

\[
\begin{align*}
\dot{\lambda} &= 800 \text{ nm} \\
h &= 1 \text{ m} & \Lambda &= 2\pi \times 200 \text{ MHz} \\
P &= 1 \text{ W} & \eta &= 0.5
\end{align*}
\]

- The three sections of the table illustrate results for several choices of bandwidth \( \Gamma_c \), height \( h \) and number of discrete components \( n_w \). As can be seen, modulation of the seed offers here a hundredfold improvement in accuracy compared with the reliance on phase noise. The accuracy is still far from the quantum limit.

### 8. Summary and conclusions

The FSF laser offers useful application for high-accuracy interferometry. It allows determination of distance by measuring frequencies, and is insensitive to material phase shifts that occur upon reflection. As we have shown, it is essential that the laser output have available a range of frequencies that include the beat frequency \( \gamma_c T \), for any \( T \) of interest. Although the reliance on noise can provide the needed frequencies, we suggest an alternative based on frequency modulation.
The basic principle is that of chirped radar, here implemented with a non-trivial modification in the form of a FSF laser. The greater chirp rate and extreme linearity of chirp make this far superior to conventional methods.

The position error $\delta h$ can be made small, for fixed error in beat frequency, by increasing the chirp rate $\gamma_c$, by making the cavity circumference $L$ smaller, or by making the frequency shift $\Delta$ larger.

Our theory provides analytic expressions for the intensity of the beat signal as a function of frequency. For any seed frequency, the beat signal has a spectral distribution sharply peaked about the value $\omega_0 = (2\gamma_c c)h$ and thus a measurement of the position of the spectral peak provides a measurement of $h$. The theory also provides a connection between the spectral width $\Gamma_{\text{res}}$ and laser parameters, specifically the effective number of discrete frequency components $n_w$ within the output spectrum of the FSF laser (i.e., the output bandwidth divided by the frequency shift $\Delta$). It is this width that ultimately restricts the possible resolution by limiting the accuracy with which the beat frequency can be determined. Under appropriate conditions the error in beat frequency determination can provide subwavelength accuracy comparable to, or exceeding, what is obtained with conventional interferometry.

To achieve the accuracy $\delta h = 100 \text{nm}$ with a frequency shift of $\Delta = 2\pi \times 10^5$ MHz one needs $n_w \approx 10^6$. This value is very large, but may be achievable with a Ti-Sapphire laser or fiber laser.

\[ F_{\pm}(\omega, T) = \int_{-\infty}^{\infty} d\tau F_{\pm}(\tau, T) \exp [i\omega \tau] \quad (A.1) \]

of the functions

\[ F_{\pm}(t_1, t_2) = \langle \exp [-i\varphi_s(t) + i\varphi_s(t + t_1) + i\varphi_s(t + t_2) \pm i\varphi_s(t + t_2 + t_1)] \rangle, \quad (A.2) \]

where $\langle \cdot \cdot \cdot \rangle$ denotes an averaging over realizations of the stochastic phase $\varphi_s(t)$. These functions incorporate all of the effects of noise upon the interferometer beat spectrum.

We assume that the time derivative of the phase $\varphi_s(t)$ is a stochastic process $\dot{\xi}(t)$,

\[ \dot{\xi}(t) = \xi(t), \quad (A.3) \]

with zero mean and exponential correlation $[\xi(t)$ is an Ornstein–Uhlenbeck process]

\[ \langle \xi \rangle = 0, \quad \langle \xi(t)\xi(t') \rangle = G \Delta \exp (-G|t - t'|). \quad (A.4) \]

We then evaluate stochastic averages of exponentiated noise, using the formula

\[ \langle \exp(i\Phi) \rangle = \exp \left[ -\frac{1}{2} \langle \Phi^2 \rangle \right]. \quad (A.5) \]

For $F_-(T, \tau)$ one obtains the result

\[ F_-(t_1, t_2) = \exp \left( -\frac{1}{2} \left[ g(t, t) + g(t + t_1, t + t_1) + g(t + t_1 + t_2, t + t_1 + t_2) + g(t + t_2, t + t_2) - 2g(t, t + t_1) + 2g(t, t + t_2 + t_1) - 2g(t + t_1 + t_2, t + t_2) - 2g(t + t_1, t + t_2) + 2g(t + t_1, t + t_2 + t_1) + 2g(t + t_1 + t_2, t + t_2) \right] \right), \quad (A.6) \]

where, with $t_<$ as the lesser of $t_1$ and $t_2$,

\[ g(t_1, t_2) \equiv \langle \varphi_s(t_1)\varphi_s(t_2) \rangle \]

\[ = \int_{-\infty}^{t_1} \, dt' \int_{-\infty}^{t_2} \, dt'' \langle \xi(t')\xi(t'') \rangle \]

\[ = GD \int_{0}^{t_1} \, dt' \int_{0}^{t_2} \, dt'' \exp (-G|t'' - t'|) \]

\[ = 2Dt_< \frac{D}{G} \left[ 1 - \exp (-Gt_1) - \exp (-Gt_2) + \exp (-G(t_2 - t_1)) \right]. \quad (A.7) \]
The functions $F_\pm(t_1, t_2)$ are even functions of $t_1$ and $t_2$. For positive $t_1$ and $t_2$ one obtains the results

$$F_+(t_1, t_2) = \exp \left[ -2Dt_1 - \frac{D}{G} \left( -2 + 2 \exp(-Gt_1) + 2 \exp(-Gt_2) + \exp(-G|t_2 - t_1|) \right) \right],$$

(A.8)

$$F_+(t_1, t_2) = \exp \left[ -2D(2t_1 - t_2) - \frac{D}{G} \left( -2 + 2 \exp(-Gt_1) + 2 \exp(-Gt_2) + \exp(-G|t_2 - t_1|) \right) \right].$$

(A.9)

These expressions, when used with formulas for $J_S(\omega)$, allow one to calculate the spectrum of beating (24) for arbitrary values of the phase fluctuation parameters $G$ and $D$, the delay time $T$ and the laser parameters.

A.2. Slow fluctuation

The preceding subsection dealt with an Ornstein–Uhlenbeck process governed by parameters $G$ and $D$. To get a simpler expression we consider further two limiting cases: (1) the phase changes slowly, $G \ll D$ and (2) the noise is delta correlated, $G \to \infty$. Bearing in mind the optimal conditions for narrow resonance (24) observation we will deal only with the function $F_-(\tau, T)$. In the first case, of slow phase fluctuations, the expression for $F_-$ becomes

$$F_-(\tau, T) = \begin{cases} \exp \left[ -1/3DG^2\tau^3(3T - \tau) \right], & \tau \leq T, \\ \exp \left[ -1/3DG^2\tau^2(3T - \tau) \right], & \tau > T. \end{cases}$$

(A.10)

For weak fluctuations, $DG^2T^3 \ll 1$, the characteristic width of $F_-(\tau, T)$ as function of $\tau$ is $\delta \tau = 1/(DG^2T^3)$. Hence in this case the Fourier component $F_-(\omega, T)$ is a narrow function of $\omega$ with characteristic width $\delta \omega = DG^2T^3 \frac{1}{T} \ll \frac{1}{T}$. As the fluctuation intensity increases the function $F_-(\tau, T)$ becomes narrower and hence the width of $F_-(\omega, T)$ increases. For $DG^2T^3 \gg 1$ we obtain a Gaussian profile for $F_-$.

$$F_-(\omega, T) = T \frac{1}{\sqrt{DG^2T^3}} \exp \left[ -\frac{\omega^2}{4DG^2T^3} \right].$$

(A.11)

The width of this is $\delta \omega = 2\sqrt{DG^2T^3} \frac{1}{T} \gg \frac{1}{T}$.

A.3. Rapid fluctuations

Here we assume the noise fluctuates rapidly, and consider the limit $G \to \infty$ when the correlation time is infinitesimal, i.e., the noise is delta correlated. Then the expression for $F_-$ becomes

$$F_-(t_1, t_2) = \exp \left[ -2Dt_1 \right].$$

(A.12)

When $\omega \neq 0$ we obtain for the spectral function $F_-(\omega, T)$ the formula

$$F_-(\omega, T) = 2 \text{Re} \left[ \frac{1 - \exp(-2DT + i\omega T)}{2 - i\omega} \exp(-2DT) \frac{1 - \exp(i\omega T)}{-i\omega} \right].$$

(A.13)

For small $DT \ll 1$ this function has the width $\delta \omega \approx \frac{1}{T}$. With increasing $DT \gg 1$ the width of $F_-(\omega, T)$ increases. The function is Lorentzian

$$F_-(\omega, T) = \frac{4D}{4D^2 + \omega^2}$$

(A.14)

with width $\delta \omega = D/2$.

References