arXiv:hep-th/9701113 v1 22 Jan 1997

On water, steam and string theory

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(Review lecture for physicists and non-physicists as part of the requirements for "Habilitation" at Bern University.)

At a pressure of 220 atmospheres and a temperature of 374 ^{0}C there is a second-order phase transition between water and steam. Understanding it requires a key concept of elementary particle physics: the renormalization group. Its basic ideas are explained with images from computer simulations of the lattice gas model. It is then briefly reviewed how the renormalization group is used to compute critical coefficients for the water-steam phase transition, in good agreement with experiment. Finally, some applications in particle physics and string theory are mentioned. The appendix contains a sample of the author's results on the curious features of renormalization group flows in theories with dynamical gravity.

1. INTRODUCTION

We all know that water can come in different phases: as a solid, a liquid or a gas. Above 100^o C, water becomes steam, and below 0^o C it becomes ice. We also know that the boiling and freezing temperatures of water depend on the pressure. On top of Jungfrau, for example, water already boils at less than 90^o C because of the lower pressure. The pressure dependence of the boiling and freezing temperatures of water can be plotted in a phase diagram (Fig. 1).

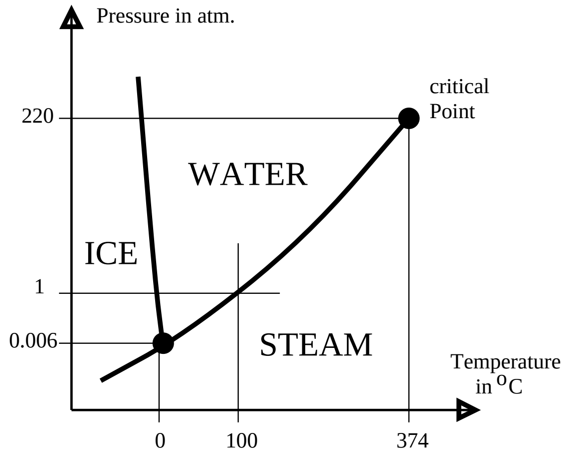

Figure 1. Phase diagram of water.

Two points in this phase diagram catch one's eye. The first one is the so-called triple point at 0^o C and 0.006 atmospheres pressure. Below this pressure the liquid phase of water disappears, and ice directly evaporates when heated up. *At* the triple point, ice, water and steam can coexist. So this is kind of interesting. But what I would like to convince you of in my talk is that another point in this phase diagram is *really* interesting, because key concepts of elementary particle physics are hidden in it. This is the so-called critical point at 374^o C and 220 atmospheres pressure.

The critical point is the point at which the phase separation curve between water and steam ends. First of all: how can the separation curve between two obviously different things such as water and steam just end? Well, when we say that water and steam are obviously different we mean that they have drastically different properties such as densities. Fig. 2 shows the density of water at 1 atm. pressure as a function of the temperature. The density jumps to less than $\frac{1}{1000}$ of its value at the point where water becomes steam. This is called a first-order phase transition.

If we now raise the pressure, we find that the densities of water and steam at boiling temperature approach each other. The critical point is the point where both become equal, so that the distinction between water and steam seems

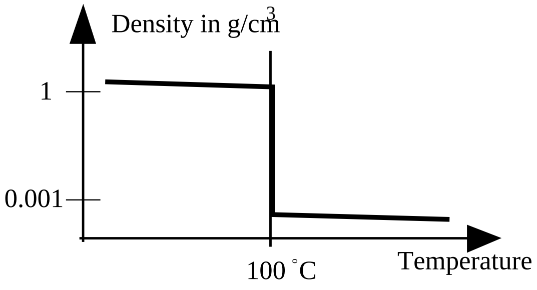

Figure 2. Density of water at 1 atm. pressure.

to disappear. Now, if we measure exactly *how* the densities of water and steam approach each other as we move along the phase separation line, we find an interesting behavior: the density is not a smooth function of the temperature, but has a singularity.

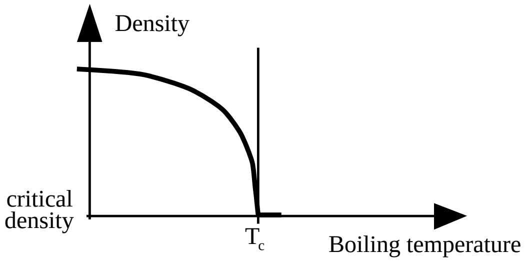

Figure 3. Density of water along the phase separation curve in Fig.1.

More precisely, close to the critical point the deviation of the density ρ from the critical density ρ_c is proportional to a power of the deviation of the temperature T from the critical temperature T_c:

$$|\rho - \rho_c| \sim |T - T_c|^\beta . \qquad (1)$$

The so-called critical coefficient β can be measured experimentally. One finds:

$$\beta \sim 0.33 \pm 0.01 . \qquad (2)$$

This is called a second-order phase transition.

What I would like to do in my talk is to first give you an idea of what exactly happens at the critical point. I will explain things with the help of images from a computer simulation of a simple toy model, the so-called lattice gas model. As a test that this is the correct idea I will show that it indeed reproduces the power law (1), and that even the coefficient β comes out right.

Those are old ideas. The key idea is that of the so-called renormalization group[1]. This renormalization group has important applications in particle physics, some of which I will mention in the end. My own results on the renormalization group in the presence of gravity and its relation to perturbative string theory can be found in my habilitation thesis and some of them are summarized in the appendix.

2. THE LATTICE GAS MODEL

One might worry that one needs to know a lot about the structure of water molecules, hydrogen bridges, and so on, in order to explain why β is about 0.33. But surprisingly, if one measures β for other substances with water-like phase diagrams like Xenon or carbon dioxide, one always finds roughly 0.33.[2] This value seems to be a universal property of "Van-der-Waals" gases, i.e., of gases with two basic properties: first, the water molecules have a hard core, so that they cannot come arbitrarily close. And second, there is a weak attractive force between the molecules.

May I now introduce to you a simple toy model of such a gas, the so-called lattice gas model. In the lattice gas model we imagine that three-dimensional space is a lattice as in Fig. 4. Only two dimensions are drawn for simplicity. At each lattice point there is either one molecule or none. So we equip each lattice point i with an occupation number n_i that is either 0 or 1. The rule that a lattice point is occupied by at most one molecule represents the hard core of the molecules.

We define the energy of a given configuration

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[1] For a review, see [1]. The calculation of critical coefficients is discussed, e.g., in [2, 3].

[2] For a classic experiment that measures β in CO_2, see [4].

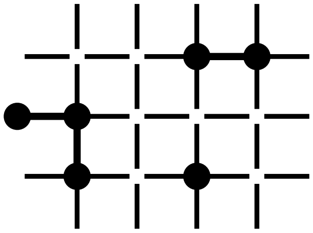

Figure 4. The lattice gas model.

of occupation numbers as

$$E \;=\; - \sum_{\text{neighbors} <ij>} n_i \;\cdot\; n_j \;. \tag{3}$$

Here, i and j are two neighboring lattice points. So the sum is a sum over all links. Put differently, for each pair of neighboring molecules the energy is decreased by -1. E.g., the energy of the configuration in Fig. 4 is -3 since there are three links that are occupied at both ends. This definition of the energy represents the attractive force between the molecules, because the molecules now try to stick together in order to minimize the energy.

We now imagine that the occupation numbers are subject to statistical fluctuations, with the total number of molecules

$$N \;=\; \sum_i n_i \tag{4}$$

held fixed. Just as in a real gas the molecules can fly from one point to another, in the lattice gas they can disappear at one point and reappear at another.

Rather than considering a closed system with fixed particle number it is actually more useful to consider a system that is part of a larger system with which it can exchange particles and with which it is in equilibrium. This is equivalent to replacing the condition (4) of fixed molecule number by adding the term

$$\mu \sum_i n_i \tag{5}$$

to the energy E. μ is the so-called chemical potential whose value is controlled by the larger system. This term punishes the smaller system for each particle it contains, thus providing a counterweight to the attraction term (3), which tries to attract as many molecules as possible.

This is the lattice gas model. You will object that the lattice gas is, at best, a rough caricature of a real gas. But you will see that the caricature is good enough to reproduce the critical coefficient $\beta = 0.33 \pm 0.01$.

3. THE ISING MODEL

To study the lattice gas model, it is useful to replace the occupation numbers n_i by new variables s_i:

$$s_i \;\equiv\; 2n_i \;-\; 1 \;=\; \pm 1 \;. \tag{6}$$

s_i is $+1$ if the lattice point i is occupied, and -1 if not. Expressed in terms of the new variables s_i, the energy (3+5) is:

$$E \;=\; -\frac{1}{4} \sum_{<ij>} s_i s_j \;-\; B \sum_i s_i \tag{7}$$

plus an unimportant constant, where $B = 1 - \frac{\mu}{2}$. This is the so-called Ising model, which is thus equivalent to the lattice gas model. In the Ising model, the occupation numbers s_i have the interpretation of spins that can point either up ($s_i = +1$) or down ($s_i = -1$), while B has the interpretation of an external magnetic field. Because of the first term in the energy, neighboring spins try to point in the same direction, while they try to point in the direction of the external magnetic field because of the second term.

What is the ground state of the Ising model - the state of lowest energy? For $B = 0$ there are two ground states: in the first one, all spins point up. In the second one, all spins point down. Let us call them “Plus” and “Minus”. For positive magnetic field the ground state is “Plus”. For negative magnetic field the ground state is “Minus”.

Imagine the system is in the state “Plus”. To reverse a single spin we need $4 \cdot \frac{1}{2}$ units of energy since the spin has 4 neighbors and for each neighboring pair we must raise the energy from $-\frac{1}{4}$ to $+\frac{1}{4}$ (see Fig. 5a). Similarly one needs $\frac{L}{2}$ units of energy to create a "minus bubble" of length L (Fig. 5b). One can say that bubbles have a surface tension in the sense that their energy is proportional to the circumference of the bubble, or to the surface of the bubble in the three-dimensional model.

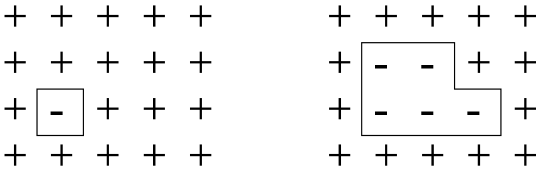

Figure 5. a) A reversed spin; b) a minus bubble of length $L = 10$.

We now imagine that the spins fluctuate. Let $\{s_i\}$ be a given configuration of all s_i. As usual in Statistical Mechanics, we consider the ensemble of all possible spin configurations $\{s_i\}$, weighted by the so-called "Gibbs factor"

$$\exp\{-\frac{E(\{s_i\})}{T}\} , \tag{8}$$

where T is the temperature. The Gibbs factor can be thought of as a relative probability that is assigned to each spin configuration. E.g., the expectation value of the spin at the point i is defined as

$$< s_i > = \frac{1}{Z} \sum_{\{s_i\}} s_i \ \exp\{-\frac{E(\{s_i\})}{T}\} . \tag{9}$$

Z is a normalization constant,

$$Z = \sum_{\{s_i\}} \exp\{-\frac{E(\{s_i\})}{T}\} , \tag{10}$$

that normalizes the probabilities such that their sum is 1. Z is called the partition sum.

The Gibbs factor (8) suppresses spin configurations with high energy. More precisely, for small temperatures all configurations are very unlikely relative to the one with lowest energy. Fig. 6 shows images of a computer simulation of the Ising model on a lattice of size 500 times 200.[3] "+" spins are black and "–" Spins are white. We see two typical spin configurations at low temperature and zero magnetic field. The system is either in the state "Plus" with only a few negative spins, or it is in the state "Minus" with only a few positive spins. Those are our two phases. Because of their interpretation in the lattice gas model (see (6)) I shall call the upper phase "Water" and the lower phase "steam".

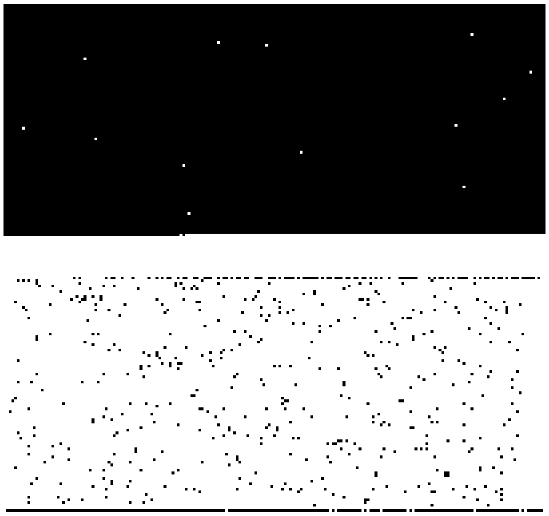

Figure 6. "Water" (above) and "steam" (below) in the Ising model at low temperature.

For very high temperatures, on the other hand, the Gibbs factor is almost one for all energies. All spin configurations are therefore equally probable. So the typical configuration is a mix of randomly distributed spins (Fig. 7) that is neither water nor steam. There are no different phases.

We can now draw the phase diagram of the Ising model in Fig. 8. The temperature is plotted to the right and the magnetic field upwards.

As we have seen there is a phase separation curve at low temperature and zero magnetic field,

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[3] A simulation program for the Ising model is available at http://penguin.phy.bnl.gov/www/xtoys/xtoys.html.

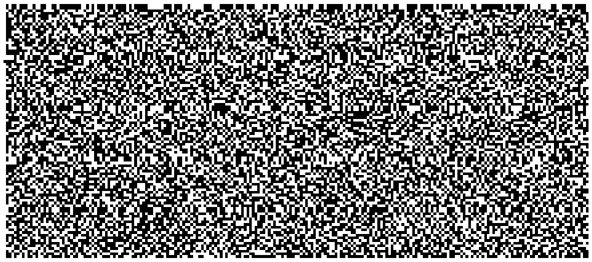

Figure 7. Ising model at very high temperature.

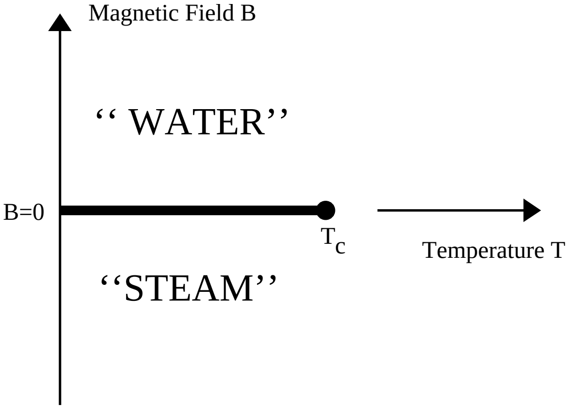

Figure 8. Phase diagram of the Ising model.

$B = 0$. The phase transition is *first* order in the sense that the spin expectation value $< s_i >$ in (9) - and thereby the density of molecules in the lattice gas model – jumps as a function of B. This is shown in Fig. 9. Starting from the state "water" at $B = 0$ (upper left corner) we apply a small magnetic field B. In a time-dependent simulation of the model, sooner or later bubbles appear despite of the surface tension that are large enough to spread and turn the system into "steam" (lower right corner) (for B constant in time). One could say that the "water boils".

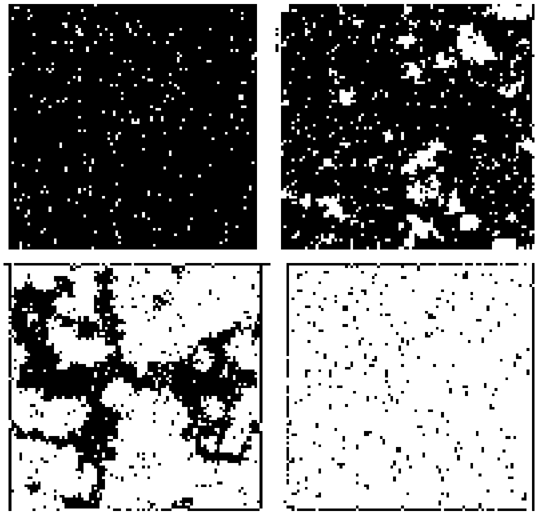

Figure 9. "Boiling" in the Ising model.

We have seen that there is no difference between "water" and "steam" for high temperature. So we assume that the phase separation curve ends in a critical point at some critical temperature T_c. The analogy with the phase diagram of water (Fig. 1) is clear.

4. CLOSE TO THE CRITICAL POINT

Suppose the Ising model is at the phase separation curve at low temperature T in the state "water" (e.g., $T = \frac{1}{2}T_c, B = 0$). We now raise the temperature, thus approaching the critical point along the separation curve. As T grows, spin configuration with higher energy get less strongly suppressed by the Gibbs factor (8). Thus, the statistical fluctuations become more important and the typical configuration contains more and bigger bubbles. Fig. 10 shows typical configurations for temperatures $T = 0.9\,T_c$ (above), $T = 0.98\,T_c$ (middle), and $T = T_c$ (below).

Let us call the average size of a droplet ξ. You can see clearly that ξ grows with the temperature. At the critical temperature (lower picture), ξ seems to diverge. There you see black droplets inside white droplets inside black droplets and so on. You cannot really distinguish whether the

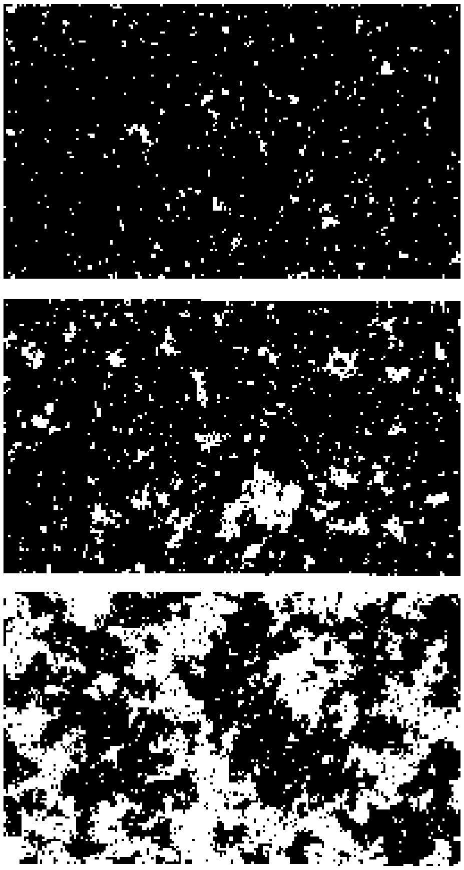

Figure 10. Approaching T_c from below: $T = 0.9\ T_c$ (above); $T = 0.98\ T_c$ (middle); $T = T_c$ (below).

picture shows black droplets in a white sea or white droplets in a black sea. The distinction between the phases disappears.

Let us now approach the critical temperature from above, again at zero magnetic field. Fig. 11 shows typical configurations for temperatures $T = \infty$ (above), $T = 1.08\ T_c$ (middle) and $T = 1.01\ T_c$ (below). At infinite temperature the spins are randomly distributed. As the temperature drops, the spins start to feel the interaction with their neighbors through the Gibbs factor. Therefore structures form. Let us now call ξ the average size of the structures, the so-called correlation length. You see again that ξ grows as we get closer to the critical temperature.

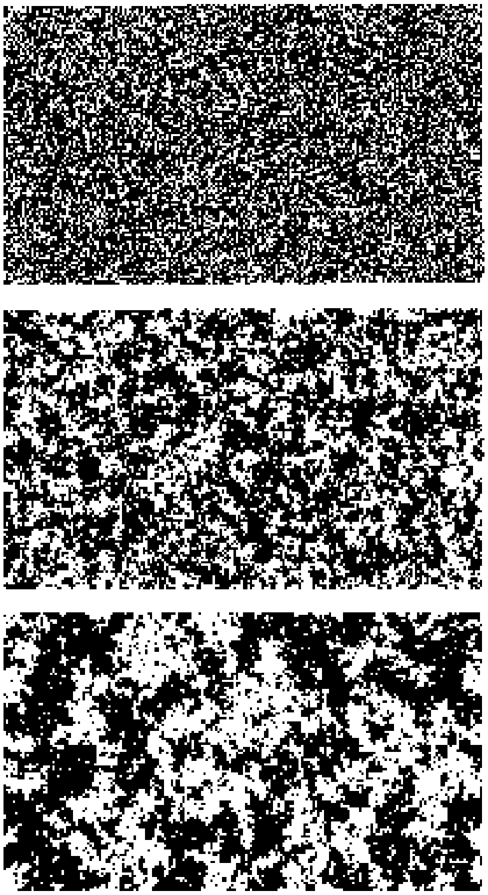

Figure 11. Approaching T_c from above: $T = \infty$ (above); $T = 1.08\ T_c$ (middle); $T = 1.01\ T_c$ (below).

At the critical point it seems again as if *the correlation length diverges*. Structures of all magnitudes can be recognized in a typical configuration (Fig. 12, above). It is easy to imagine that the structures you see are only part of even larger structures that do not fit on the screen.

This divergence of the correlation length can

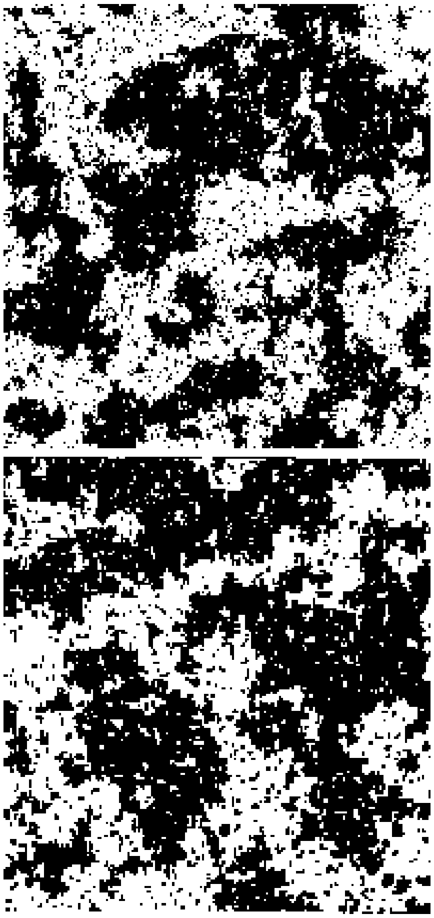

Figure 12. Ising model at the critical point (above); enlarged Ising model at the critical point (below).

also be measured experimentally in water and other substances near the critical point. One finds that the correlation length diverges as an inverse power of the deviation of the temperature from the critical temperature:

$$\xi \ \sim \ \frac{1}{|T - T_c|^{\nu}} \ , \tag{11}$$

with a certain coefficient ν. The relation (11) is similar to the relation (1). For the coefficient ν one measures:

$$\nu \ \sim \ 0.63 \pm 0.01 \ . \tag{12}$$

We will explain this behavior later, together with the behavior (1).

5. SCALE TRANSFORMATIONS

Another property of the critical point can also be seen in Fig. 12: *scale invariance*. By this, the following is meant: Fig. 12, below, shows a portion of another typical configuration of the Ising model near the critical point, but magnified by a factor of 2. This of course makes the picture more coarse-grained. But merely from the distribution of the sizes of the structures you cannot determine well which of the pictures is the magnified one.

How does the Ising model behave under scale transformations when T is near, but not *at* the critical temperature? Let us consider Fig. 11 again. It seems like the lower picture could be a magnified portion of the middle picture. So it seems as if, near the critical temperature, *a scale transformation is equivalent to a change in temperature.*

Since this is a key point, let us make more precise what we mean by a scale transformation. One can define it in two steps. The first step is to make a picture like Fig. 11, below, more coarse-grained. E.g., one can combine 3 × 3 lattice cells into one large cell throughout the picture. When the majority of the 9 small cells is black, the large cell is black; otherwise it is white (Fig. 13).

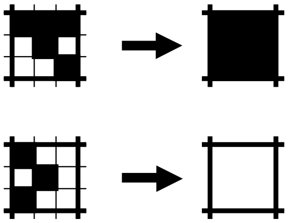

Figure 13. Increasing the lattice spacing.

The second step consists in down-sizing the picture by a factor 3. We call both steps together a scale transformation by a factor of 3: it is as if we regarded the picture from 3 times as far away. The statement is that the resulting picture can be interpreted as a portion of a typical configuration of the Ising model at a different temperature T'. T' is farther away from T than the original temperature: the correlation length has decreased to $\frac{1}{3}$ of its size.

Now imagine an Ising model of infinite extent at a temperature T very close to the critical temperature. Let us call T the "actual", or "bare" temperature. We consider some small portion of the model. We now do a scale transformation as described above by a factor e^τ. This results in an Ising model at a new temperature T' that depends on τ (as opposed to T). Let us call this fictitious temperature T' the "renormalized" temperature $T_{ren}(\tau)$. We have

$$T_{ren}(\tau = 0) \;=\; T \;. \tag{13}$$

This dependence of the renormalized temperature $T_{ren}(\tau)$ on a change of scale by e^τ is what we call the "renormalization group flow" of $T_{ren}(\tau)$. The critical temperature T_c, at which the system is scale invariant, is called a fixed point of this flow.

In Thermodynamics and in daily life one usually talks only about the bare temperature T. On the other hand, in particle physics one usually talks about renormalized quantities (such as charges), since those are the ones that can be observed.

Next I would like to mention how the idea of the renormalization group flow can be used to explain the power laws (1) and (11) and to determine the coefficients β and ν.

6. CRITICAL COEFFICIENTS

Let us quickly put these statements into formulas. We define the deviation of the temperature T_{ren} from the critical temperature as t:

$$t(\tau) \;=\; T_{ren}(\tau) \;-\; T_c \;. \tag{14}$$

Scale invariance at the critical point means that $t = 0$ is a fixed point of the flow:

$$\frac{d}{d\tau} t(\tau) \;=\; 0 \quad \text{für} \quad t \;=\; 0 \;. \tag{15}$$

When t is nonzero but small, the "flow velocity" $\frac{dt}{d\tau}$ can be expanded in powers of t:

$$\frac{d}{d\tau}\, t \;=\; \frac{1}{\nu}\, t \;+\; c\, t^2 \;+\; d\, t^3 \;+\; ... \tag{16}$$

with coefficients ν, c, d that need to be determined. The right-hand side of the equation is the so-called "beta function" $\beta(t)$. The flow of t is shown in Fig. 14. The arrows represent the flow velocity $\beta(t)$. The renormalized temperature moves away from T_c as the scale is increased.

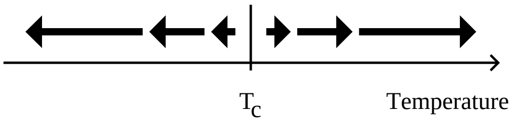

Figure 14. Flow of the renormalized temperature.

For very small t, the terms t^2, t^3, etc. in the beta function in (16) can be neglected. (16) then becomes a linear differential equation that can easily be solved:

$$t(\tau) \;\equiv\; |T_{ren}(\tau) - T_c| \;\sim\; |T - T_c|\; e^{\frac{1}{\nu}\tau} \;. \tag{17}$$

From this, a relation between the correlation length ξ and the actual temperature T can be derived: as discussed in the context of Fig. 11, the correlation length shrinks under a scale transformation by, say, a factor $e^\tau = 3$ as follows:

$$\xi \;\rightarrow\; \frac{1}{3}\, \xi \;. \tag{18}$$

From (17) we read off that such a scale transformation imitates the following change in the actual temperature:

$$|T - T_c| \;\rightarrow\; |T - T_c|\; 3^{\frac{1}{\nu}} \;. \tag{19}$$

(Because the scale transformation and the change in temperature then lead to the same fictitious temperature T_{ren}.) From (18) and (19) we get, near the critical point, the proportionality

$$\xi \;\sim\; \frac{1}{|T - T_c|^\nu} \;, \tag{20}$$

because ξ then transforms as in (18) under the change of temperature (19). This is the relation (11), which is thereby explained by the renormalization group. But ν still needs to be determined.

One now replaces the three-dimensional lattice again by a continuous space. One also replaces the spin variable s_i by a continuous field $\phi(x)$ that describes the average value of the spins in the vicinity of the point x. In the lattice gas model, ϕ is the deviation of the density of molecules from the critical density $\rho_c = \frac{1}{2}$:

$$\phi(x) \;=\; \rho(x) \;-\; \rho_c \;. \tag{21}$$

Next, one tries to describe the lattice gas model at the critical point and at large scales by an effective field theory for the scalar field ϕ. E.g., one replaces the partition sum in (10),

$$\sum_{\{s_i\}} \exp\{-\frac{E(\{s_i\})}{T}\} \;, \tag{22}$$

by a path integral

$$\int [d\phi(x)] \;\; \exp\{-S[\phi(x)]\} \;, \tag{23}$$

where we formally have to do one integral for each point x in space. $S[\phi(x)]$ is a certain effective action for the field $\phi(x)$ that I will come to later. Let me only mention now that the advantage of this continuum formulation is that it is much easier to do computations with it than with the lattice model. In particular, one can test whether it is really true what we have read off from Fig. 11: is near the critical temperature and at large scales a scale transformation by the factor e^τ really equivalent to a change of the temperature $T \rightarrow T(\tau)$?

One finds that this is indeed the case, *provided that* one simultaneously allows for a dimension d_ϕ of the field ϕ. This means that ϕ is rescaled under scale transformations as follows:

$$\phi(x) \;\rightarrow\; \phi_{ren}(x) \;=\; e^{d_\phi \tau} \; \phi(x) \;. \tag{24}$$

Combining (17), (21) and (24), one finds similarly as in deriving (20):

$$|\rho \;-\; \rho_c| \;\sim\; |T - T_c|^{\nu d_\phi} \;. \tag{25}$$

This explains the experimentally observed behavior (1) with $\beta = \nu d_\phi$.

It remains to compute the coefficients ν and d_ϕ. I cannot do this calculation in my talk and only mention that it turns out to be difficult in three dimensions but can relatively easily be done near four dimensions with the help of the effective field theory. So one formally computes ν and d_ϕ in $4 - \epsilon$ dimensions, where ϵ is small, and then tries to set $\epsilon = 1$ in the result. One finds:

$$\nu \;=\; \frac{1}{2} \;+\; \frac{1}{12}\epsilon \;+\; \frac{7}{162}\epsilon^2 \;+\; ... \tag{26}$$

$$d_\phi \;=\; 1 \;-\; \frac{1}{2}\epsilon \;+\; \frac{1}{108}\epsilon^2 \;+\; ... \tag{27}$$

Therefore

$$\beta \;=\; \nu d_\phi \;=\; \frac{1}{2} \;-\; \frac{1}{6}\epsilon \;+\; \frac{1}{162}\epsilon^2 \;+\; ... \tag{28}$$

Those are perturbation expansions in ϵ. Such expansions are useful only when they converge. In this case we seem to be lucky: the coefficients quickly become smaller, and the series seems to converge even in the case $\epsilon = 1$, corresponding to three dimensions (at least so it seems to this order in ϵ). If we plug in $\epsilon = 1$, we get:

$$\nu \;\sim\; 0.63 \quad , \quad \beta \;\sim\; 0.34 \;. \tag{29}$$

This is in nice agreement with the experimental data (12) and (2).

7. UNIVERSALITY

It remains to give a reason why Xenon, carbon dioxide, water, and even the primitive lattice gas model all produce the same critical coefficients at the critical point. The renormalization group explains this as follows: The previously mentioned effective action for the field $\phi(x)$ is of the form

$$S[\phi(x)] \;=\; \int d^3x \; \{ \sum_{i=1,2,3} (\frac{\partial\phi}{\partial x_i})^2 \;-\; V(\phi)\}, \tag{30}$$

where the first term measures the density fluctuations of the gas, and the second term is a potential

$$V(\phi) \;=\; r\phi^2 \;+\; \sum_{k\geq 4} g_k \phi^k \tag{31}$$

that looks as in Fig. 15 for small ϕ and $r < 0$. r is roughly $(T - T_c)$, and the coefficients g_k depend on whether one describes H_2O, CO_2, Xe or

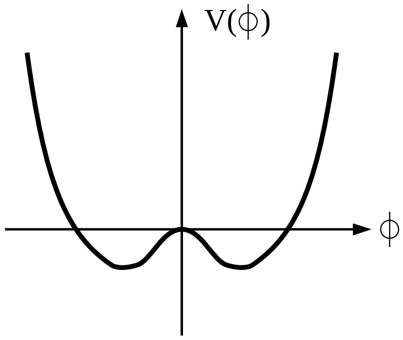

Figure 15. Effective Potential.

the lattice gas model (this action describes these systems only at the phase separation line).

One finds that all the coefficients g_k flow under scale transformations. But they flow in such a way that all points in the space of coupling constants flow at large distances towards a single fixed line, as schematically shown in Fig. 16. In Fig. 14, *only* this line was drawn.

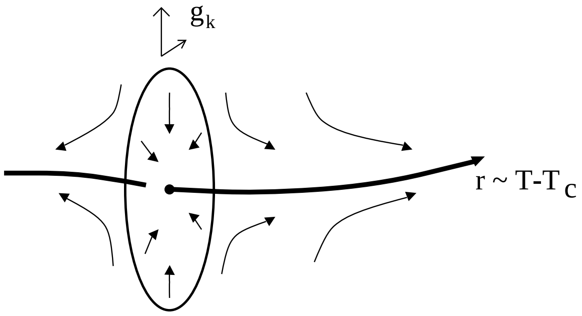

Figure 16. Flow of coupling constants.

On this line there is a fixed point. It obviously corresponds to the critical point. E.g., in $4-\epsilon$ dimensions it lies, to lowest order in ϵ, at

$$r = -\frac{1}{6}\epsilon \quad , \quad g_4 = \frac{1}{144}\epsilon \quad , \tag{32}$$

while the g_k with $k > 4$ are of order ϵ^2. The only relevant parameter, i.e. the only parameter that is observable at large distances is the parameter along the fixed line, $(T - T_c)$. The initial (bare) values of all other parameters g_k are irrelevant in the sense that they flow to the same points at large distances. Water, hydrogen, Xenon and the lattice gas model all have the same critical coefficients because at large scales they are all described by the same one-parameter family of effective potentials - those that correspond to the fixed line.

8. ELEMENTARY PARTICLE THEORY

I have told you that water and steam at the critical point are described by a three-dimensional field theory of a scalar field. Now, there are other fields in nature. An example is the electromagnetic field which is - among other things - responsible for the attractive force between an electron and a positron, which is proportional to the square of the elementary charge e and inversely proportional to the square of the distance (Fig. 17).

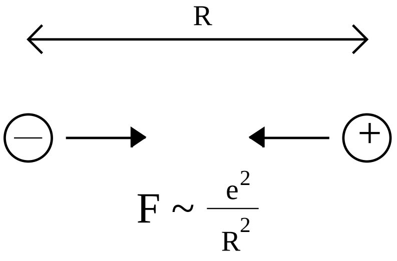

Figure 17. Force between electrons and positrons.

Out of the elementary charge e, Planck's constant $\hbar$ and the speed of light c, a dimensionless constant can be formed, the so-called fine structure constant α:

$$\alpha = \frac{e^2}{\hbar c} \sim \frac{1}{137} \; . \tag{33}$$

The interaction of the electromagnetic field with electrons and positrons is described by a four-dimensional field theory called quantum electrodynamics. It turns out that α flows under scale transformations by the factor e^{τ}, similarly as the temperature before. One can compute the following flow equation:

$$\frac{d}{d\tau}\alpha \;\sim\; -\,\alpha^2 \tag{34}$$

with a certain constant of proportionality. Here, the limit $\tau \to \infty$ corresponds to *large* scales. The corresponding flow diagram is shown qualitatively in Fig. 18. The arrows indicate how α changes under a change of scale. You see that α decreases at large scales and increases at small scales [5].

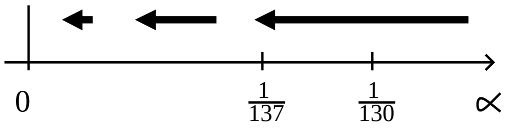

Figure 18. Flow of α; the arrows indicate the flow as the scale is *increased*.

This effect can actually be measured. When electrons and positrons are collided with high energies so that they come close together, one finds that they scatter as if α was already $\frac{1}{130}$ at scales of $10^{-17}m$.

Now, there are not only electric charges in nature but also color charges. Quarks exist in three different "colors": red, green and blue. There are also "color forces" between colored objects, with a corresponding interaction strength α_S. α_S is of order 1 at scales of order $10^{-15}m$ (this is roughly the size of a nucleus).

The interaction of the corresponding "gluon" fields with quarks is described by another field theory, quantum chromodynamics. It turns out that α_S also flows under scale transformations. When α_S is small it obeys a differential equation of the form

$$\frac{d}{d\tau}\alpha_S \;\sim\; +\,\alpha_S^2\ . \tag{35}$$

It is similar to the one for α, but the constant of proportionality is different and in particular has a *reversed sign.* Therefore the arrows in the flow diagram (Fig. 19) point in the other direction. You see that α_S grows at large scales and *decreases* at small scales [6].

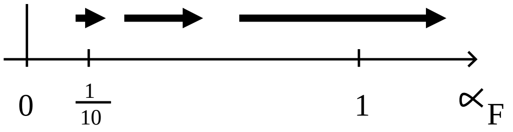

Figure 19. Flow of α_S under increasing the scale.

This effect can also be measured. Experimentally, it seems as if α_S is only about $\frac{1}{10}$ at scales of $10^{-19}m$. Quarks seem to feel the color forces only weakly at small distances. This is called "asymptotic freedom".

9. STRING THEORY

There are many other interesting applications of the renormalization group in elementary particle theory, but I want to conclude by mentioning its importance in the most ambitious branch of particle theory: in string theory. String theory is ambitious in that it promises to unify conventional elementary particle theory with Einstein's theory of gravity.[4]

What is string theory? I can only show two pictures in this talk. Usually it is assumed that elementary particles are pointlike, in the sense that their classical trajectories are described by a vector $\vec{x}(\sigma)$ that depends on a real parameter σ (Fig 20). In string theory one assumes that $\vec{x}$ depends on *two* real parameters σ_1 and

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[4] For an introduction to string theory, see [7].

σ_2 which thereby parametrize a surface - the so-called "world-sheet" of the string (Fig. 21).

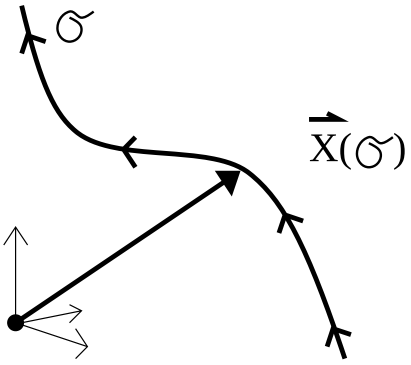

Figure 20. Trajectory of a point particle.

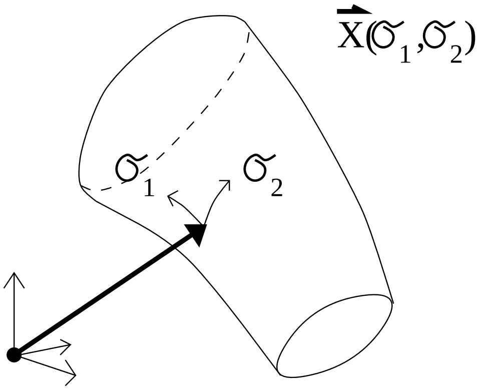

Figure 21. World-sheet of a string.

It has proven to be very useful to reinterpret the coordinates $\vec{x}(\sigma_1, \sigma_2)$ as fields that live on the world-sheet. The motion of the string is then described by a two-dimensional field theory. It can be shown that the corresponding action is, in the simplest case, of the form

$$\int d^2\sigma \sum_{\alpha,i} (\frac{\partial x^i}{\partial \sigma_\alpha})^2, \tag{36}$$

where both $\alpha \in \{1, 2\}$ and the index i of space must be summed over. This action is similar to (30) without potential - but the coordinates x^i are now the *fields*.

This action can be generalized in many ways. World-sheets that live in a curved space rather than in a flat space, e.g., are described by replacing

$$\sum_{\alpha,i} (\frac{\partial x^i}{\partial \sigma_\alpha})^2 \quad \rightarrow \quad \sum_{\alpha,i,j} \frac{\partial x^i}{\partial \sigma_\alpha} \frac{\partial x^j}{\partial \sigma_\alpha} \; g_{ij}(\vec{x}) \tag{37}$$

where $g_{ij}(\vec{x})$ is the metric of the curved space.

Just like the potential in (30) introduces a self-interaction of the field ϕ, $g_{ij}(\vec{x})$ introduces a generally complicated interaction between the fields x^i. And just like the coupling constants g_k in Fig. 16 flow under scale transformations, one finds that the metric $g_{ij}(\vec{x})$ changes under scale transformations on the world-sheet. The corresponding flow equation is quite famous [8]:

$$\frac{d}{d\tau} \; g_{ij}(\vec{x}) \;\sim\; R_{ij}(\vec{x}) \tag{38}$$

plus corrections, where $R_{ij}(\vec{x})$ is the curvature tensor of space at the point $\vec{x}$. In particular, at a fixed point of the renormalization group flow the metric obeys Einstein's field equations $R_{ij} = 0$ (in vacuum) in lowest approximation. In this sense, string theory automatically contains Einstein's theory of gravity.[5]

Those two-dimensional field theories can be generalized such that, in addition to a metric, they contain electromagnetic and other fields. The fixed point equations then become the classical equations of motion of these interacting fields.

There are many such fixed points. Their corresponding properties directly translate into predictions for elementary particle physics that can be compared with observation, such as the number of families of quarks and leptons, the observable gauge group, the number of Higgs fields, etc. An important part of (perturbative) string theory consists of finding the fixed points of the renormalization group flow in the general two-dimensional field theory and to compute their properties.

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[5] That string theory contains gravity was found in [9].

To do this, one often uses methods that are related to those that one uses for the computation of the critical coefficients $\beta \sim 0.33$ and $\nu \sim 0.63$ for the phase transition between water and steam: the methods of the renormalization group and of "conformal", i.e. scale invariant field theory [10]. The reason for this relation between string theory and the theory of the phase transition is that water and steam at the critical point provide a prototype of a nontrivial scale invariant field theory.

This is what I meant when I said in the beginning that the critical point is the most interesting point in the phase diagram of water (Fig. 1).

APPENDIX: FLOW WITH GRAVITY

The renormalization group comes into its own in two dimensions, where field theories often have a rich structure of fixed points and flows between them. These flows become particularly interesting when the theories are coupled to gravity. Here, a sample of the author's results about renormalization group flows in two-dimensional field theories on surfaces with fluctuating metric and topology is given. For further results, explanations and references the author refers to his habilitation thesis, the main part of which consists of the publications in [11, 12].

A.1. Problem and motivation

Why is it interesting to study the flow "in the presence of gravity" in the first place? The motivation is two-fold.

First, many physical systems such as QCD or the Ising model of my talk are conjectured to be described by random-surfaces - the surfaces swept out by flux tubes or the surfaces of droplets (see e.g. [13]). In such cases one is often interested in predicting fixed points, critical exponents or phase diagrams. In other words, one is interested in properties of the renormalization group flow of the corresponding two-dimensional field theories coupled to gravity. As is explained here for the simplest case of "phantom surfaces" that do not "feel" self-intersections, gravity has interesting effects: it modifies critical coefficients and phase diagrams, and leads to curious phenomena such as oscillating flows and quantum mechanical flows.

Second, it turns out that the flow trajectories in the presence of gravity are time–dependent classical solutions of string theory.[6] The reason is simple. In the conformal gauge approach to quantum gravity, the parameter ϕ in the conformal factor e^{ϕ} of the world-sheet metric becomes the time coordinate of the target spacetime [15, 16]. Thus, time translations in target space correspond to overall scale transformations, that is, renormalization group transformations, on the world-sheet. This suggests a rather fascinating interpretation of *perturbative* string theory, that can perhaps be extended to nonperturbative string theory (which contains not only strings but also higher-dimensional extended objects).

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[6] See, e.g., [14].

We thus study the following problem: consider a renormalizable two–dimensional field theory with coupling constants λ^i on a surface with fixed background metric $g_{\alpha\beta}$. The coupling constants will typically flow: the same theory that is described at some scale μ by coupling constants $\vec{\lambda}$ is described at the new scale μe^{τ} by new coupling constants $\vec{\lambda}(\tau)$.

The dependence of $\vec{\lambda}$ on τ is determined by the flow equations

$$\dot{\lambda}^i \;=\; \beta^i(\lambda^j) \tag{39}$$

where

$$\beta^i \;=\; d^i\lambda^i \;+\; c^i_{jk}\lambda^j\lambda^k \;+\; ... \tag{40}$$

are the beta functions of the theory, with scaling dimensions d^i and given coefficients c^i_{jk}.[7] How is this flow $\lambda^i(\tau)$ modified if the theory is coupled to gravity, i.e., if the metric $g_{\alpha\beta}$ is taken to be a dynamical variable?

A.2. "Gravitational dressing"

The subject of fixed points of the flow (39) (i.e., of conformal field theories) coupled to gravity was studied in [15]. The "gravitational dressing" of the linear part of the beta functions (1.2) was derived in light-cone gauge in [17], and in conformal gauge in [16]. The dressed dimensions $\tilde{d}^i$ are

$$\tilde{d}^i \;=\; \frac{Q}{\alpha} - \frac{1}{\alpha}\sqrt{Q^2 + 4d_i} \tag{41}$$

with certain coefficients Q, α that depend on the central charge c of the theory at $\vec{\lambda} = 0$.[8] One has to restrict oneself to the vicinity of fixed points with central charge $c \leq 1$.

In [11a], the conformal gauge approach is extended to quadratic order in λ. This can in particular be used to determine the modification of the quadratic piece c^i_{jk} in (40) by gravity.[9] One finds in the case $d^i = 0$:

$$c^i_{jk} \quad\rightarrow\quad \tilde{c}^i_{jk} \;=\; |\frac{2}{Q\alpha}|\; c^i_{jk} \tag{42}$$

in agreement with the independent light-cone gauge computation [18]. A modification of the cubic coefficient can also be derived but does not seem to be universal [20]. The supersymmetric analogs of these results have also been found [21].

The modification of both d^i and c^i_{jk} in (40) can be stated in an alternative way that turns out to be more useful when one includes topology fluctuations: gravity does not modify the beta functions in (40) at all, but instead replaces the time derivative in the flow equation (39) by a second-order derivative operator [12]:

$$\dot{\vec{\lambda}} \quad\rightarrow\quad \frac{\alpha^2}{4}\,\ddot{\vec{\lambda}} \;-\; \frac{\alpha}{2}Q\,\dot{\vec{\lambda}} \;+\; \text{order}(\dot{\lambda}^2)\;, \tag{43}$$

where α is negative and - to lowest order -

$$\begin{aligned} Q^2 \;&=\; \frac{1}{3}|c(\vec{\lambda}) - 25| \;+\; \text{order}(\dot{\lambda}^2) & (44)\\ &\approx\; \frac{1}{3}|c - 25|\;. & (45)\end{aligned}$$

$c(\vec{\lambda})$ is the Zamolodchikov function [22]. This gives - to the relevant order - the second-order differential equation

$$\frac{\alpha^2}{4}\,\ddot{\vec{\lambda}} \;-\; \frac{\alpha}{2}Q\,\dot{\vec{\lambda}} \;=\; \vec{\beta}\;. \tag{46}$$

If one now also requires $\vec{\lambda}$ to obey a first-order differential equation

$$\dot{\vec{\lambda}} \;=\; \tilde{d}^i\lambda^i \;+\; \tilde{c}^i_{jk}\lambda^j\lambda^k\;, \tag{47}$$

and plugs this into (46), then $\tilde{d}$ and $\tilde{c}$ are the "modified coefficients" mentioned above. Here one expands to linear order in λ if $d_i \neq 0$, and to quadratic order in λ if $d_i = 0$; higher orders are not universal.

(43) and (44) are an example of the relation between renormalization group trajectories in theories coupled to gravity and classical solutions of string theory: they can be thought of as the string equations of motion. (43) is the tachyon or graviton equation if set equal to $\vec{\beta}$, while (44) is the dilaton equation.

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[7] In this section, the limit $\tau \rightarrow \infty$ means the *ultraviolet*.

[8] $Q = \sqrt{\frac{1}{3}|25 - c|}, \alpha = -\frac{Q}{2} + \frac{1}{2}\sqrt{Q^2 - 8}$ for $c \leq 1$. For $c \geq 25$ one would have $\alpha = -\frac{Q}{2} + \frac{1}{2}\sqrt{Q^2 + 8}$.

[9] combining (2.4) and (3.1) of [11a]; see also [19].

A.3. Sine-Gordon model with gravity

One can use the conformal gauge approach to study, for example, the flow in the sine-Gordon model with interaction term $m \cos px$, coupled to gravity [11a]. To the order considered, the resulting phase diagram (Fig. 1) is in complete agreement with numerical matrix model results [23]:

The Kosterlitz-Thouless transition takes place in the presence of gravity, at the same momentum p as without gravity (at the boundary between regions VI and III). In addition, there is a new region at half the Kosterlitz-Thouless momentum (region IV). There appears to be a transition to negative cosmological constant, although the interpretation has not yet been clarified.

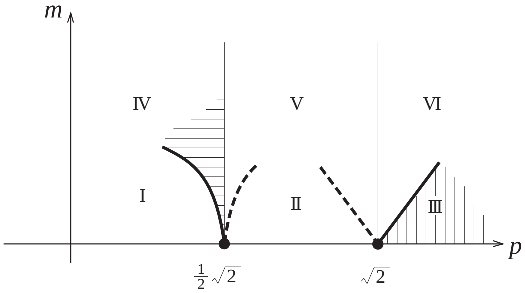

Figure 22. Phase diagram of the sine-Gordon model coupled to gravity.

A.4. Topology fluctuations

So far, the genus of the two-dimensional manifold has been assumed to be zero, corresponding to a sphere. How is the flow modified if not only the metric of the manifold fluctuates but also its topology?

One then has to distinguish between effects that come from two types of pinched surfaces: pinched necks and pinched handles. It turns out that, while metric fluctuations modify the left hand side of of the flow equation (39) to (46), pinched necks modify - in addition - the right-hand side (the beta functions). Pinched handles, on the other hand, make the flow quantum mechanical.

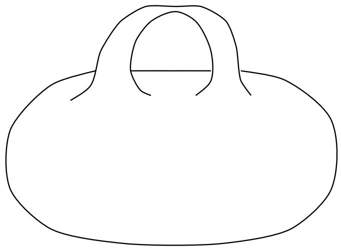

Figure 23. A pinched handle.

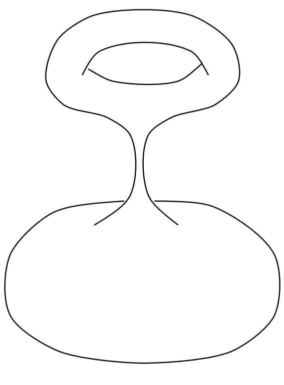

Figure 24. A pinched neck.

Consider as an example a scalar matter field that lives on a circle of radius R [11b]. One finds that, at genus one, random surfaces with pinched necks induce a beta function for the radius R of the circle due to the “Fischler-Susskind mechanism” [24]. The radius has a fixed point at the self-dual value, $R = \sqrt{2}$ in units where $\alpha' = 2$. The deviation of R from this value acquires an anomalous dimension,

$$d_R \;=\; -\,\frac{1}{24}\,\kappa^2 \qquad (48)$$

where κ^2 is the topological coupling constant.

At higher genus there are further contributions to the beta functions for both R and κ^2 that can be computed in conformal gauge. A highly nontrivial consistency check yields precise agreement with numerical (matrix model) results up to genus three.[10]

Pinched handles lead to a curious phenomenon if the matter theory has an isolated "massless" state; in the case at hand, this is the cosmological constant. Due to a bilocal logarithmic divergence, the "classical" RG trajectory $\lambda^i(\tau)$ must then be replaced by a quantum mechanical analog: to leading order, a Gaussian distribution of theories must be considered whose square–width σ^2 also becomes a running coupling constant. σ^2 obeys a first-order differential equation like λ [11b]:

$$\frac{\alpha^2}{4}\,(\sigma^2)^{\cdot\cdot} - \frac{\alpha}{2}Q\,(\sigma^2)^{\cdot} \,\propto\, \kappa^2 \tag{49}$$

with a beta function proportional to κ^2.

Those and all further modifications of the flow can be summarized as follows: Topology fluctuations are accounted for by (i) "quantizing" the flow and (ii) adding the higher-genus vertices of closed string field theory to the beta functions. This is another example of the relation between the flow with dynamical gravity and perturbative string theory.

A.5. Large central charge

So far, we have discussed models with central charge $c \leq 1$ coupled to gravity. Another interesting case is that of supersymmetric models with central charge $\hat{c} \geq 9$ coupled to supergravity. For simplicity, one may start by formally considering bosonic theories with $c \geq 25$ coupled to gravity (which as such make no sense) and simply ignore the tachyon instability. Let us also restrict ourselves to surfaces of genus zero (classical string theory).

For $c \geq 25$ ($\hat{c} \geq 9$), the conformal factor corresponds to a Minkowskian, rather than Euclidean time coordinate [16, 14]. This has interesting consequences. First, some minus signs get flipped, and instead of (46) we now get

$$-\,\frac{\alpha^2}{4}\,\ddot{\vec{\lambda}} \,+\, \frac{\alpha}{2}Q\,\dot{\vec{\lambda}} \,=\, \vec{\beta}\,. \tag{50}$$

The flow towards the *infrared* can now be thought of as the damped motion of a particle in a potential, the generalized Zamolodchikov function [12]. (α is now positive and the sign of τ is still such that $\tau \to \infty$ corresponds to the ultraviolet). For $c \to \infty$ (infinite damping), the flow without gravity is recovered.

A novelty is that now there can be situations where the flow towards the infrared performs damped oscillations around infrared fixed points. In a theory without gravity, such oscillations would be forbidden by the fact that the coupling constants at one scale uniquely determine the coupling constants at a different scale. But when the scale is dynamical and integrated over, such an argument cannot be made. In fact, there seems to be no reason to impose an additional condition like (47) for Minkowskian ϕ.

It turns out that for $c = 25$ ($\hat{c} = 9$), corresponding to critical string vacua, the oscillations decay power-like rather than exponentially.

Let us mention one more aspect [12]. Without gravity, the renormalization group flow of two-dimensional field theories decreases the central charge. The generic theory that starts from an ultraviolet fixed point with large central charge, say $c > 25$ ($\hat{c} > 9$) will eventually - without fine-tuning of parameters - end up in the infrared at $c = 0$.

It turns out that *with* gravity the lowest IR fixed points that such a theory can end up at are those with $c = 25$ ($\hat{c} = 9$) (corresponding to critical string vacua). RG trajectories can pass through fixed points with lower c but then diverge.

This is another example of the relation between the flow with gravity and string theory: there are two classes of renormalization group trajectories in the presence of gravity, corresponding to string solutions with either Minkowskian or Euclidean space-time signature. Fixed points with $c > 25$ ($\hat{c} > 9$) lie in the former while fixed points with $c \leq 1$ ($\hat{c} \leq 1$) lie in the latter class.

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[10] This agreement was obtained in discussion with Igor Klebanov.

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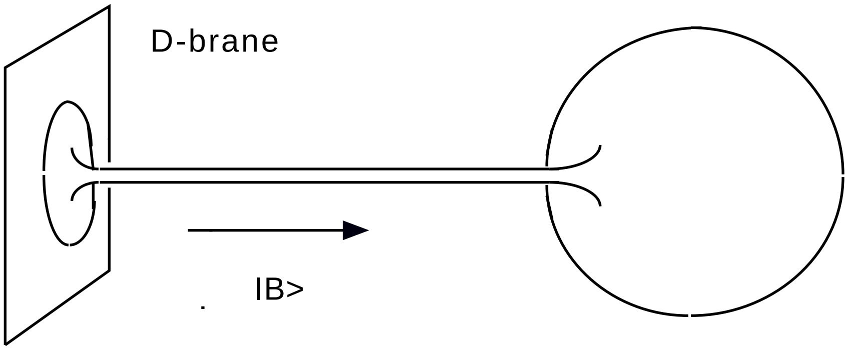

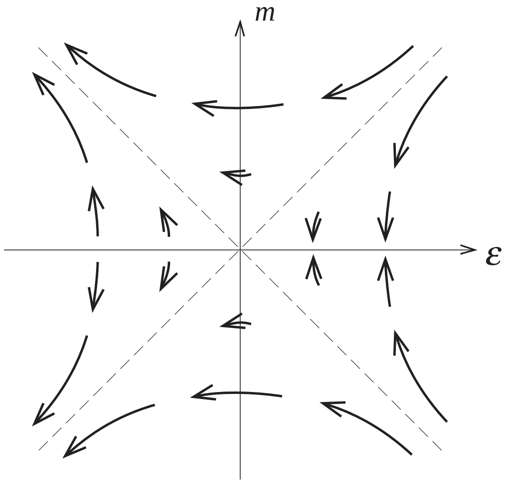

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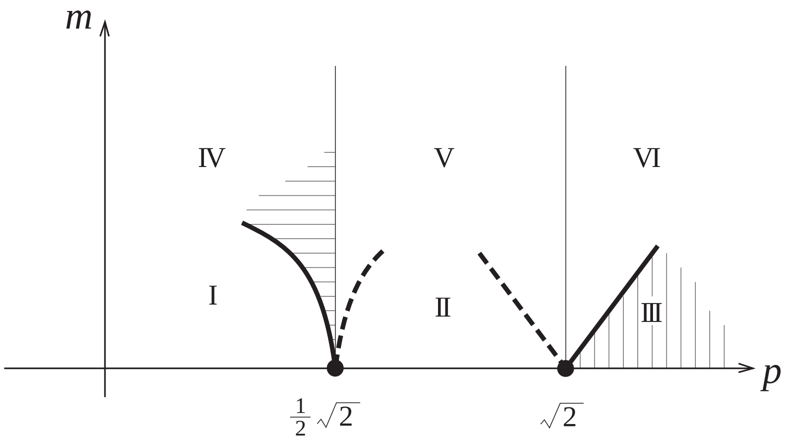
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