Chapter 5.12: Scale Invariance near critical points

Fractal dimensions

$$V = L^D$$

$$D = \frac{\log V}{\log L}$$

Scaling dimension:

near critical point, each quantity changes with a characteristic powerlaw under rescaling with λ

Free energy is self-similar function of $t = T - T_c$ and $h = B - B_c$

$$G(t,h) = \lambda G(\lambda^s t, \lambda^r h)$$

$$m = -\frac{\partial G}{\partial h} = \lambda^{r+1} m(\lambda^s t, \lambda^r h)$$

$$\chi(t,h) = \lambda^{2r+1}\chi(\lambda^s t, \lambda^r h)$$

$$C_h(t,h) = \lambda^{2s+1} C_h(\lambda^s t, \lambda^r h)$$

Scaling for $h = 0$ and/or $t = 0$

$$
\begin{aligned}
C_h(t,0) &= |t|^{-(2s+1)/s}\, C_h(\pm 1, 0) \\
m(t,0) &= (-t)^{-(r+1)/s} m(-1,0) \\
\chi(t,0) &= |t|^{-(2r+1)/s}\, \chi(\pm 1, 0) \\
m(0,h) &= |h|^{-(r+1)/r}\, m(0,\pm 1)
\end{aligned}
$$

$$
\begin{aligned}
\alpha &= \frac{2s+1}{s} \\
\beta &= -\frac{r+1}{s} \\
\gamma &= \frac{2r+1}{s} \\
\delta &= -\frac{r}{r+1}
\end{aligned}
$$

Hence:

$$\alpha + 2\beta + \gamma = 2$$

$$\gamma = \beta(\delta - 1)\ .$$

Mathematical tool for scale invariance: **The renormalization group (RG)**

- Changing length or energy scales will result in self-similar model

- Microscopic details become less important as length scales are increased

- At very long length scales, short wave length excitations are “lost”

- Integrating out: partial sum over lost degrees of freedom gives new effective model

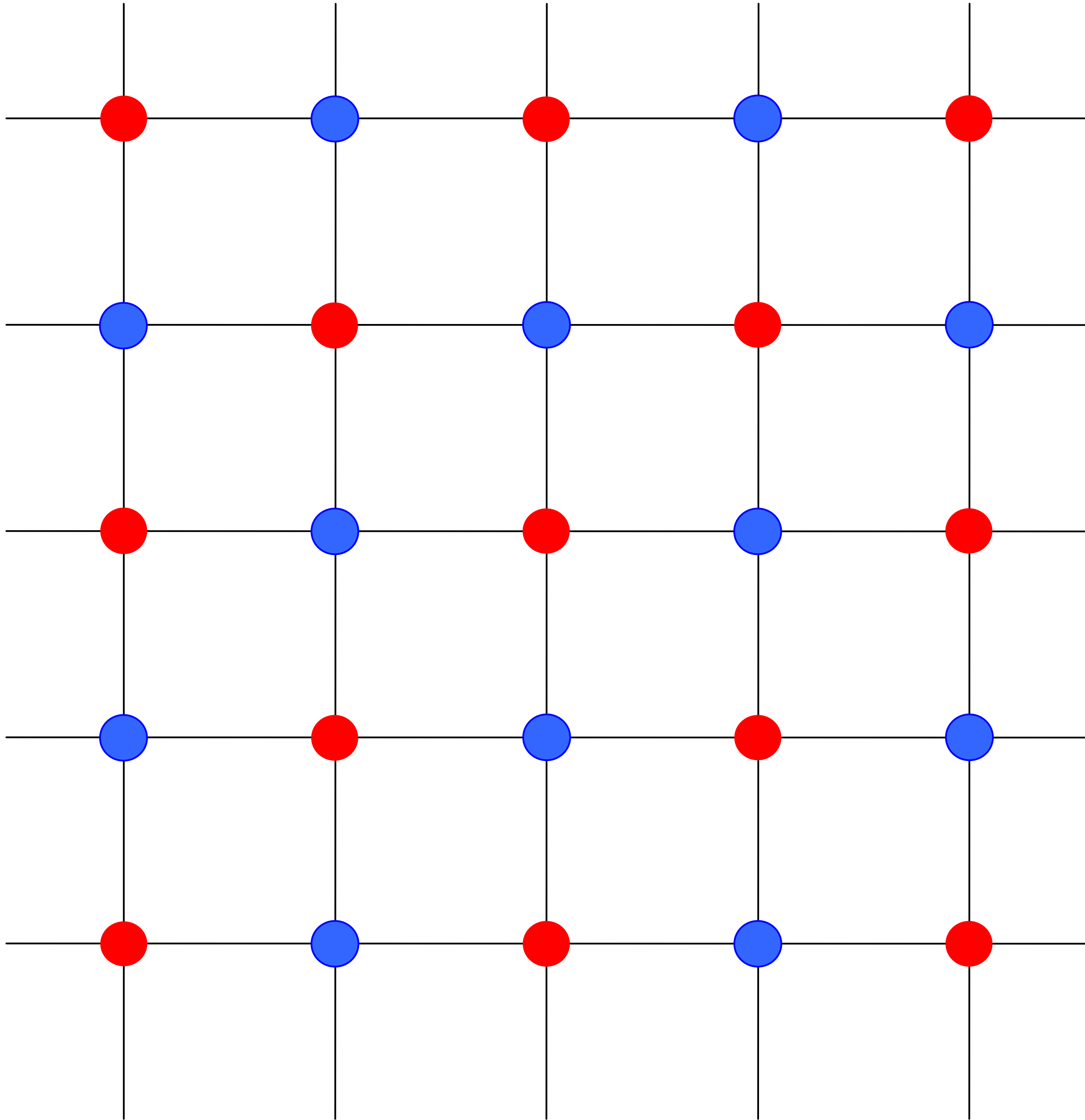
Renormalization group equations: rescaling of parameters under change of “cut-off”

Example: The 2D Ising model at B=0

$$H_{\text{Ising}} = -J \sum_{\langle i,j \rangle} S_i S_j$$

Partition function $Z = \sum_{\{S_j\}} \exp(\beta J \sum_{\langle i,j \rangle} S_i S_j)$

A and B sublattices



Ansatz

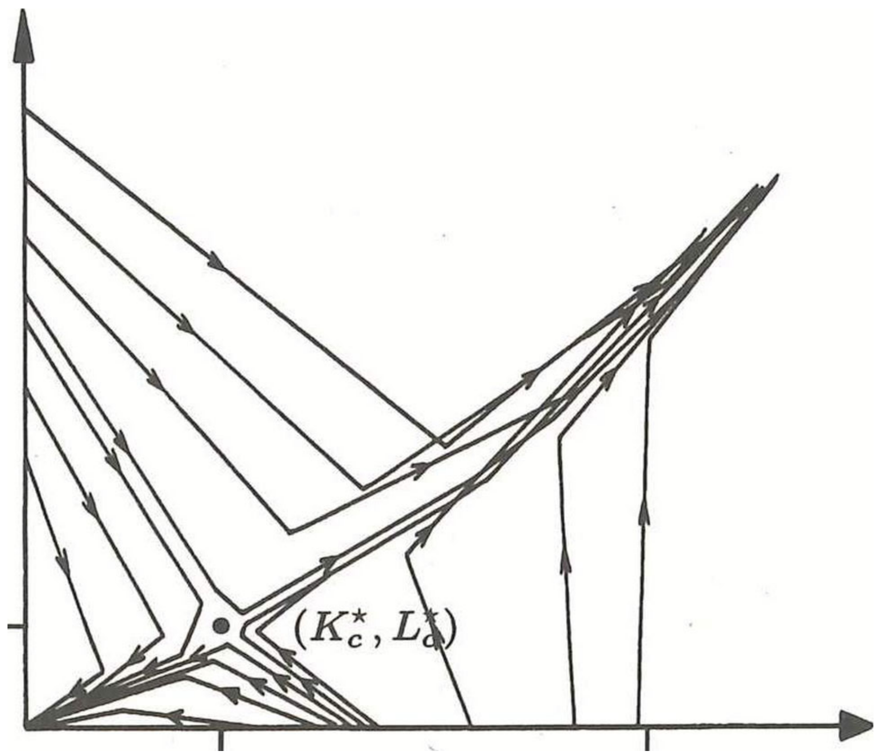
$$Z = \sum_{\{S_j\}_A} \prod_{i_B} 2\cosh \beta J(S_{i+x} + S_{i-x} + S_{i+y} + S_{i-y}) = \sum_{\{S_j\}_A} \prod_{i_B} \exp(-\beta H'(S_{i-x}, S_{i-x}, S_{i+y}, S_{i-y}))$$

$$H'(S_1, S_2, S_3, S_4) = \varepsilon' + \frac{J'}{2}(S_1 S_2 + S_2 S_3 + S_3 S_4 + S_1 S_4) + J_2'(S_1 S_3 + S_2 S_4) + M'(S_1 S_2 S_3 S_4)$$

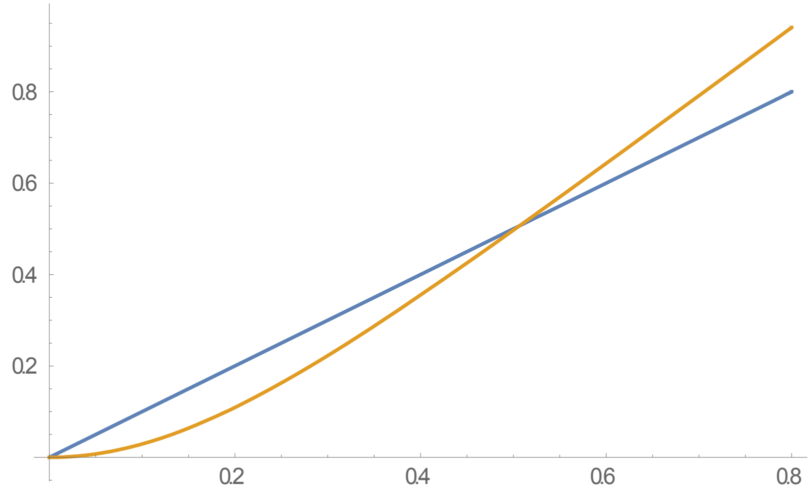
Solution

$$\beta J' = \frac{1}{4}\ln(\cosh 4\beta J) + \beta J_2$$

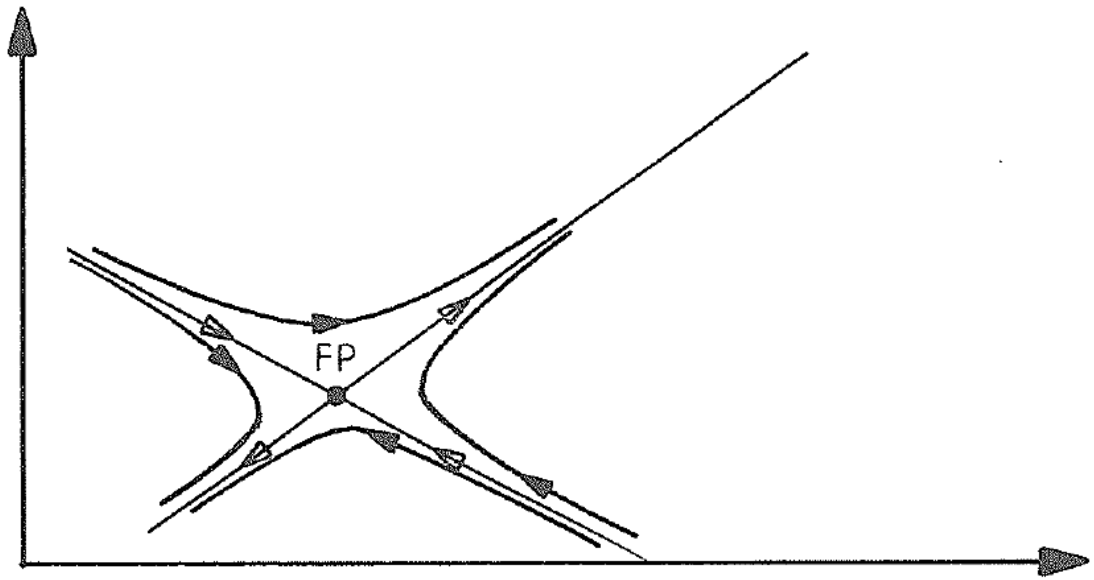
$$\beta J_2' = \frac{1}{8}\ln(\cosh 4\beta J)$$



Critical point $\beta J_c = \frac{3}{8}\ln(\cosh 4\beta J_c)$



Flow near fixed point: Linearization



Generalized description of the RG approach:

- rescale energy, momentum or distance cutoff: $\frac{\Lambda}{b} \longrightarrow \Lambda$ $\Leftrightarrow k \longrightarrow bk\,; \quad x \longrightarrow \frac{x}{b}$

- Scale invariant functions are rescaled according to scaling dimension $\phi(x) \longrightarrow b^{\Delta}\phi(x)$ $\Leftrightarrow \phi(k) \longrightarrow b^{\Delta-d}\phi(k)$

- All coupling constant are redefined under rescaling. $\vec{g} \longrightarrow \vec{g}' \quad : \quad \vec{g}' = R(\vec{g}, b)$
- Repeated transformation form are possible (“group”) $R(R(\vec{g}, b), b') = R(\vec{g}, bb')$

- Finally, find RG flow equations $\frac{d\vec{g}}{dl} = R(\vec{g})$ where $l = ln(b) \qquad d\vec{g} = \vec{g}' - \vec{g}$