Chapter 4.5: Bose-Einstein and Fermi-Dirac distribution

$$\ln \mathcal{Z} = \sum_r \ln Z_r$$

Fermions: $Z_r = 1 + z e^{-\beta \varepsilon_r}$ $\quad n_r = 0, 1$

Bosons: $Z_r = \sum_{n_r=0}^{\infty} e^{-\beta(\varepsilon_r - \mu) n_r} = \frac{1}{1 - z e^{-\beta \varepsilon_r}}$ $\quad$ if $\varepsilon_r - \mu > 0$

Average occupation numbers $\langle n_r \rangle$

$$P(n_r) = \frac{e^{-\beta(\varepsilon_r - \mu) n_r}}{Z_r} = \frac{e^{-\alpha n_r} e^{-\beta \varepsilon_r n_r}}{Z_r} =$$ probability that level r is occupied with n_r particles

$$\langle n_r \rangle = \sum_{n_r} \frac{n_r e^{-\beta \varepsilon_r n_r - \alpha n_r}}{Z_r} = -\left(\frac{\partial \ln Z_r}{\partial \alpha} \right)_\beta = -\frac{1}{Z_r} \left(\frac{\partial Z_r}{\partial \alpha} \right)_\beta$$

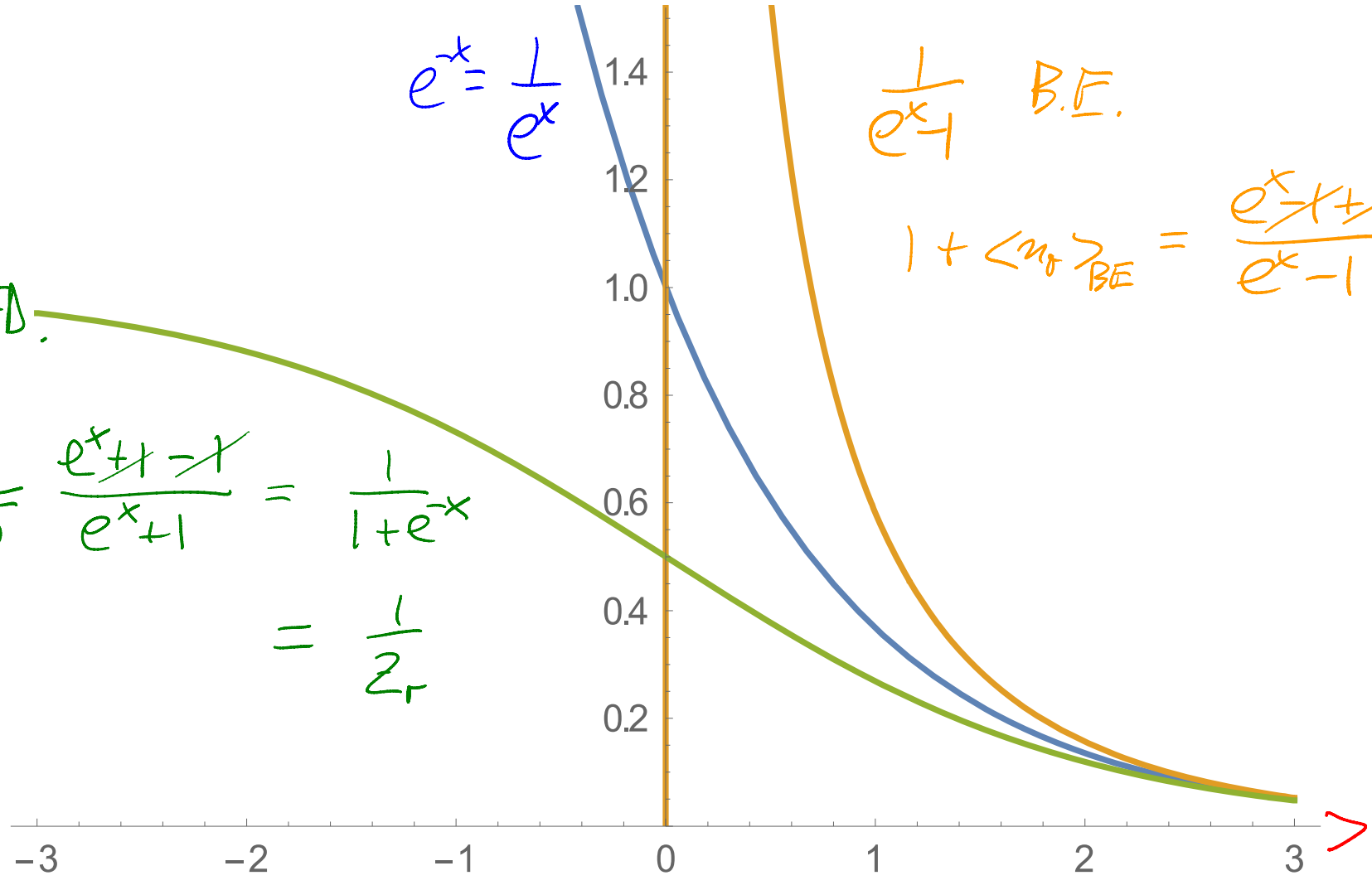
$z = e^{-\alpha}$

$\frac{\partial z}{\partial \alpha} = -z$

Fermions: $-\frac{1}{1 + z e^{-\beta \varepsilon_r}} (-z e^{-\beta \varepsilon_r}) = \frac{1}{e^{\beta \varepsilon_r} z^{-1} + 1} = \langle n_r \rangle_{FD}$ Fermi-Dirac distribution

Bosons: $-(1 - z e^{-\beta \varepsilon_r})(z e^{-\beta \varepsilon_r}) \frac{-1}{(1 - z e^{\beta \varepsilon_r})^2} = \frac{1}{e^{\beta \varepsilon_r} z^{-1} - 1}$

$= \langle n_r \rangle_{BE}$ Bose Einstein distribution



$e^{-x} = \frac{1}{e^x}$

$\frac{1}{e^x - 1}$ B.E.

$1 + \langle n_r \rangle_{BE} = \frac{e^x - 1 + 1}{e^x - 1} = \frac{1}{1 - e^{-x}}$

$= Z_r$

$\frac{1}{e^x + 1}$ F.D.

$\langle n_r \rangle_{FD} = \frac{e^x + 1 - 1}{e^x + 1} = \frac{1}{1 + e^{-x}}$

$= \frac{1}{Z_r}$

$x = \beta(\varepsilon_r - \mu)$

μ is fixed by $N = \sum_r \langle n_r \rangle$

$\ln \mathcal{Z} = \sum_r \ln Z_r$

Energy, pressure, and entropy in terms of occupation numbers

$$N = -\left(\frac{\partial \Phi}{\partial \mu}\right)_T = -\left(\frac{\partial \ln \mathcal{Z}}{\partial \alpha}\right)_\beta = -\sum_r \left(\frac{\partial \ln Z_r}{\partial \alpha}\right)_\beta = \sum_r \langle n_r \rangle$$

reminder: $g(\varepsilon) = \sum_r \delta(\varepsilon - \varepsilon_r)$ allows to determine integral

$$E = -\left(\frac{\partial \ln \mathcal{Z}}{\partial \beta}\right)_{\alpha,V} = -\sum_r \left(\frac{\partial \ln Z_r}{\partial \beta}\right)_\alpha = \sum_r \langle n_r \rangle \varepsilon_r$$

$\frac{\partial \ln Z_r}{\partial \beta} = \langle n_r \rangle \varepsilon_r$ check yourself

homogeneous

$$p = -\left(\frac{\partial \Phi}{\partial V}\right)_{\alpha,T} = -\frac{\Phi}{V} = \sum_r p_r$$

$p_r = \frac{k_B T \ln Z_r}{V}$ since $\Phi = -k_B T \ln \mathcal{Z}$

$= \mp \frac{k_B T}{V} \ln\left(1 \mp z e^{-\beta \varepsilon_r}\right)$

$$S = -\left(\frac{\partial \Phi}{\partial T}\right)_{\alpha,V} = k_B[\ln \mathcal{Z} + \beta(E - \mu N)] = \mp \frac{k_B T}{V} \ln\left(1 \mp \langle n_r \rangle\right)$$

+ Bos. − Ferm.

see exercise 16 $= \sum_r s_r$ s_r = entropy per single particle level