Chapter 4.4: Noninteracting Fermions and Bosons

Reminder: Each basis state is fully symmetrized/anti-symmetrized superposition.

Impossible to specify which particle is in which state. Instead specify
occupation numbers n_r = number of particles in state $|r\rangle$

Basis states:

Bosons

$|n_{r_1}, n_{r_2}, n_{r_3}, n_{r_4}, \ldots\ldots\rangle_B$ = fully symmetric superposition of all possible states
fully specified $\{n_{r_1}, n_{r_2}, \ldots\}$

$n_r = 0, \ldots, \infty$

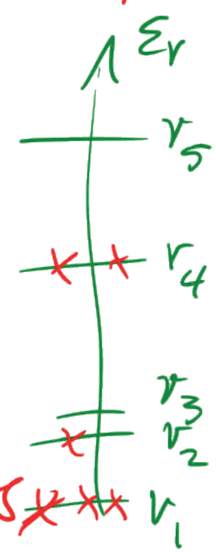
Fermions

$|n_{r_1}, n_{r_2}, n_{r_3}, n_{r_4}, \ldots\ldots\rangle_F$ = slater determinant

$n_r = 0, 1$

if particles are indepent $E = n_{r_1}\varepsilon_{r_1} + n_{r_2}\varepsilon_{r_2} + \ldots$

if non-interacting



for noninteracting $E = \sum_r n_r \varepsilon_r$

Calculation for non-interacting particles

is simpler for $\mathcal{Z}$ than for $Z_N = \sum_{\{r\}} e^{-\beta E(\{r\})}$

$$\mathcal{Z} = \sum_{N=0}^{\infty} e^{-\alpha N} \sum_{\{r\}} e^{-\beta E(\{r\})}$$

$\sum_{\{r\}}$ = sum over all quantum #'s of particles

$$= \sum_{N=0}^{\infty} e^{-\alpha N} \sum_{\{n_r\}}{}' e^{-\beta \sum_r \varepsilon_r n_r}$$

$= \sum'_{\{n_r\}}$ = sum over all allowed occupation #'s restricted so $\sum_r n_r = N$

e.g. for $N=1$

$\sum'_{\{r\}} e^{-\beta \sum_r n_r \varepsilon_r} = Z_1$

$$= \sum_{\{n_r\}} e^{-\alpha N} e^{-\beta \sum_r \varepsilon_r n_r}$$

$$= \sum_{\{n_r\}} e^{-\beta \sum_r (\varepsilon_r - \mu) n_r}$$

$\sum_N \sum'_{\{n_r\}} = \sum_{\{n_r\}}$

unrestricted

$$= \sum_{\{n_r\}} \prod_r e^{-\beta(\varepsilon_r - \mu) n_r}$$

$\alpha = -\beta\mu$

Distributive rules

$a_1 a_2 a_3 + a_1 a_2 b_3 + a_1 b_2 a_3 + a_1 b_2 b_3$

$+ b_1 a_2 a_3 + b_1 a_2 b_3 + b_1 b_2 a_3 + b_1 b_2 b_3$

$= (a_1+b_1)(a_2+b_2)(a_3+b_3)$

$r=1,2,3$

$a_r = e^{-\beta(\varepsilon_r-\mu)\cdot 0}$

$n_r=0,1$

$b_r = e^{-\beta(\varepsilon_r-\mu)1}$

$$\sum_{\{n_r\}} \prod_{r=1,2,3} e^{-\beta(\varepsilon_r-\mu)n_r} = \prod_{r=1,2,3} \sum_{n_r=0}^{1} e^{-\beta(\varepsilon_r-\mu)n_r}$$

$$\mathcal{Z} = \sum_{\{n_r\}} e^{-\beta\sum_r(\varepsilon_r-\mu)n_r} = \sum_{\{n_r\}}\prod_r e^{-\beta(\varepsilon_r-\mu)n_r} = \prod_r \sum_{n_r} e^{-\beta(\varepsilon_r-\mu)n_r} = \prod_r Z_r$$

partition function each level r

$$Z_r = \sum_{n_r} e^{-\beta(\varepsilon_r-\mu)n_r}$$

Fermions $n_r = 0, 1$

Bosons $n_r = 0, \dots \infty$

$$\ln \mathcal{Z} = \sum_r \ln Z_r$$

Bosons $Z_r = \sum_{n_r=0}^{\infty} e^{-\beta(\varepsilon_r-\mu)n_r} = \frac{1}{1-e^{-\beta(\varepsilon_r-\mu)}}$

Noninteracting partition functions

Fermions $Z_r = \sum_{n_r=0}^{1} e^{-\beta(\varepsilon_r-\mu)n_r} = 1+e^{-\beta(\varepsilon_r-\mu)}$ (geom. sum)

Bosons

$$\ln Z_B = -\sum_r \underbrace{\ln(1-e^{-\beta(\varepsilon_r-\mu)})}_{Z_r} = -\sum_r \ln(1-ze^{-\beta\varepsilon_r})$$

$z = e^{\beta\mu}$

$\ln Z_r = -\ln(1-ze^{-\beta\varepsilon_r})$

Fermions

$$\ln Z_F = \sum_r \underbrace{\ln(1+e^{-\beta(\varepsilon_r-\mu)})}_{Z_r} = \sum_r \ln(1+ze^{-\beta\varepsilon_r})$$

$\ln Z_r = \ln(1+ze^{-\beta\varepsilon_r})$

Analogous $z \to -z$

$\ln Z_B(z) = -\ln Z_F(z)$

expansion $\ln(1+\delta) \approx \delta$

$\ln Z_{classical} = \sum_r ze^{-\beta\varepsilon_r} = zZ_1$