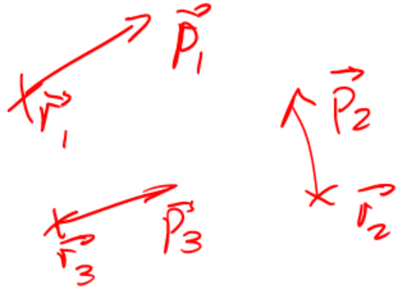
Chapter 2.1: Statistics



$j = 1, \ldots N$

$6N$ variable

Probability distribution

Every classical microstate $\{\vec{r}_1, \vec{p}_1, \vec{r}_2, \vec{p}_2, \vec{r}_3, \vec{p}_3, \ldots, \vec{r}_N, \vec{p}_N\}$ has probability density $P(\{\vec{r}_j, \vec{p}_j\})$, where

$$\int P(\{\vec{r}_i, \vec{p}_i\}) d^3\vec{r}_1 d^3\vec{p}_1 d^3\vec{r}_2 d^3\vec{p}_2 \ldots d^3\vec{r}_N d^3\vec{p}_N = 1$$

Phase space integration: $\prod_j d^3\vec{r}_j d^3\vec{p}_j = d\Gamma^{6N}$

$d\Gamma$ has dim. of momenta and positions

$P(\Gamma)d\Gamma$ is dimensionless

the probability to find state in $[\Gamma, \Gamma + d\Gamma[$

Goal: $S, T, E, \ldots$

A given probability distribution may not be diagonal in any desired basis (e.g. position)

Quantum states

can be in superposition of a large number of basis states.

A probability distribution must define a preferred basis $|r\rangle$ with quantum numbers r and probability P_r

Def. 2.1: The density matrix $\hat{\rho}$ with $\mathrm{tr}\,\hat{\rho} = 1$ and positive definite eigenvalues $P_r \geq 0$ defines a quantum mechanical probability distribution. In general the system is in a mixed state.

in eigenbasis: $\hat{\rho} = \sum_r P_r |r\rangle\langle r|$ $\quad \mathrm{tr}\,\rho = \sum_r P_r = 1$

in general not given in diagonal form

pure state: definite state $|\psi\rangle = |r=1\rangle$ $\quad P_1 = 1$

in this case $\hat{\rho}^2 = \hat{\rho}$ $\quad P_{rest} = 0$

all other cases: mixed state

is represented by operator $\hat{\rho}$ not by a vector

(Not superposition)

Measurements

$\Lambda(\{\vec{r}_j, \vec{p}_j\})$ or $\langle r|\hat{\Lambda}|r\rangle$

Def. 2.2: Statistical expectation values of a quantity $\hat{\Lambda}$ are obtained by using the probability distribution

$$\langle\Lambda\rangle = \int d\Gamma P(\{\vec{r}_j, \vec{p}_j\})\Lambda(\{\vec{r}_j, \vec{p}_j\})$$

average value

$$\langle\hat{\Lambda}\rangle = \sum_r \langle r|\hat{\Lambda}|r\rangle P_r = \mathrm{tr}\hat{\Lambda}\hat{\rho}$$

$$= \sum_{rr'} \langle r|\hat{\Lambda}|r'\rangle\langle r'|r\rangle P_{r'} = \sum_r \langle r|\hat{\Lambda}\hat{\rho}|r\rangle = \mathrm{tr}\,\hat{\Lambda}\hat{\rho}$$

also fluctuations are interesting

Example find particle at x

pure state $|\psi_r(x)|^2 = |\langle r|x\rangle|^2 = \langle r|x\rangle\langle x|r\rangle = \langle r|\hat{P}_x|r\rangle$

"QM expectation value" of $\hat{\Lambda} = \hat{P}_x$ $\qquad \hat{P}_x = |x\rangle\langle x| \mathrel{\hat{=}}$ measurement

thermal/statistical expectation

$$\langle P_x\rangle = \sum_r \langle r|\hat{P}_x|r\rangle P_r = \sum_r \langle r|x\rangle\langle x|r\rangle P_r = \sum_r |\psi_r(x)|^2 P_r$$

Fluctuations

Def. 2.3: The standard deviation $\Delta\Lambda$ of a quantity $\hat{\Lambda}$ is given by

mean: $\lambda = \langle \hat{\Lambda} \rangle \in \mathbb{C}$ or $\mathbb{R}$

$$\Delta\Lambda^2 = \left\langle \left(\hat{\Lambda} - \langle \hat{\Lambda} \rangle\right)^2 \right\rangle \geq 0$$

"root mean squared" (RMS)

$$\Delta\Lambda = \sqrt{\langle (\hat{\Lambda} - \langle \hat{\Lambda} \rangle)^2 \rangle}$$

$$\Delta\Lambda^2 = \langle \hat{\Lambda}^2 - 2\hat{\Lambda}\langle\hat{\Lambda}\rangle + \langle\hat{\Lambda}\rangle^2 \rangle$$

$$= \langle \hat{\Lambda}^2 \rangle - 2\langle\hat{\Lambda}\rangle\langle\hat{\Lambda}\rangle + \langle\Lambda\rangle^2$$

$\langle \lambda \rangle = \lambda \quad \forall \lambda \in \mathbb{C}$

$$= \langle \Lambda^2 \rangle - \langle \Lambda \rangle^2$$

$\Delta\Lambda^2 \geq 0$

$\Rightarrow \langle \Lambda^2 \rangle \geq \langle \Lambda \rangle^2$

but sometimes $\langle \Lambda \rangle \in \mathbb{C}$ exists

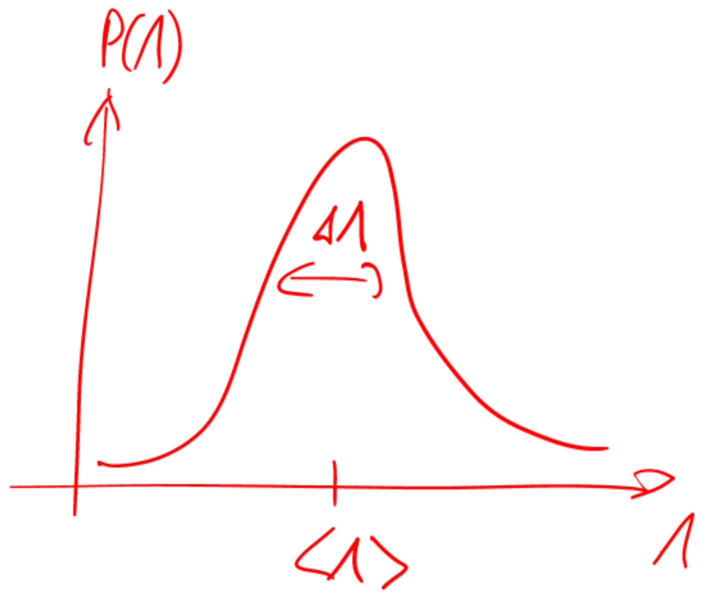
but $\langle \Lambda^2 \rangle$ may diverge

Def. 2.4: The central limit theorem holds for a sum $\Lambda = \sum_{j=1}^{n} Y_j$ of very many independent statistical variables Y_j which have the same probability distribution $P_Y(Y_j)$ and finite ΔY. In the limit $n \to \infty$ the probability distribution for Λ approaches a Gaussian distribution

"normal distribution"

e.g. random walk

$$P(\Lambda) = \frac{1}{\sqrt{2\pi}\Delta\Lambda} \exp\left(-\frac{1}{2} \left(\frac{\Lambda - \langle\Lambda\rangle}{\Delta\Lambda}\right)^2 \right)$$



where $\langle\Lambda\rangle = n\langle Y\rangle$ and $\Delta\Lambda = \sqrt{n}\Delta Y$

$P(\Lambda)d\Lambda$ is probability to measure $[\Lambda, \Lambda + d\Lambda[$

$$\frac{\Delta\Lambda}{\langle\Lambda\rangle} = \frac{1}{\sqrt{n}} \frac{\Delta Y}{\langle Y\rangle}$$

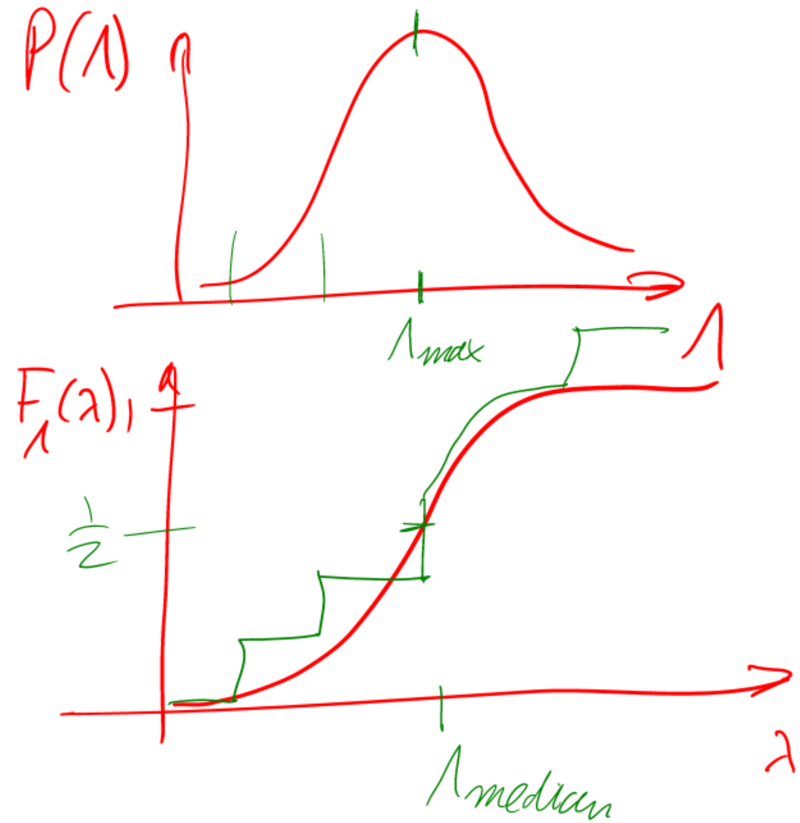
fluctuation decrease with more measurements

examples: Binomial distribution: random walk, Repeated measurements

Def. 2.5: A cumulative distribution function $F_\Lambda(\lambda) = P(\Lambda \le \lambda)$ is the probability that the random variable Λ is measured with a value less than λ. Two variables are identically distributed iff $F_{\Lambda_1}(\lambda) = F_{\Lambda 2}(\lambda) \forall \lambda$. They are independent iff $F_{\Lambda_1,\Lambda_2}(\lambda_1,\lambda_2) = P(\Lambda_1 \le \lambda_1 \wedge \Lambda_2 \le \lambda_2) = F_{\Lambda 1}(\lambda_1) F_{\Lambda 2}(\lambda_2)$

probability densities

$$F_\Lambda(\lambda) = \int_{-\infty}^{\lambda} P(\Lambda)\, d\Lambda$$



in general

$$\Lambda_{median} \neq \Lambda_{max} \neq \langle \Lambda \rangle$$

Example: wealth distribution