Tuning the Order of the Nonequilibrium Quantum Phase Transition in a Hybrid Atom-Optomechanical System

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A quantum many-body hybrid system is considered formed by a nanomembrane in an optical cavity, the outcoupled light of which provides a lattice for an ultracold atomic Bose gas. Depending on the applied laser intensity, the optomechanical coupling of the membrane motion to transitions between two internal atomic states can be tuned. Above a critical intensity, a nonequilibrium quantum phase transition occurs to a symmetry-broken phase with a sizeable occupation of the energetically higher internal states and a displaced membrane. Its order can be changed by tuning the transition frequency. For symmetric coupling, it is continuous below a certain transition frequency and discontinuous above. For an asymmetric coupling, a first-order phase transition occurs.

Intriguing collective behavior emerges when different quantum many-body systems with complementary advantages are combined. In recently realized hybrids, state-of-the-art optomechanical systems are merged with atom-optical quantum systems in form of a cloud of \textsuperscript{87}Rb atoms into a single atom-optomechanical set-up. In this way, as theoretically proposed \cite{1} and later experimentally realized \cite{2–5}, the vibrational motion of a nanomembrane in an optical cavity is coupled to the spatial motion of a distant cloud of cold \textsuperscript{87}Rb atoms that reside in the optical lattice of the out-coupled light field. In order to overcome the resolved sideband cooling limit and cool the nanomechanical oscillator close to its quantum mechanical ground state, the atom gas can be used as a coolant \cite{1–4}, collective effects in the quantum many-body system lead to collective atomic motion with an instability \cite{5} and a continuous quantum phase transition \cite{6} to a state with a spatially shifted optical lattice. Besides, investigations such as indirect measurement, a way to induce atom-membrane entanglement and coherent state transfer are in the focus of interest \cite{7–11}.

For a near resonant coupling mechanism, this motional coupling scheme limits the frequency of the nanosillator to feasible trap frequencies in the optical lattice, which is in the sub-MHz regime. An eligible candidate to circumvent this problem is the so-called internal state coupling scheme \cite{12} which allows a better tunability in order to reach a resonant coupling regime. Here, the mechanical motion of the membrane is indirectly coupled to transitions between internal states, e.g., Zeeman or hyperfine ground states, of the atomic gas via translating the phase shift of the light due to the membrane displacement into a polarization rotation using a polarizing beam splitter (PBS). This allows a near resonant coupling mechanism for membrane frequencies beyond the sub-MHz regime. Yet, a vast number of methods to perturb the internal states in the atom gas and influence the nanosillator are applicable. These can be used to induce membrane cooling \cite{12, 13}, a displacement squeezed membrane \cite{14}, or to realize a harmonic oscillator with negative mass by the internal states \cite{15}.

On the other hand, the collective nature of the hybrid system mediates long-range interactions in the atom gas similar to those in a spinor dipolar Bose-Einstein condensate (BEC) \cite{16, 17}. It is well known that long-range interactions in quantum gases induce long-range order and new phases of matter. A renowned hybrid quantum system includes an ultracold cloud of atoms inside a high finesse cavity that is transversely pumped by a laser \cite{18, 19}. The collective coupling of the atoms to a single cavity mode induces long-range atom-atom interactions over the whole cavity. Above a critical interaction strength, this leads to a second-order nonequilibrium quantum phase transition (NQPT) to a self-organized superradiant state with a checkerboard pattern \cite{18–22}.

In this work, we investigate a hybrid atom-optomechanical system in the “membrane-in-the-middle-setup” \cite{12}. The light field is adiabatically eliminated, leading to an effective coupling between the membrane and the transition between two states in the atom gas, see Fig. 1. In a mean-field description, the atomic part is reduced to a single-site problem with a Gaussian ansatz for the condensate profile. Tuning the atom-membrane coupling by alteration of the laser intensity, the system undergoes a NQPT. Moreover, we show that, by tuning the transition frequency, the order of the phase transition can be changed from second to first order and vice versa. In case of a discontinuous phase transition, the system exhibits a hysteresis which can be measured on the basis of polarization measurements of the atom gas.

Model. We consider an ensemble of \textit{N} ultracold \textsuperscript{87}Rb atoms placed in an external optical lattice. The atoms exhibit three relevant internal states $\tau \in \{-, +, e\}$ that are arranged in a \textit{A}-type level scheme. The two lowest states are energetically separated by $\Omega_a$ which can be tuned via an external magnetic field ($\hbar = 1$). A potential realization are the two hyperfine states $| \tau \rangle = |F = 2, m_F = -2\rangle$ and $|+\rangle = |F = 2, m_F = 0\rangle$ of the $5^2S_{1/2}$ ground state.
The transition between $|+\rangle$ and $|e\rangle$ is driven by an applied $\sigma_-$ circularly polarized laser with a finite detuning $\Delta$. The passing beam is directed to a PBS which divides the circularly polarized light into linearly polarized $\pi_x$ and $\pi_y$ light beams on two equal length arms (measured for an undisplaced membrane), see Fig. 1. The vertical path consists of a fixed mirror which reflects light with conserved polarization $\pi_x$. In the horizontal path, a nanomembrane is placed inside a low-finesse cavity, which reflects $\pi_y \to \pi_y$ light when undisplaced. Here, a single vibrational mode of the membrane with frequency $\Omega_m$ is considered. The outcoupled light of both arms is directed back onto the atoms mediating the effective atom-membrane coupling. In a quasi-static picture, a finite displacement of the membrane induces a finite phase shift on the propagating horizontal $\pi_y$ beam leading to a rotated polarization after the light has passed the PBS backwards. The emergent $\sigma_+$ photon with frequency $\omega_+$ now impinges on an atom and may induce a two-photon transition between the states $|-\rangle \leftrightarrow |+\rangle$, when the resonance condition $\Omega_m - \Omega_A$ is met, such that $\omega_+ = \omega_L + \Omega_m \simeq \omega_L + \Omega_m$. The back-action of the atoms on the membrane is induced by a transition of the atoms between the states $|\pm\rangle$. The emitted $\sigma_+$ photons pass the PBS with 50% probability horizontally and hit the membrane. This changes the radiation pressure on the membrane.

In the bad-cavity limit, the light field and $|e\rangle$ can be eliminated in a Born-Markov approximation. In the mean-field regime, the hybrid atom-membrane system is effectively described by the equations of motion [23]

$$i\dot{\alpha} = (\Omega_m - i\Gamma_m)\alpha - \sqrt{N}\lambda_{\text{ex}} \int dz \cos(2z)|\psi_+|^2 - \sqrt{N}\lambda \int dz \cos(2z)\text{Re}\{\psi_+^*\psi_-\}, \quad (1a)$$

$$i\dot{\psi}_- = \left[ N \sum_{\tau=\pm} g_{\tau}|\psi_+|^2 + \frac{\Omega_m}{2} - \omega_R \partial_z^2 - \frac{V}{2} \cos(2z) \right] \psi_- - \sqrt{N}\lambda \text{Re}(\alpha) \cos(2z)\psi_+ + \psi_- \right], \quad (1b)$$

with the first equation describing the motion of the membrane, and the latter two the motion of the two species condensate. Here, $\alpha = |a\rangle/\sqrt{N}$ is the scaled mean value of the ladder operator $a$ that annihilates an excitation on the membrane, $\psi_\pm$ are the condensate wave functions of the corresponding internal state and $N$ denotes the number of atoms. Fluctuations due to light field and thermal fluctuation are neglected. The atoms are placed in an optical lattice with depth $V$ and local atom-atom interactions with strength $g_{\tau\tau'}$ are included. Moreover, $\omega_R = \omega_0^2/2m$ is the recoil frequency of the atoms with mass $m$. The atom-membrane coupling consists of two different processes. First, the coupling induces transitions between the internal states $\tau = -$ and $\tau = +$ with coupling constant $\lambda$. Second, a term arises which couples the occupation number of atoms in the state $\tau = +$ to the membrane displacement with strength $\lambda_{\text{ex}}$ [6]. In fact, the coupling constants are not independent of each other as $\lambda_{\text{ex}}/\lambda = 2\mu_+/\mu_- = \chi/2$, where $\mu_+$ is the atomic transition dipole moment for the corresponding internal state. Hence, we choose the parametrization $\lambda_{\text{ex}} = \lambda \chi/2$ and refer to the case $\chi = 0$ as symmetric coupling.

For a deep optical lattice, $V \gg \omega_R$, the condensate profile is well described by a sum of Gaussians residing in the individual lattice wells. When the wave function overlap between neighboring sites is small, the problem reduces to a single-site problem. It is then reasonable to make the ansatz

$$\psi_\tau(t, z) = c_\tau(t) \left( \frac{1}{\pi \sigma(t)^2} \right)^{1/4} e^{-z^2/2(\sigma(t))^2+i\beta(t)z^2}, \quad (2)$$

with a constant number of atoms, i.e., $|c_-|^2 + |c_+|^2 = 1$, the condensate width $\sigma$, and $\beta$ is used to produce the dynamics for $\sigma$. For a binary BEC, the ground state is either in a miscible or immiscible state, where the condensate profile of the different internal states are either mixable or avoid each other. This feature depends on the ratio of the coupling constants $\eta = g_{++}g_{-}/g_{+}^2$. The condensate is mixable for $\eta > 1$ and non-mixable for $\eta < 1$ [24, 25]. Moreover, it has been shown that a transition from an immiscible to a miscible state is induced by a linear interconversion coupling [26] which is similar to the present internal state coupling. For simplicity, we assume that the width is the same for both particle species and restrict this work to the case $g_{\tau\tau'} \equiv g$, i.e., $\eta = 1$. The validity of this ansatz is verified by numerical estimations of the condensate profile [23].

Next, we perform a cumulant expansion [6] of the Eqs. (1) to find the equations of motion for the respective variational parameters in Eq. (2). That is, we calculate the (i) 0th and (ii) 2nd cumulants by multiplying Eqs. (1b)
with the effective potential energy and (1c) (i) with \( \psi \) which one is \( \dot{\sigma} = 4\omega_R\beta \sigma \), and
\[
\dot{\sigma} = -i\partial_\sigma E - \Gamma_m \alpha ,
\]
\[
\dot{\tau} = -i\partial_\tau E ,
\]
\[(4\omega_R)^{-1}\dot{\sigma} = -\partial_\sigma E ,
\]
with the effective potential energy
\[
E = \Omega_m |\alpha|^2 + \frac{\Omega_a}{2} \left[ |c_+|^2 - |c_-|^2 \right] + \left[ \frac{\omega_R}{2\sigma^2} + \frac{Ng}{\sqrt{8\pi\sigma}} \right] \text{Re}(\alpha) - \frac{V}{2e\sigma^2} - \sqrt{\Lambda} \left[ |\alpha|^2 + 2\text{Re}(c_+^*c_-) \right] e^{-\sigma^2}. \tag{4}
\]

**Tuning the order of the Quantum Phase Transition.** In the presence of damping, the system will eventually relax to a steady state. Thus, each of the parameters can be split into their steady state value and deviations from the steady state. Here, we are mainly interested in the steady state properties of the combined hybrid system. That is, we make the ansatz \( \alpha(t) = \alpha_0, c_-(t) = \sqrt{1 - \gamma_0^2}, c_+(t) = \gamma_0, \sigma(t) = \sigma_0 \). By inserting this ansatz in Eq. (3), the relation \( \alpha_0 = \sqrt{\Lambda} \left[ 2\gamma_0 \sqrt{1 - \gamma_0^2} + \chi_0^2 \right] e^{-\sigma_0^2/2} / (\Omega_m - i\Gamma_m) \) is found, whereas the condensate parameters \( \gamma_0, \sigma_0 \) are determined via \( \gamma_0^2 = \Omega_m \right\}

For a qualitative understanding of the role of both the atom-membrane coupling \( \lambda \) and the asymmetry \( \chi \), we study the potential energy surface in Eq. (4). The optimal choice of parameters \( (\alpha_0, \gamma_0, \sigma_0) \) is characterized by a globally minimized potential energy \( E_{\min} = E[\alpha_0, \gamma_0, \sigma_0] \) (note that \( \alpha_0 \) is fixed by \( \gamma_0 \) and \( \sigma_0 \)). Rather than minimizing the energy potential with respect to all three parameters, we minimize with respect to \( \alpha_0, \sigma_0 \) for a given \( \gamma \), which is taken as an order parameter, and study the resulting potential energy surface \( E(\gamma) = E[\alpha_0(\gamma), \gamma, \sigma_0(\gamma)] \). The global symmetry properties of the hybrid system are determined by \( \gamma \) via the influence of the occupations of the condensate species \( (c_+(t) = \gamma, c_-(t) = \sqrt{1 - \gamma^2}) \).

Figure 2 shows the normalized energy surface \( E(\gamma, \lambda) = \left[ E(\gamma) - E(0) \right] / \max_{|\gamma| \leq 1} \left| E(\gamma) - E(0) \right| \). The dashed curves show the value \( \gamma_0 \) that minimizes the energy potential. Below a certain coupling rate \( \lambda \leq \lambda_c \), the potential is minimized for \( \gamma_0 = 0 \) in which all atoms occupy the state \( |\tau = -\rangle \). At \( \lambda = \lambda_c \), the system undergoes a NQPT that is characterized by a non-vanishing \( \gamma_0 \neq 0 \) with different characteristics. The case \( \chi = 0 \) is shown in Fig. 2(a) and (b). In Fig. 2(a), the NQPT is of second order with a critical exponent \( \gamma_0 \sim (\lambda - \lambda_c)^{1/2} \). By tuning the transition frequency \( \Omega_m \), the NQPT becomes a symmetric first-order phase transition, see Fig. 2(b). Here, the bistable phase corresponds to the two states \( (|\tau = -\rangle \pm |\tau = +\rangle) / \sqrt{2} \). For a non-vanishing asymmetry \( \chi > 0 \), an asymmetric first-order phase transition occurs at the critical coupling, where the left branch \( (|\tau = -\rangle - |\tau = +\rangle) / \sqrt{2} \) is energetically raised, see Fig. 2(c).

In order to quantify which type of NQPT occurs and to specify the critical coupling rate, we perform a Landau expansion of the potential energy \( E(\gamma) = \sum_{n \geq 2} a_n \gamma^n \) in the order parameter around \( \gamma = 0 \). For symmetric coupling \( \chi = 0 \), only even orders in \( \gamma \) turn out to contribute and the critical coupling rate for the second-order NQPT \( \lambda_{\text{NQPT}} \) is defined by \( a_2 = 0 \), whereas critical coupling for the symmetric first-order NQPT \( \lambda_{\text{NQPT}} \) is approximately estimated via \( 13a_2a_4 = 4a_4^2 \). Moreover, the order of the NQPT is determined by the modulus of \( a_4 \). At the point \( a_4 = 0 \), a transition from a second- to a symmetric first-order NQPT occurs, which allows us to define a critical
transition frequency

$$\Omega_c = \frac{\omega_\sigma^2}{32\omega_R \sigma_0^2}, \quad (5)$$

where $$\omega_\sigma^2 = 4\omega_R [3\omega_R/\sigma_0^4 + Ng/\sqrt{2\pi} \sigma_0^4 + V(1-2\sigma_0^2)e^{-\sigma_0^2}]$$ is the frequency of the condensate breathing mode. For $$\Omega_a < \Omega_c$$, the NQPT is continuous, whereas it becomes discontinuous for $$\Omega_a > \Omega_c$$. This fact is shown in Fig. 3 as a function of the lattice depth $$V$$ in (a) and the interaction strength $$gN$$ in (b). The asymmetric coupling $$\chi \neq 0$$ gives rise to terms of odd order in $$\gamma$$ in the Landau expansion and the NQPT is always discontinuous. Here, the critical coupling $$\lambda_c = \lambda_a$$ is approximated by the smallest solution for $$\lambda$$ of the equation $$4a_2a_4 = a_3^2$$.

**Hysteresis.** A characteristic feature of a first-order phase transition is the occurrence of hysteresis, shown in Fig. 4(a) and (b) for $$\chi = 0$$ and $$\chi = 0.5$$, respectively. To obtain hysteresis, we take the equations of motion (3) with $$\gamma(t) = c_4(t)$$, in the limit of infinite time, i.e., $$\lim_{t \to \infty} \gamma(t) = \gamma_\infty$$. On the forward path, the coupling strength $$\lambda$$ is adiabatically increased and the system is initially placed in the minimum at $$\gamma = 0$$. The system stays there until it becomes unstable at $$\lambda = \lambda_f$$ and jumps to the adjacent minimum at $$\gamma_0 \neq 0$$. This point is defined by $$a_2 = 0$$ and coincides with the critical coupling rate $$\lambda_{c2}$$ in the symmetric regime. Afterwards, the steady state solution $$\gamma_0$$ is followed as $$\lambda$$ increases.

On the backward path, the system follows the minimum at $$\gamma_\infty \neq 0$$ until this point becomes unstable at $$\lambda_b$$ and jumps to the solution at $$\gamma_\infty = 0$$. For the symmetric case (a), this jumping point is given by the relation $$3a_2a_0 - a_3^2 = 0$$, whereas it is given by the relation $$9a_3^2 - 32a_2a_4 = 0$$ for the asymmetric case (b).

In the picture of potential energy surfaces, the reason of the hysteresis behavior is the existence two stable local minima at $$\gamma = 0$$ and $$\gamma \neq 0$$ in the coexistence region $$\lambda_b \leq \lambda \leq \lambda_f$$ as indicated in Fig. 4(c) for $$\chi = 0.5$$.

**FIG. 3.** Order of the NQPT as a function of the atomic transition frequency $$\Omega_a$$ and (a) the lattice depth $$V$$ or (b) the interaction strength $$gN$$. The dashed lines show the critical transition frequency $$\Omega_a = \Omega_c$$ according to Eq. (5). The fixed parameter in (a) is the interaction strength $$gN = \omega_R$$ and in (b) the potential depth $$V = 100 \omega_R$$.

**FIG. 4.** Hysteresis and potential curves. Hysteresis curve ($$\gamma_\infty$$, solid) as a function of the coupling parameter $$\lambda$$ for (a) the symmetric 1st order ($$\chi = 0$$) and (b) the asymmetric 1st order PT ($$\chi = 0.5$$). The dashed line shows the stationary order parameter $$\gamma_0$$. (c) Potential curves for different coupling strengths as indicated by the arrows in (b). Parameters chosen for (a) as in Fig. 2(b) and for (b), (c) as in Fig. 2(c).

**Experimental Realization.** An experimental realization of the discussed coupling scheme has not yet been established. Current experimental set-ups use the motional coupling scheme [1, 2]. This system also exhibits a NQPT from a localized symmetric state to a symmetry-broken quantum many-body state with a shifted cloud-membrane configuration [6]. Here, the relevant energy scale of the condensate, which corresponds to the breathing mode frequency, is the energy to excite particles to a higher motional band, see Eq. (5). Both of these frequencies scale with the lattice depth, which makes a direct observation of the transition from a second- to a first-order NQPT impossible. This considerably improves for the internal state coupling scheme as considered here. For the proposed system, the direct observation is possible by either measuring the membrane eigenfrequency or the condensate width $$\sigma_0$$ [6]. In the case of a first-order NQPT, these quantities exhibit a jump at the critical coupling rate, rather than a continuous behavior as in the case of a second-order NQPT. Moreover, a direct measurement of the condensate polarization $$\gamma_0$$ ($$\gamma_\infty$$) can detect the NQPT in a straightforward way. Furthermore, the asymmetry parameter $$\chi$$ can be tuned by applying an additional perpendicular laser field that drives the transition from $$|\rangle$$ to $$+\rangle$$, giving rise to a term $$\delta(c_+^c c_- + c_-^c c_+)$$ in the potential energy (4). Compensating an additional force on the membrane that scales with $$\sqrt{N}\lambda$$ and tuning the parameter $$\delta$$ allows an indirect variation of the asymmetry parameter $$\chi$$.

**Conclusion.** We have shown that the hybrid atom-optomechanical system not only undergoes a nonequilibrium quantum phase transition between phases of different collective behavior, but also that the order of the phase transition can be tuned in a straightforward manner. The steady state of an ultracold atomic condensate in an optical lattice, whose internal states are coupled to a single mechanical vibrational mode of a distant mem-
brane, has been analyzed, based on a Gross-Pitaevskii-like mean-field approach by a time-dependent Gaussian variational ansatz. Mediated by the light field of a common laser, the coupling between the atoms and the membrane is tuned by changing the laser intensity. Below a certain critical coupling \( \lambda_c \), all the atoms occupy the energetically lower state \( |\rangle \) and at the critical point a nonequilibrium quantum phase transition occurs. This phase is characterized by a sizeable steady-state occupation of the energetically higher state \( |+\rangle \) and a displaced membrane. Its order is determined by the state-dependent atom-membrane coupling and the transition frequency \( \Omega_a \). For an asymmetric coupling, \( \chi \neq 0 \), an asymmetric first-order phase transition occurs with a preferred polarization orientation. Instead, for a symmetric coupling, \( \chi = 0 \), the phase transition is continuous for transition frequencies below a certain critical value \( \Omega_c \) and discontinuous above. Moreover, hysteresis is obtained by adiabatically tuning the coupling strength in the regime of a first-order phase transition. The transition between a first- and second-order is observable by tuning readily accessible parameters in the internal state coupling scheme.

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