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Hartree–Fock mean-field theory for trapped dirty bosons

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Abstract. Here we work out in detail a non-perturbative approach to the dirty boson problem, which relies on the Hartree–Fock theory and the replica method. For a weakly interacting Bose gas within a trapped confinement and a delta-correlated disorder potential at finite temperature, we determine the underlying free energy. From it we determine via extremization self-consistency equations for the three components of the particle density, namely the condensate density, the thermal density, and the density of fragmented local Bose–Einstein condensates within the respective minima of the random potential landscape. Solving these self-consistency equations in one and three dimensions in two other publications has revealed how these three densities change for increasing disorder strength.

Keywords: Bose–Einstein condensation (theory), disordered systems (theory)

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Contents

1. Introduction

In the dirty boson problem, the combined effect of disorder and two-particle interaction yields an intriguing interplay between localization and superfluidity [1]. Experimentally, the dirty boson problem was first studied with superfluid helium in porous media like aerosol glasses (Vycor), where the pores are modeled by statistically distributed local scatterers [2–5]. In Bose gases disorder appears either naturally as, e.g. in magnetic wire traps [6–10], where imperfections of the wire itself can induce local disorder, or it may be created artificially and controllably as, e.g. by using laser speckle fields [11–15]. A set-up more in the spirit of condensed matter physics relies on a Bose gas with impurity atoms of another species trapped in a deep optical lattice, so the latter represent randomly distributed scatterers [16, 17]. Furthermore, an incommensurate optical lattice can provide a pseudo-random potential for an ultracold Bose gas [18–20].

Theoretically, the dirty boson problem can be treated, in principle, via two complementary approaches. The first one applies the Bogoliubov theory [21] and treats disorder, quantum, and thermal fluctuations perturbatively, which is only valid in systems with sufficiently small random potential and interaction strength at low enough temperatures [22]. With this it was found that a weak random disorder potential leads to a depletion of both the condensate and the superfluid density due to the localization of bosons in the respective minima of the random potential. This seminal Huang-Meng theory was later on extended in different research directions. Results for the shift of the velocity of sound as well as for its damping due to collisions with the external field are worked out in [23]. Furthermore, the original special case of a delta-correlated random potential was generalized to experimentally more realistic disorder correlations with a finite correlation length, which model, for instance, the pore size dependence of Vycor glass. A Gaussian correlation was discussed in [24], whereas laser speckles are treated in [25, 26]. Also the disorder-induced shift of the critical temperature for the homogeneous case was analyzed in [27, 28], which also has implications for a harmonic confinement [29]. Furthermore, it was shown in [30–32] that dirty dipolar Bose gases yield even at zero temperature characteristic directional dependences for thermodynamic quantities due to the anisotropy emerging of superfluidity. The recent perturbative work [33, 34] studies even in detail the impact of the external random potential upon the quantum fluctuations. Despite all these many theoretical predictions of the Huang-Meng theory, which also affect the collective excitations frequencies of harmonically trapped dirty bosons [35], so far no experiment has tested them quantitatively.

On the other hand, the dirty boson problem was also tackled non-perturbatively in different ways. A major result is that increase in the disorder strength at zero temperature yields a first-order quantum phase transition from a superfluid to a Bose-glass phase, where in the latter all particles reside in the respective minima of the random potential. This prediction is achieved for three dimensions by solving the underlying Gross–Pitaevskii equation with a random phase approximation [36], as well as by a stochastic self-consistent mean-field approach using two chemical potentials, one for the condensate and one for the excited particles [37, 38]. Dual to that, the non-perturbative approach of [39, 40] investigates energetically shape and size of the local minicondensates in the disorder landscape and deduces from that, for a decreasing disorder strength, when the Bose-glass phase becomes unstable and goes over into the superfluid. At finite temperatures the location of superfluid, Bose-glass, and normal phase in the phase diagram was qualitatively analyzed in [41] on the basis of a Hartree–Fock mean-field theory with the replica method. Also Monte-Carlo (MC) simulations have been applied to study the dirty boson problem. Diffusion MC in [42] obtained the surprising result that a strong enough disorder yields a superfluid density larger than the condensate density. Furthermore, worm algorithm MC [43, 44] was able to determine the dynamic critical exponent of the quantum phase transition from the Bose-glass to the superfluid in two dimensions.

All those previous theoretical investigations mainly focus on the possible emergence of the Bose-glass phase and its elusive properties for homogeneous dirty bosons. Experimentally, however, ultracold quantum gases have to be confined with the help of a harmonic trapping potential. Therefore, in case of trapped dirty bosons, there is a lack of knowledge concerning the Bose-glass region, where the bosons within the harmonic trap localize in the respective minima of the superimposed random potential. The present paper works out in detail a theoretical approach how to describe this localization of bosons within a harmonic confinement in a systematic way. To this end we extend the Hartree–Fock mean-field theory of [41] for a three-dimensional weakly interacting homogeneous Bose gas in a delta-correlated disorder potential to the experimentally relevant trapping confinement via a semi-classical approximation and to a general number of spatial dimensions. By doing so, we work out, in particular, all the respective technical details which were omitted for brevity in [41]. In the following we start in section 2 with introducing the functional integral representation of the partition function for a trapped weakly interacting Bose gas in a disorder potential at finite temperature. Applying the replica method in section 3 allows to eliminate the random potential right away at the expense of introducing disorder-induced interactions between different replica fields, which are nonlocal in both space and time. Then we work out a Hartree–Fock mean-field theory for this model in section 4. After specializing to replica symmetry in section 5, we restrict ourselves to a delta-correlated disorder potential and contact interaction potential for the dirty boson model in section 6. The underlying free energy is obtained in section 7. From it we determine via extremization the underlying self-consistency equations for the three components of the particle density, namely the condensate density, the thermal density, and the density of fragmented local Bose–Einstein condensates within the respective minima of the random potential landscape. The case of three dimensions is treated in section 8, whereas one spatial dimension is dealt with in section 9. Note that the two-dimensional case is not treated in this paper, since our mean-field theory turns out to diverge in two dimensions, so both a regularization and a subsequent renormalization is needed, which goes beyond the scope of the present paper. Furthermore, we introduce the statistical description of a disorder potential, which is central for describing the dirty boson problem, as well as the disorder ensemble average in appendix A. Finally, appendix B defines the order parameters for the superfluid and the Bose-glass phase via off-diagonal long-range order of corresponding correlation functions.

2. Bose model

We start by considering the model of an n-dimensional Bose gas in an arbitrary trap $V(\mathbf{x})$ and a general interaction potential $V^{(\mathrm{int})}(\mathbf{x}-\mathbf{x}')$ at finite temperature T in n spatial dimensions. The starting point is the functional integral for the grand-canonical partition function

$$\mathcal{Z} = \oint \mathcal{D}\psi^* \oint \mathcal{D}\psi \mathrm{e}^{-\mathcal{A}[\psi^*,\psi]/\hbar}, \tag{1}$$

where the integration is performed over all Bose fields $\psi^*(\mathbf{x},\tau), \psi(\mathbf{x},\tau)$ which are periodic in imaginary time τ, i.e. $\psi(\mathbf{x},\tau)=\psi(\mathbf{x},\tau+\hbar\beta)$. The Euclidean action is given in standard notation by

$$\mathcal{A}[\psi^*,\psi] = \int_0^{\hbar\beta} d\tau \int d\mathbf{x} \left\{ \psi^*(\mathbf{x},\tau)\left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2M}\Delta + V(\mathbf{x}) + U(\mathbf{x}) - \mu\right]\psi(\mathbf{x},\tau) + \frac{1}{2}\int d\mathbf{x}'\psi^*(\mathbf{x},\tau)\psi(\mathbf{x},\tau)V^{(\mathrm{int})}(\mathbf{x}-\mathbf{x}')\psi^*(\mathbf{x}',\tau)\psi(\mathbf{x}',\tau)\right\}, \quad (2)$$

where M denotes the particle mass, μ the chemical potential, $\beta = 1/k_B T$ the reciprocal temperature, and T the temperature. Furthermore, $U(\mathbf{x})$ denotes a generally correlated disorder landscape, whose statistical properties are explained in detail in appendix A.

Note that, in order to guarantee the normal ordering within the functional integral, we should work with adjoint fields $\psi^*(\mathbf{x},\tau^+)$ with a shifted imaginary time $\tau^+ = \tau + \eta$ with $\eta \to 0^+$ which is infinitesimally later than the imaginary time τ of the fields $\psi(\mathbf{x},\tau)$. However, for the sake of simplicity, we mainly use in the following the notation $\psi^*(\mathbf{x},\tau)$ and emphasize the normal ordering only when it is indispensable.

3. Replica method

A standard method to deal with disorder problems is the replica method [45–47]. Instead of treating the actual problem, one looks at $\mathcal{N}$ copies of the system, then analytically continues the replicated system to the limit $\mathcal{N} \to 0$. As the concrete realization of the disorder potential $U(\mathbf{x})$ is not known, the free energy of the system Ω is defined as the free energy for fixed disorder potential averaged over all its realizations

$$\Omega = -\frac{1}{\beta}\overline{\ln \mathcal{Z}}, \quad (3)$$

where $\overline{\bullet}$ corresponds to the disorder average over many realizations. In general it is not possible to explicitly evaluate expression (3), as $\overline{\ln \mathcal{Z}} \neq \ln \overline{\mathcal{Z}}$. The replica method is provided by investigating the $\mathcal{N}$th power of the grand-canonical partition function $\mathcal{Z}$ in the limit $\mathcal{N} \to 0$, which yields for the replicated partition function $\mathcal{Z}^{\mathcal{N}} = 1 + \mathcal{N}\ln\mathcal{Z} + \ldots$. Thus, we deduce for the free energy (3)

$$\Omega = -\frac{1}{\beta}\lim_{\mathcal{N}\to 0}\frac{\overline{\mathcal{Z}^{\mathcal{N}}} - 1}{\mathcal{N}}. \quad (4)$$

The fact that all $\mathcal{N}$ replicas are identical simplifies the calculation further as we will show below. The $\mathcal{N}$-fold replication of the partition function of the disordered Bose gas in equation (1) and a subsequent averaging with respect to the disorder potential $U(\mathbf{x})$ results in:

$$\overline{\mathcal{Z}^{\mathcal{N}}} = \left\{\prod_{\alpha=1}^{\mathcal{N}} \oint \mathcal{D}\psi_\alpha^* \oint \mathcal{D}\psi_\alpha\right\} \exp\left\{\frac{-1}{\hbar}\int_0^{\hbar\beta} d\tau \int d\mathbf{x} \sum_{\alpha=1}^{\mathcal{N}} \left\{\psi_\alpha^*(\mathbf{x},\tau)\left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2M}\boldsymbol{\Delta} + \mathbf{V}(\mathbf{x}) - \mu\right]\psi_\alpha(\mathbf{x},\tau) + \frac{1}{2}\int d\mathbf{x}'\psi_\alpha^*(\mathbf{x},\tau)\psi_\alpha(\mathbf{x},\tau)V^{(\mathrm{int})}(\mathbf{x}-\mathbf{x}')\psi_\alpha^*(\mathbf{x}',\tau)\psi_\alpha(\mathbf{x}',\tau)\right\}\right\} \times \overline{\exp\left\{\int d\mathbf{x}\frac{-1}{\hbar}\int_0^{\hbar\beta} d\tau \sum_{\alpha=1}^{\mathcal{N}} \psi_\alpha^*(\mathbf{x},\tau)\psi_\alpha(\mathbf{x},\tau)U(\mathbf{x})\right\}}, \quad (5)$$

where $\psi_\alpha^*(\mathbf{x},\tau)$, $\psi_\alpha(\mathbf{x},\tau)$ are the replica fields with the replica index α. The remaining disorder ensemble average of the exponential function can be performed exactly on a formal level explained in appendix A. Indeed, comparing expressions (5) and (A.11) shows that averaging with respect to the disorder potential $U(\mathbf{x})$ corresponds to the generating functional (A.16) with the auxiliary current field:

$$j(\mathbf{x}) = \frac{-1}{\hbar}\int_0^{\hbar\beta} \mathrm{d}\tau \sum_{\alpha=1}^{\mathcal{N}} \psi_\alpha^*(\mathbf{x},\tau)\psi_\alpha(\mathbf{x},\tau). \tag{6}$$

Therefore, the disordered Bose gas is described by the disorder averaged, replicated grand-canonical partition function

$$\overline{\mathcal{Z}^{\mathcal{N}}} = \left\{\prod_{\alpha=1}^{\mathcal{N}} \oint \mathcal{D}\psi_\alpha^* \oint \mathcal{D}\psi_\alpha\right\} \mathrm{e}^{-\mathcal{A}^{(\mathcal{N})}[\psi^*,\psi]/\hbar}, \tag{7}$$

with the following replica action

$$\begin{aligned}\mathcal{A}^{(\mathcal{N})}[\psi^*,\psi] = &\int_0^{\hbar\beta} \mathrm{d}\tau \int \mathrm{d}\mathbf{x} \sum_{\alpha=1}^{\mathcal{N}} \Bigg\{\psi_\alpha^*(\mathbf{x},\tau)\left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2M}\boldsymbol{\Delta} + V(\mathbf{x}) - \mu\right]\psi_\alpha(\mathbf{x},\tau)\\ &+\frac{1}{2}\int \mathrm{d}\mathbf{x}'\psi_\alpha^*(\mathbf{x},\tau)\psi_\alpha(\mathbf{x},\tau)\,V^{(\mathrm{int})}(\mathbf{x}-\mathbf{x}')\psi_\alpha^*(\mathbf{x}',\tau)\psi_\alpha(\mathbf{x}',\tau)\Bigg\}\\ &+\sum_{i=2}^{\infty}\frac{1}{i!}\left(\frac{-1}{\hbar}\right)^{i-1}\int_0^{\hbar\beta}\mathrm{d}\tau_1\cdots\int_0^{\hbar\beta}\mathrm{d}\tau_i\int\mathrm{d}\mathbf{x}_1\cdots\int\mathrm{d}\mathbf{x}_i\\ &\times\sum_{\alpha_1=1}^{\mathcal{N}}\cdots\sum_{\alpha_i=1}^{\mathcal{N}} D^{(i)}(\mathbf{x}_1,\ldots,\mathbf{x}_i)\left|\psi_{\alpha_1}(\mathbf{x}_1,\tau_1)\right|^2\cdots\left|\psi_{\alpha_i}(\mathbf{x}_i,\tau_i)\right|^2,\end{aligned} \tag{8}$$

where $D^{(i)}(\mathbf{x}_1,\ldots,\mathbf{x}_i)$ denote the respective cumulants of the disorder potential, see appendix A. For any experimental realistic disorder potential the dominant cumulant is of second order, as we assume, without loss of generality, that the first cumulant vanishes according to (A.1). Therefore, it is physically justified to restrict ourselves in the following to the second cumulant, i.e. only $D^{(2)}(\mathbf{x}_1-\mathbf{x}_2) = D(\mathbf{x}_1-\mathbf{x}_2)$ contributes to the replicated action (8):

$$\begin{aligned}\mathcal{A}^{(\mathcal{N})}[\psi^*,\psi] = &\int_0^{\hbar\beta} \mathrm{d}\tau \int \mathrm{d}\mathbf{x} \sum_{\alpha=1}^{\mathcal{N}} \Bigg\{\psi_\alpha^*(\mathbf{x},\tau)\left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2M}\boldsymbol{\Delta} + V(\mathbf{x}) - \mu\right]\psi_\alpha(\mathbf{x},\tau)\\ &+\frac{1}{2}\int \mathrm{d}\mathbf{x}'\psi_\alpha^*(\mathbf{x},\tau)\psi_\alpha(\mathbf{x},\tau)\,V^{(\mathrm{int})}(\mathbf{x}-\mathbf{x}')\psi_\alpha^*(\mathbf{x}',\tau)\psi_\alpha(\mathbf{x}',\tau)\Bigg\}\\ &-\frac{1}{2\hbar}\int_0^{\hbar\beta}\mathrm{d}\tau\int_0^{\hbar\beta}\mathrm{d}\tau'\int\mathrm{d}\mathbf{x}\int\mathrm{d}\mathbf{x}'\sum_{\alpha=1}^{\mathcal{N}}\sum_{\alpha'=1}^{\mathcal{N}} D(\mathbf{x}-\mathbf{x}')\psi_\alpha^*(\mathbf{x},\tau)\psi_\alpha(\mathbf{x},\tau)\psi_{\alpha'}^*(\mathbf{x}',\tau')\psi_{\alpha'}(\mathbf{x}',\tau').\end{aligned} \tag{9}$$

Thus, we conclude that, in this case, disorder leads to a residual attractive interaction between the replica fields $\psi_\alpha^*(\mathbf{x},\tau)$, $\psi_\alpha(\mathbf{x},\tau)$ which is, in general, bilocal in both space and imaginary time.

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4. Hartree–Fock mean-field equations

Now we apply standard methods for developing a self-consistent mean-field approximation [48, 49] in order to derive Hartree–Fock mean-field equations for the Bose gas in a random potential. To this end we use the Bogoliubov approximation, i.e. we split the Bose fields $\psi^*_\alpha(\mathbf{x},\tau)$, $\psi_\alpha(\mathbf{x},\tau)$ into the background fields $\Psi^*_\alpha(\mathbf{x},\tau)$, $\Psi_\alpha(\mathbf{x},\tau)$ describing the condensate wave function, plus the fluctuations $\delta\psi^*_\alpha(\mathbf{x},\tau)$, $\delta\psi_\alpha(\mathbf{x},\tau)$ describing the non-condensed fractions:

$$\psi^*_\alpha(\mathbf{x},\tau)=\Psi^*_\alpha(\mathbf{x},\tau)+\delta\psi^*_\alpha(\mathbf{x},\tau),\;\; \psi_\alpha(\mathbf{x},\tau)=\Psi_\alpha(\mathbf{x},\tau)+\delta\psi_\alpha(\mathbf{x},\tau). \tag{10}$$

Thus, the replica action (9) decomposes according to $\mathcal{A}^{(\mathcal{N})}[\psi^*,\psi]=\sum_{k=0}^{4}\mathcal{A}^{(\mathcal{N},k)}[\delta\psi^*,\delta\psi]$, where $\mathcal{A}^{(\mathcal{N},k)}[\delta\psi^*,\delta\psi]$ denotes all terms that contain fluctuations $\delta\psi^*_\alpha(\mathbf{x},\tau)$, $\delta\psi_\alpha(\mathbf{x},\tau)$ to the kth power. Then, we approximate the higher nonlinear terms $k=3$ and $k=4$ within a Gaussian factorization, where expectation values are calculated with respect to a fluctuation action $\tilde{\mathcal{A}}^{(\mathcal{N},2)}[\delta\psi^*,\delta\psi]$ which is determined self-consistently below:

$$\langle\bullet\rangle=\frac{\left\{\prod_{\alpha=1}^{\mathcal{N}}\oint\mathcal{D}\delta\psi^*_\alpha\oint\mathcal{D}\delta\psi_\alpha\right\}\bullet\,\mathrm{e}^{-\tilde{\mathcal{A}}^{(\mathcal{N},2)}[\delta\psi^*,\delta\psi]/\hbar}}{\left\{\prod_{\alpha=1}^{\mathcal{N}}\oint\mathcal{D}\delta\psi^*_\alpha\oint\mathcal{D}\delta\psi_\alpha\right\}\mathrm{e}^{-\tilde{\mathcal{A}}^{(\mathcal{N},2)}[\delta\psi^*,\delta\psi]/\hbar}}. \tag{11}$$

As we restrict ourselves to a Hartree–Fock mean-field theory, we only keep normal correlations $\langle\delta\psi_\alpha(\mathbf{x},\tau)\,\delta\psi^*_{\alpha'}(\mathbf{x}',\tau')\rangle$ and neglect all anomalous correlations of the form $\langle\delta\psi_\alpha(\mathbf{x},\tau)\,\delta\psi_{\alpha'}(\mathbf{x}',\tau')\rangle$ or $\langle\delta\psi^*_\alpha(\mathbf{x},\tau)\,\delta\psi^*_{\alpha'}(\mathbf{x}',\tau')\rangle$. With this we obtain for the cubic terms in the fluctuations:

$$\begin{aligned}\delta\psi^*_\alpha(\mathbf{x},\tau)\,\delta\psi_\alpha(\mathbf{x},\tau)\,\delta\psi_{\alpha'}(\mathbf{x}',\tau')\approx{}&\langle\delta\psi^*_\alpha(\mathbf{x},\tau^+)\,\delta\psi_\alpha(\mathbf{x},\tau)\rangle\,\delta\psi_{\alpha'}(\mathbf{x}',\tau')\\&+\langle\delta\psi^*_\alpha(\mathbf{x},\tau)\,\delta\psi_{\alpha'}(\mathbf{x}',\tau')\rangle\,\delta\psi_\alpha(\mathbf{x},\tau),\end{aligned} \tag{12}$$

together with its complex conjugate and, correspondingly, the fourth order terms in the fluctuations reduce to:

$$\begin{aligned}&\delta\psi^*_\alpha(\mathbf{x},\tau)\,\delta\psi_\alpha(\mathbf{x},\tau)\,\delta\psi^*_{\alpha'}(\mathbf{x}',\tau')\,\delta\psi_{\alpha'}(\mathbf{x}',\tau')\\&\approx\langle\delta\psi^*_\alpha(\mathbf{x},\tau^+)\,\delta\psi_\alpha(\mathbf{x},\tau)\rangle\,\delta\psi^*_{\alpha'}(\mathbf{x}',\tau')\,\delta\psi_{\alpha'}(\mathbf{x}',\tau')+\langle\delta\psi^*_{\alpha'}(\mathbf{x}',\tau'^+)\,\delta\psi_{\alpha'}(\mathbf{x}',\tau')\rangle\,\delta\psi^*_\alpha(\mathbf{x},\tau)\,\delta\psi_\alpha(\mathbf{x},\tau)\\&+\langle\delta\psi^*_\alpha(\mathbf{x},\tau)\,\delta\psi_{\alpha'}(\mathbf{x}',\tau')\rangle\,\delta\psi_\alpha(\mathbf{x},\tau)\,\delta\psi^*_{\alpha'}(\mathbf{x}',\tau')+\langle\delta\psi_\alpha(\mathbf{x},\tau)\,\delta\psi^*_{\alpha'}(\mathbf{x}',\tau')\rangle\,\delta\psi^*_\alpha(\mathbf{x},\tau)\,\delta\psi_{\alpha'}(\mathbf{x}',\tau')\\&-\langle\delta\psi^*_\alpha(\mathbf{x},\tau^+)\,\delta\psi_\alpha(\mathbf{x},\tau)\rangle\,\langle\delta\psi^*_{\alpha'}(\mathbf{x}',\tau'^+)\,\delta\psi_{\alpha'}(\mathbf{x}',\tau')\rangle-\langle\delta\psi^*_\alpha(\mathbf{x},\tau)\,\delta\psi_{\alpha'}(\mathbf{x}',\tau')\rangle\,\langle\delta\psi_\alpha(\mathbf{x},\tau)\,\delta\psi^*_{\alpha'}(\mathbf{x}',\tau')\rangle.\end{aligned} \tag{13}$$

Here we have used τ^+ as an imaginary time which is infinitesimally later than τ in order to guarantee the normal ordering of the fluctuations within the respective expectation values. Therefore, the Gaussian factorization procedure for a Hartree–Fock mean-field theory leads to the following approximation of the replica action (9):

$$\mathcal{A}^{(\mathcal{N})}[\psi^*,\psi]\approx\tilde{\mathcal{A}}^{(\mathcal{N},0)}[\delta\psi^*,\delta\psi]+\tilde{\mathcal{A}}^{(\mathcal{N},1)}[\delta\psi^*,\delta\psi]+\tilde{\mathcal{A}}^{(\mathcal{N},2)}[\delta\psi^*,\delta\psi], \tag{14}$$

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where $\tilde{\mathcal{A}}^{(\mathcal{N},k)}[\delta\psi^*, \delta\psi]$ denotes the kth-order terms of the replica action (9). To make our notation concise, we express in all those terms the fluctuations in (12) and (13) by the following mean-fields:

$$Q_{\alpha\alpha'}(\mathbf{x},\tau;\mathbf{x}',\tau') = \Psi_\alpha(\mathbf{x},\tau)\,\Psi^*_{\alpha'}(\mathbf{x}',\tau') + \langle\delta\psi_\alpha(\mathbf{x},\tau)\,\delta\psi^*_{\alpha'}(\mathbf{x}',\tau')\rangle, \tag{15}$$

$$Q^*_{\alpha\alpha'}(\mathbf{x},\tau;\mathbf{x}',\tau') = Q_{\alpha'\alpha}(\mathbf{x}',\tau';\mathbf{x},\tau), \tag{16}$$

$$\Sigma_\alpha(\mathbf{x},\tau) = Q_{\alpha\alpha}(\mathbf{x},\tau;\mathbf{x},\tau^+). \tag{17}$$

With this the first term of the replica action (14), which is independent of the fluctuations $\delta\psi^*_\alpha(\mathbf{x},\tau)$, $\delta\psi_\alpha(\mathbf{x},\tau)$, reads:

$$\begin{aligned}\tilde{\mathcal{A}}^{(\mathcal{N},0)}[\delta\psi^*,\delta\psi] = &\int_0^{\hbar\beta}\mathrm{d}\tau\int\mathrm{d}\mathbf{x}\sum_{\alpha=1}^{\mathcal{N}}\Bigg\{\Psi^*_\alpha(\mathbf{x},\tau)\left[\hbar\frac{\partial}{\partial\tau}-\frac{\hbar^2}{2M}\boldsymbol{\Delta}+V(\mathbf{x})-\mu\right]\Psi_\alpha(\mathbf{x},\tau)\\ &-\frac{1}{2}\int\mathrm{d}\mathbf{x}'\,V^{(\mathrm{int})}(\mathbf{x}-\mathbf{x}')[\Psi^*_\alpha(\mathbf{x},\tau)\Psi_\alpha(\mathbf{x},\tau)\Psi^*_\alpha(\mathbf{x}',\tau)\Psi_\alpha(\mathbf{x}',\tau)+\Sigma_\alpha(\mathbf{x},\tau)\,\Sigma_\alpha(\mathbf{x}',\tau)\\ &-2\Sigma_\alpha(\mathbf{x},\tau)\,\Psi^*_\alpha(\mathbf{x}',\tau)\Psi_\alpha(\mathbf{x}',\tau)+Q_{\alpha\alpha}(\mathbf{x},\tau;\mathbf{x}',\tau)Q^*_{\alpha\alpha}(\mathbf{x},\tau;\mathbf{x}',\tau)\\ &-Q_{\alpha\alpha}(\mathbf{x},\tau;\mathbf{x}',\tau)\Psi_\alpha(\mathbf{x}',\tau)\Psi^*_\alpha(\mathbf{x},\tau)-Q^*_{\alpha\alpha}(\mathbf{x},\tau;\mathbf{x}',\tau)\Psi_\alpha(\mathbf{x},\tau)\Psi^*_\alpha(\mathbf{x}',\tau)]\Bigg\}\\ &+\frac{1}{2\hbar}\int_0^{\hbar\beta}\mathrm{d}\tau\int_0^{\hbar\beta}\mathrm{d}\tau'\int\mathrm{d}\mathbf{x}\int\mathrm{d}\mathbf{x}'\sum_{\alpha=1}^{\mathcal{N}}\sum_{\alpha'=1}^{\mathcal{N}}D(\mathbf{x}-\mathbf{x}')\\ &\times\{\Psi^*_\alpha(\mathbf{x},\tau)\Psi_\alpha(\mathbf{x},\tau)\Psi^*_{\alpha'}(\mathbf{x}',\tau')\Psi_{\alpha'}(\mathbf{x}',\tau')+\Sigma_\alpha(\mathbf{x},\tau)\,\Sigma_{\alpha'}(\mathbf{x}',\tau')\\ &-2\Sigma_\alpha(\mathbf{x},\tau)\,\Psi^*_{\alpha'}(\mathbf{x}',\tau')\Psi_{\alpha'}(\mathbf{x}',\tau')+Q_{\alpha\alpha'}(\mathbf{x},\tau;\mathbf{x}',\tau')Q^*_{\alpha\alpha'}(\mathbf{x},\tau;\mathbf{x}',\tau')\\ &-Q_{\alpha\alpha'}(\mathbf{x},\tau;\mathbf{x}',\tau')\Psi_{\alpha'}(\mathbf{x}',\tau')\Psi^*_\alpha(\mathbf{x},\tau)-Q^*_{\alpha\alpha'}(\mathbf{x},\tau;\mathbf{x}',\tau')\Psi_\alpha(\mathbf{x},\tau)\Psi^*_{\alpha'}(\mathbf{x}',\tau')\}.\end{aligned} \tag{18}$$

Furthermore, the second term of decomposition (14), i.e. $\tilde{\mathcal{A}}^{(\mathcal{N},1)}[\delta\psi^*, \delta\psi]$, is linear in the fluctuations $\delta\psi^*_\alpha(\mathbf{x},\tau)$, $\delta\psi_\alpha(\mathbf{x},\tau)$ and turns out to vanish. Indeed, following the field-theoretic background field method [50, 51] it can be shown that the first-order terms $\tilde{\mathcal{A}}^{(\mathcal{N},1)}[\delta\psi^*, \delta\psi]$ can be neglected here as they would vanish later on from extremising $\tilde{\mathcal{A}}^{(\mathcal{N},0)}[\delta\psi^*, \delta\psi]$ with respect to the background fields $\Psi^*_\alpha(\mathbf{x},\tau)$, $\Psi_\alpha(\mathbf{x},\tau)$. The third term of decomposition (14) is quadratic in the fluctuations:

$$\begin{aligned}\tilde{\mathcal{A}}^{(\mathcal{N},2)}[\delta\psi^*,\delta\psi] = &\int_0^{\hbar\beta}\mathrm{d}\tau\int\mathrm{d}\mathbf{x}\sum_{\alpha=1}^{\mathcal{N}}\Bigg\{\delta\psi^*_\alpha(\mathbf{x},\tau)\left[\hbar\frac{\partial}{\partial\tau}-\frac{\hbar^2}{2M}\boldsymbol{\Delta}+V(\mathbf{x})-\mu\right]\delta\psi_\alpha(\mathbf{x},\tau)\\ &+\frac{1}{2}\int\mathrm{d}\mathbf{x}'\,V^{(\mathrm{int})}(\mathbf{x}-\mathbf{x}')[2\Sigma_\alpha(\mathbf{x},\tau)\,\delta\psi^*_\alpha(\mathbf{x}',\tau)\,\delta\psi_\alpha(\mathbf{x}',\tau)\\ &+Q_{\alpha\alpha}(\mathbf{x},\tau;\mathbf{x}',\tau)\,\delta\psi_\alpha(\mathbf{x}',\tau)\,\delta\psi^*_\alpha(\mathbf{x},\tau)+Q^*_{\alpha\alpha}(\mathbf{x},\tau;\mathbf{x}',\tau)\,\delta\psi_\alpha(\mathbf{x},\tau)\,\delta\psi^*_\alpha(\mathbf{x}',\tau)]\Bigg\}\\ &-\frac{1}{2\hbar}\int_0^{\hbar\beta}\mathrm{d}\tau\int_0^{\hbar\beta}\mathrm{d}\tau'\int\mathrm{d}\mathbf{x}\int\mathrm{d}\mathbf{x}'\sum_{\alpha=1}^{\mathcal{N}}\sum_{\alpha'=1}^{\mathcal{N}}D(\mathbf{x}-\mathbf{x}')\\ &\times\{2\,\Sigma_\alpha(\mathbf{x},\tau)\,\delta\psi^*_{\alpha'}(\mathbf{x}',\tau')\,\delta\psi_{\alpha'}(\mathbf{x}',\tau')+Q_{\alpha\alpha'}(\mathbf{x},\tau;\mathbf{x}',\tau')\,\delta\psi_{\alpha'}(\mathbf{x}',\tau')\,\delta\psi^*_\alpha(\mathbf{x},\tau)\\ &+Q^*_{\alpha\alpha'}(\mathbf{x},\tau;\mathbf{x}',\tau')\,\delta\psi_\alpha(\mathbf{x},\tau)\,\delta\psi^*_{\alpha'}(\mathbf{x}',\tau')\}.\end{aligned} \tag{19}$$

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Inserting expression (14) together with above results (18) and (19), into formula (7) leads to the replicated effective potential:

$$V_{\text{eff}}^{(\mathcal{N})} = -\frac{1}{\beta}\ln \overline{\mathcal{Z}^{\mathcal{N}}}, \tag{20}$$

which is given by:

$$V_{\text{eff}}^{(\mathcal{N})} = \frac{\tilde{\mathcal{A}}^{(\mathcal{N},0)}[\delta\psi^*, \delta\psi]}{\hbar\beta} - \frac{1}{\beta}\ln\left\{\left[\prod_{\alpha=1}^{\mathcal{N}}\oint \mathcal{D}\delta\psi_\alpha^* \oint \mathcal{D}\delta\psi_\alpha\right] \mathrm{e}^{-\tilde{\mathcal{A}}^{(\mathcal{N},2)}[\delta\psi^*,\delta\psi]/\hbar}\right\}. \tag{21}$$

It represents a functional of all mean-fields: $V_{\text{eff}}^{(\mathcal{N})} = V_{\text{eff}}^{(\mathcal{N})}[\Psi^*, \Psi, Q^*, Q, \Sigma]$. Extremising expression (21) with respect to the mean-fields $Q^*_{\alpha\alpha'}(\mathbf{x},\tau;\mathbf{x}',\tau')$, $Q_{\alpha\alpha'}(\mathbf{x},\tau;\mathbf{x}',\tau')$, and $\Sigma_\alpha(\mathbf{x},\tau)$ reproduces their definitions (15)–(17), where the expectation values turn out to be calculated with respect to the fluctuation action (19). Furthermore, an extremisation of the replicated effective potential (21) with respect to the background fields $\Psi^*_\alpha(\mathbf{x},\tau)$, $\Psi_\alpha(\mathbf{x},\tau)$ leads to the Gross–Pitaevskii equation:

$$\begin{aligned}
&\left\{\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2M}\boldsymbol{\Delta} + V(\mathbf{x}) - \mu\right\}\Psi_\alpha(\mathbf{x},\tau) - \int \mathrm{d}\mathbf{x}'\, V^{(\text{int})}(\mathbf{x}-\mathbf{x}')\\
&\quad\times[\Psi_\alpha(\mathbf{x},\tau)\Psi^*_\alpha(\mathbf{x}',\tau)\Psi_\alpha(\mathbf{x}',\tau) - \Sigma_\alpha(\mathbf{x},\tau)\,\Psi_\alpha(\mathbf{x}',\tau) - Q_{\alpha\alpha}(\mathbf{x},\tau;\mathbf{x}',\tau)\Psi_\alpha(\mathbf{x}',\tau)]\\
&= \frac{1}{\hbar}\int_0^{\hbar\beta}\mathrm{d}\tau'\int\mathrm{d}\mathbf{x}'\sum_{\alpha'=1}^{N} D(\mathbf{x}-\mathbf{x}')\{\,Q_{\alpha\alpha'}(\mathbf{x},\tau;\mathbf{x}',\tau')\Psi_{\alpha'}(\mathbf{x}',\tau')\\
&\quad+[\Sigma_{\alpha'}(\mathbf{x}',\tau') - \Psi_{\alpha'}(\mathbf{x}',\tau')\Psi^*_{\alpha'}(\mathbf{x}',\tau')]\Psi_\alpha(\mathbf{x},\tau)\}
\end{aligned} \tag{22}$$

and its complex conjugate.

5. Replica symmetry

Now we apply the replica symmetry, where we assume that all the respective replica indices α contribute in the same way. Furthermore, the dirty boson problem is translationally invariant in imaginary time. With this we get for the background

$$\Psi_\alpha(\mathbf{x},\tau) = \Psi(\mathbf{x}), \quad \Psi^*_\alpha(\mathbf{x},\tau) = \Psi^*(\mathbf{x}), \quad \Sigma_\alpha(\mathbf{x},\tau) = \Sigma(\mathbf{x}), \tag{23}$$

and for the mean fields

$$Q_{\alpha\alpha'}(\mathbf{x},\tau;\mathbf{x}',\tau') = Q\left(\mathbf{x}-\mathbf{x}', \frac{\mathbf{x}+\mathbf{x}'}{2}; \tau-\tau'\right)\delta_{\alpha\alpha'} + q\left(\mathbf{x}-\mathbf{x}', \frac{\mathbf{x}+\mathbf{x}'}{2}; \tau-\tau'\right) + \Psi^*(\mathbf{x})\Psi(\mathbf{x}), \tag{24}$$

and its complex conjugate. In (24) we perform a Fourier–Matsubara decomposition with respect to the differences in space and time, i.e. $\mathbf{x}-\mathbf{x}'$ and $\tau-\tau'$. Furthermore, we assume within a semi-classical approximation that the dependence on the center of mass coordinate $(\mathbf{x}+\mathbf{x}')/2$ is smooth, so we get

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$$Q\left(\mathbf{x}-\mathbf{x}',\frac{\mathbf{x}+\mathbf{x}'}{2};\tau-\tau'\right)=\int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^n}\,\mathrm{e}^{\mathrm{i}\mathbf{k}(\mathbf{x}-\mathbf{x}')}\frac{1}{\hbar\beta}\sum_{m=-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\omega_m(\tau-\tau')}Q_m\left(\mathbf{k},\frac{\mathbf{x}+\mathbf{x}'}{2}\right),\tag{25}$$

$$q\left(\mathbf{x}-\mathbf{x}',\frac{\mathbf{x}+\mathbf{x}'}{2};\tau-\tau'\right)=\int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^n}\,\mathrm{e}^{\mathrm{i}\mathbf{k}(\mathbf{x}-\mathbf{x}')}\frac{1}{\hbar\beta}\sum_{m=-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\mathrm{i}\omega_m(\tau-\tau')}q_m\left(\mathbf{k},\frac{\mathbf{x}+\mathbf{x}'}{2}\right),\tag{26}$$

and their complex conjugates, where $\omega_m=2\pi m/\hbar\beta$ denote the bosonic Matsubara frequencies and $\mathbf{k}$ the wave vector.

Using this ansatz, we have to evaluate the expectation values in the mean-field equations (15)–(17) and (22). To this end we note that the fluctuation action (19) is of the general form

$$\tilde{\mathcal{A}}^{(\mathcal{N},2)}[\delta\psi^*,\delta\psi]=\int_0^{\hbar\beta}\mathrm{d}\tau\int_0^{\hbar\beta}\mathrm{d}\tau'\int\mathrm{d}\mathbf{x}\int\mathrm{d}\mathbf{x}'\sum_{\alpha=1}^{\mathcal{N}}\sum_{\alpha'=1}^{\mathcal{N}}$$
$$\frac{1}{2}(\delta\psi_\alpha^*(\mathbf{x},\tau),\ \delta\psi_\alpha(\mathbf{x},\tau))\,G_{\alpha\alpha'}^{-1}\left(\mathbf{x}-\mathbf{x}',\frac{\mathbf{x}+\mathbf{x}'}{2};\tau-\tau'\right)\begin{pmatrix}\delta\psi_{\alpha'}(\mathbf{x}',\tau')\\ \delta\psi_{\alpha'}^*(\mathbf{x}',\tau')\end{pmatrix},\tag{27}$$

where the semi-classical Fourier–Matsubara transformation of the integral kernel

$$G_{\alpha\alpha'}^{-1}\left(\mathbf{x}-\mathbf{x}',\frac{\mathbf{x}+\mathbf{x}'}{2};\tau-\tau'\right)=\int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^n}\,\mathrm{e}^{\mathrm{i}\mathbf{k}(\mathbf{x}-\mathbf{x}')}\frac{1}{\hbar\beta}\sum_{m=-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\omega_m(\tau-\tau')}\,G_{\alpha\alpha'}^{-1}\left(\mathbf{k},\omega_m,\frac{\mathbf{x}+\mathbf{x}'}{2}\right),\tag{28}$$

decomposes according to

$$G_{\alpha\alpha'}^{-1}\left(\mathbf{k},\omega_m,\frac{\mathbf{x}+\mathbf{x}'}{2}\right)=\begin{pmatrix}a\left(\mathbf{k},\omega_m,\frac{\mathbf{x}+\mathbf{x}'}{2}\right) & 0\\ 0 & a^*\left(\mathbf{k},\omega_m,\frac{\mathbf{x}+\mathbf{x}'}{2}\right)\end{pmatrix}\delta_{\alpha\alpha'}+\begin{pmatrix}b\left(\mathbf{k},\omega_m,\frac{\mathbf{x}+\mathbf{x}'}{2}\right) & 0\\ 0 & b^*\left(\mathbf{k},\omega_m,\frac{\mathbf{x}+\mathbf{x}'}{2}\right)\end{pmatrix},\tag{29}$$

with the abbreviations

$$a\left(\mathbf{k},\omega_m,\frac{\mathbf{x}+\mathbf{x}'}{2}\right)=-\mathrm{i}\hbar\omega_m+\epsilon(\mathbf{k})-\mu+\Sigma\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right)\int d^nx''\,V^{(\mathrm{int})}(\mathbf{x}'')+V^{(\mathrm{int})}(\mathbf{k})\Psi^*\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right)\Psi\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right)$$
$$+V\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right)-\frac{1}{\hbar}\int\frac{\mathrm{d}\mathbf{k}'}{(2\pi)^n}D(\mathbf{k}')Q_m\left(\mathbf{k}-\mathbf{k}',\frac{\mathbf{x}+\mathbf{x}'}{2}\right)-\mathcal{N}\beta D(\mathbf{k})\Sigma\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right)\delta_{m,0}$$
$$+\int\frac{\mathrm{d}\mathbf{k}'}{(2\pi)^n}V^{(\mathrm{int})}(\mathbf{k}')\left[q_m\left(\mathbf{k}-\mathbf{k}',\frac{\mathbf{x}+\mathbf{x}'}{2}\right)+Q_m\left(\mathbf{k}-\mathbf{k}',\frac{\mathbf{x}+\mathbf{x}'}{2}\right)\right],\tag{30}$$

$$b\left(\mathbf{k},\omega_m,\frac{\mathbf{x}+\mathbf{x}'}{2}\right)=-\frac{1}{\hbar}\left[\int\frac{\mathrm{d}\mathbf{k}'}{(2\pi)^n}D(\mathbf{k}')q_m\left(\mathbf{k}-\mathbf{k}',\frac{\mathbf{x}+\mathbf{x}'}{2}\right)+\hbar\beta D(\mathbf{k})\Psi^*\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right)\Psi\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right)\delta_{m,0}\right],\tag{31}$$

and the free dispersion $\epsilon(\mathbf{k})=\hbar^2\mathbf{k}^2/2M$. Furthermore, $D(\mathbf{k})$ and $V^{(\mathrm{int})}(\mathbf{k})$ are the Fourier transforms of the disorder correlation function $D(\mathbf{x})$ and the two-particle interaction potential $V^{(\mathrm{int})}(\mathbf{x})$, respectively: $D(\mathbf{x})=\int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^n}\,D(\mathbf{k})\mathrm{e}^{\mathrm{i}\mathbf{k}\mathbf{x}}$, $V^{(\mathrm{int})}(\mathbf{x})=\int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^n}\,V^{(\mathrm{int})}(\mathbf{k})\mathrm{e}^{\mathrm{i}\mathbf{k}\mathbf{x}}$.

The corresponding Green function follows from solving

$$\int_0^{\hbar\beta} \mathrm{d}\tau \int \mathrm{d}\mathbf{x} \sum_{\alpha=1}^{\mathcal{N}} G^{-1}_{\alpha_1\alpha}\left(\mathbf{x}-\mathbf{x}_1, \frac{\mathbf{x}+\mathbf{x}_1}{2}; \tau-\tau_1\right) G_{\alpha\alpha_2}\left(\mathbf{x}_2-\mathbf{x}, \frac{\mathbf{x}+\mathbf{x}_2}{2}; \tau_2-\tau\right)$$
$$= \hbar\delta(\mathbf{x}_1-\mathbf{x}_2)\delta(\tau_1-\tau_2)\delta_{\alpha_1\alpha_2}, \tag{32}$$

which reduces with a semi-classical Fourier–Matsubara transformation to the algebraic identity:

$$\sum_{\alpha=1}^{\mathcal{N}} G^{-1}_{\alpha_1\alpha}\left(\mathbf{k}, \omega_m, \frac{\mathbf{x}+\mathbf{x}'}{2}\right) G_{\alpha\alpha_2}\left(\mathbf{k}, \omega_m, \frac{\mathbf{x}+\mathbf{x}'}{2}\right) = \hbar\,\delta_{\alpha_1\alpha_2}. \tag{33}$$

Thus, the corresponding Green function, which contains expectation values according to

$$G_{\alpha\alpha'}\left(\mathbf{x}-\mathbf{x}', \frac{\mathbf{x}+\mathbf{x}'}{2}; \tau-\tau'\right) = \begin{pmatrix} \langle\delta\psi_\alpha(\mathbf{x},\tau)\delta\psi^*_{\alpha'}(\mathbf{x}',\tau')\rangle & 0 \\ 0 & \langle\delta\psi^*_\alpha(\mathbf{x},\tau)\delta\psi_{\alpha'}(\mathbf{x}',\tau')\rangle \end{pmatrix}, \tag{34}$$

is determined from

$$\langle\delta\psi_\alpha(\mathbf{x},\tau)\delta\psi^*_{\alpha'}(\mathbf{x}',\tau')\rangle = g_1\left(\mathbf{x}-\mathbf{x}', \frac{\mathbf{x}+\mathbf{x}'}{2}; \tau-\tau'\right)\delta_{\alpha\alpha'} + g_2\left(\mathbf{x}-\mathbf{x}', \frac{\mathbf{x}+\mathbf{x}'}{2}; \tau-\tau'\right), \tag{35}$$

with the contributions:

$$g_1\left(\mathbf{x}-\mathbf{x}', \frac{\mathbf{x}+\mathbf{x}'}{2}; \tau-\tau'\right) = \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^n}\,\mathrm{e}^{\mathrm{i}\mathbf{k}(\mathbf{x}-\mathbf{x}')} \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}\omega_m(\tau-\tau')}}{\beta a(\mathbf{k},\omega_m,\frac{\mathbf{x}+\mathbf{x}'}{2})}, \tag{36}$$

$$g_2\left(\mathbf{x}-\mathbf{x}', \frac{\mathbf{x}+\mathbf{x}'}{2}; \tau-\tau'\right) = \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^n}\,\mathrm{e}^{\mathrm{i}\mathbf{k}(\mathbf{x}-\mathbf{x}')} \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}\omega_m(\tau-\tau')}}{\beta\mathcal{N}}$$
$$\times\left[\frac{1}{\mathcal{N}b(\mathbf{k},\omega_m,\frac{\mathbf{x}+\mathbf{x}'}{2}) + a(\mathbf{k},\omega_m,\frac{\mathbf{x}+\mathbf{x}'}{2})} - \frac{1}{a(\mathbf{k},\omega_m,\frac{\mathbf{x}+\mathbf{x}'}{2})}\right]. \tag{37}$$

Comparing equations (15)–(17) and (24)–(26) with (35)–(37) yields:

$$Q_m\left(\mathbf{k}, \frac{\mathbf{x}+\mathbf{x}'}{2}\right) = \frac{\hbar}{a(\mathbf{k},\omega_m,\frac{\mathbf{x}+\mathbf{x}'}{2})}, \tag{38}$$

$$q_m\left(\mathbf{k}, \frac{\mathbf{x}+\mathbf{x}'}{2}\right) = \frac{\hbar}{\mathcal{N}}\left[\frac{1}{\mathcal{N}b(\mathbf{k},\omega_m,\frac{\mathbf{x}+\mathbf{x}'}{2}) + a(\mathbf{k},\omega_m,\frac{\mathbf{x}+\mathbf{x}'}{2})} - \frac{1}{a(\mathbf{k},\omega_m,\frac{\mathbf{x}+\mathbf{x}'}{2})}\right], \tag{39}$$

and their complex conjugates. Equations (38) and (39) represent, due to expressions (30) and (31), two coupled integral mean-field equations for the quantities $Q_m(\mathbf{k}, \frac{\mathbf{x}+\mathbf{x}'}{2})$

and $q_m(\mathbf{k}, \frac{\mathbf{x}+\mathbf{x}'}{2})$. As it is not possible to solve them analytically for a general disorder potential and a general interaction potential, we specialize now to a δ-correlated disorder potential and a contact interaction potential.

6. Delta-correlated disorder and contact interaction potential

Now we elaborate a solution of our mean-field equations for the special case of a δ-correlated disorder potential, which is defined in equation (A.4), i.e. we have

$$D(\mathbf{k}) = D, \tag{40}$$

where D denotes the disorder strength. Furthermore, we choose a contact interaction potential

$$V^{(\mathrm{int})}(\mathbf{x}-\mathbf{x}') = g\delta(\mathbf{x}-\mathbf{x}'), \tag{41}$$

where g denotes the interaction coupling strength. In this case formulas (30) and (31) reduce to:

$$a\left(\mathbf{k}, \omega_m, \frac{\mathbf{x}+\mathbf{x}'}{2}\right) = -\mathrm{i}\hbar\omega_m + \epsilon(\mathbf{k}) - \mu + 2g\Sigma\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right) + V\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right) - \frac{D}{\hbar}Q_m\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right) - \mathcal{N}\beta D\Sigma\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right)\delta_{m,0}, \tag{42}$$

and

$$b\left(\mathbf{k}, \omega_m, \frac{\mathbf{x}+\mathbf{x}'}{2}\right) = -\frac{D}{\hbar}\left[q_m\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right) + \hbar\beta\Psi^*\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right)\Psi\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right)\delta_{m,0}\right], \tag{43}$$

where we have introduced the abbreviation

$$Q_m\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right) = \int \frac{\mathrm{d}\mathbf{k}'}{(2\pi)^n} Q_m\left(\mathbf{k}', \frac{\mathbf{x}+\mathbf{x}'}{2}\right), \tag{44}$$

$$q_m\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right) = \int \frac{\mathrm{d}\mathbf{k}'}{(2\pi)^n} q_m\left(\mathbf{k}', \frac{\mathbf{x}+\mathbf{x}'}{2}\right). \tag{45}$$

Expressions (38) and (39) yield then together with expressions (44) and (45) algebraic mean-field equations, which we can solve. Inserting expressions (42) and (43) into equations (38) and (39) and taking $\mathbf{x} = \mathbf{x}'$ in expressions (44) and (45), with the Schwinger integral [52]

$$\frac{1}{a^\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty ds\, s^{\nu-1}\,\mathrm{e}^{-as}, \tag{46}$$

and formula [53, 8.310.1], we obtain the following self-consistency equations:

$$Q_m(\mathbf{x}) = \Gamma\left(1-\frac{n}{2}\right)\hbar\left(\frac{M}{2\pi\hbar^2}\right)^{n/2}\left[-\mathrm{i}\hbar\omega_m - \mu + 2g\Sigma(\mathbf{x}) + V(\mathbf{x}) - \frac{D}{\hbar}Q_m(\mathbf{x}) - \mathcal{N}\beta D\Sigma(\mathbf{x})\,\delta_{m,0}\right]^{\frac{n}{2}-1}, \tag{47}$$

$$\begin{aligned} q_m(\mathbf{x}) = &-\Gamma\left(1-\frac{n}{2}\right)\frac{\hbar}{\mathcal{N}}\left(\frac{M}{2\pi\hbar^2}\right)^{n/2}\left[-\mathrm{i}\hbar\omega_m - \mu + 2g\Sigma(\mathbf{x}) + V(\mathbf{x}) - \frac{D}{\hbar}Q_m(\mathbf{x}) - \mathcal{N}\beta D\Sigma(\mathbf{x})\,\delta_{m,0}\right]^{\frac{n}{2}-1} \\ &+\Gamma\left(1-\frac{n}{2}\right)\frac{\hbar}{\mathcal{N}}\left(\frac{M}{2\pi\hbar^2}\right)^{n/2}\Bigg\{-\mathrm{i}\hbar\omega_m - \mu + 2g\Sigma(\mathbf{x}) + V(\mathbf{x}) - \frac{D}{\hbar}Q_m(\mathbf{x}) - \mathcal{N}\beta D\Sigma(\mathbf{x})\,\delta_{m,0} \\ &-\mathcal{N}\frac{D}{\hbar}\left[q_m(\mathbf{x}) + \hbar\beta\Psi^*(\mathbf{x})\Psi(\mathbf{x})\,\delta_{m,0}\right]\Bigg\}^{\frac{n}{2}-1}. \end{aligned} \tag{48}$$

From the above expressions, we conclude $Q_m^*(\mathbf{x}) = Q_{-m}(\mathbf{x})$ and $q_m^*(\mathbf{x}) = q_{-m}(\mathbf{x})$. With this we read off from equations (25) and (26) that $Q^*(\mathbf{x};\tau'-\tau) = Q(\mathbf{x};\tau-\tau')$ and $q^*(\mathbf{x};\tau'-\tau) = q(\mathbf{x};\tau-\tau')$, respectively.

The expressions for $Q_m(\mathbf{x})$ and $q_m(\mathbf{x})$ in equations (47) and (48) turn out to diverge in two spatial dimensions because of the prefactor $\Gamma\left(1-\frac{n}{2}\right)$. This means that our theory in its actual form is not valid in the two-dimensional case. In order to get valid self-consistency equations also in two dimensions, one way would be to choose a disorder potential with a finite correlation length, e.g. a Lorentzian-correlated potential. Then this finite correlation length would provide a regularization that would yield together with a renormalization, finite self-consistency equations. As the treatment of a Lorentzian-correlated disorder potential lies out of the scope of the present paper, we will restrict ourselves later on to the study of the one- and the three-dimensional cases.

We note in equations (47) and (48) that the terms containing the parameter β are always multiplied by the number of replicas $\mathcal{N}$. This is important because it means that in the zero temperature case, i.e. $\beta \to \infty$, those terms will be eliminated in the replica limit $\mathcal{N} \to 0$, and otherwise they would diverge.

In [41] the replica limit is taken as soon as the replica number $\mathcal{N}$ appears at different steps of the calculation. In our work, and contrary to [41], until this level of the calculation no replica limit was performed. We are taking this limit as late as possible in order to avoid any loss of terms due to the performance of the replica limit in the earlier steps of the calculation.

Note that in the replica limit $\mathcal{N} \to 0$, equations (47) and (48) yield

$$Q_m(\mathbf{x}) = \Gamma\left(1-\frac{n}{2}\right)\hbar\left(\frac{M}{2\pi\hbar^2}\right)^{n/2}\left[-\mathrm{i}\hbar\omega_m - \mu + 2g\Sigma(\mathbf{x}) + V(\mathbf{x}) - \frac{D}{\hbar}Q_m(\mathbf{x})\right]^{\frac{n}{2}-1} \tag{49}$$

and

$$q(\mathbf{x}) = D\Gamma\left(2-\frac{n}{2}\right)\left(\frac{M}{2\pi\hbar^2}\right)^{n/2}\frac{\left[q(\mathbf{x}) + \Psi^*(\mathbf{x})\Psi(\mathbf{x})\right]}{\left[-\mu + 2g\Sigma(\mathbf{x}) + V(\mathbf{x}) - \frac{D}{\hbar}Q_0(\mathbf{x})\right]^{2-\frac{n}{2}}}, \tag{50}$$

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where $q(\mathbf{x}) = q_0(\mathbf{x})/\hbar\beta$ and $q_m(\mathbf{x}) = 0$ for $m \neq 0$.

Now we insert the replica-symmetric solution ansatz (24) and (25) also in the other mean-field equations (17) and (22). In this way we obtain in the replica limit $\mathcal{N} \to 0$ the mean-field

$$\Sigma(\mathbf{x}) = q(\mathbf{x}) + n_0(\mathbf{x}) + \lim_{\eta\to 0^+} \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega_m\eta} \frac{Q_m(\mathbf{x})}{\hbar\beta} \tag{51}$$

and the Gross–Pitaevskii equation

$$\left[-\mu + 2g\Sigma(\mathbf{x}) + V(\mathbf{x}) - gn_0(\mathbf{x}) - \frac{D}{\hbar} Q_0(\mathbf{x}) - \frac{\hbar^2}{2M}\Delta\right] \sqrt{n_0(\mathbf{x})} = 0, \tag{52}$$

where we have set $n_0(\mathbf{x}) = \Psi^*(\mathbf{x})\Psi(\mathbf{x})$.

7. Thermodynamic properties

Now we return to the replicated effective potential (21) and evaluate it for the special case of a δ-correlated disorder potential (40) and contact interaction potential (41) at the replica-symmetric background fields (23) and (24) by taking into account equation (25). Thus, the replicated effective potential decomposes according to $V_{\mathrm{eff}}^{(\mathcal{N})} = V_{\mathrm{eff}}^{(\mathcal{N},1)} + V_{\mathrm{eff}}^{(\mathcal{N},2)}$. The first term reads

$$\begin{aligned} V_{\mathrm{eff}}^{(\mathcal{N},1)} = \mathcal{N} \int \mathrm{d}\mathbf{x} \Bigg\{ &-g\Sigma^2(\mathbf{x}) - \frac{g}{2}\Psi^{*2}(\mathbf{x})\Psi^2(\mathbf{x}) + \Psi^*(\mathbf{x}) \left[-\mu - \frac{\hbar^2}{2M}\boldsymbol{\Delta} + 2g\Sigma(\mathbf{x}) + V(\mathbf{x}) - \frac{D}{2\hbar} Q_0^*(\mathbf{x}) - \frac{D}{2\hbar} Q_0(\mathbf{x})\right] \\ &\times \Psi(\mathbf{x}) + \frac{D}{2\hbar}\left[Q_0^*(\mathbf{x}) + Q_0(\mathbf{x})\right] \Psi^*(\mathbf{x})\Psi(\mathbf{x}) + \frac{D}{2\beta\hbar^2} \lim_{\eta\to 0^+} \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega_m\eta} \left[Q_m^*(\mathbf{x}) + Q_m(\mathbf{x})\right] q_m(\mathbf{x}) \\ &+ \frac{D}{2\hbar^2\beta} \lim_{\eta\to 0^+} \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega_m\eta} Q_m(\mathbf{x}) Q_{-m}^*(\mathbf{x}) \Bigg\} + \frac{\mathcal{N}^2\beta D}{2} \int \mathrm{d}\mathbf{x} \Bigg\{ \left[\Sigma(\mathbf{x}) - \Psi^*(\mathbf{x})\Psi(\mathbf{x})\right]^2 \\ &+ \frac{1}{(\beta\hbar)^2} \lim_{\eta\to 0^+} \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega_m\eta} q_m(\mathbf{x}) q_{-m}^*(\mathbf{x}) - \Psi^{*2}(\mathbf{x})\Psi^2(\mathbf{x}) \Bigg\}, \end{aligned} \tag{53}$$

where, again, the normal ordering is explicitly emphasized and the second term is given by the tracelog of the integral kernel (27):

$$V_{\mathrm{eff}}^{(\mathcal{N},2)} = \frac{1}{2\beta} \mathrm{Tr} \ln G^{-1}. \tag{54}$$

With the help of the Fourier–Matsubara transformation (28) the latter reduces to

$$V_{\mathrm{eff}}^{(\mathcal{N},2)} = \frac{1}{2\beta} \int \mathrm{d}\mathbf{x} \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^n} \lim_{\eta\to 0^+} \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega_m\eta} \ln\left[\mathrm{Det}\, G_{\alpha\alpha'}^{-1}(\mathbf{k},\omega_m,\mathbf{x})\right], \tag{55}$$

where the determinant of the matrix (29) yields

$$\begin{aligned} \mathrm{Det}\, G_{\alpha\alpha'}^{-1}(\mathbf{k},\omega_m,\mathbf{x}) = &\left[a(\mathbf{k},\omega_m,\mathbf{x}) a^*(\mathbf{k},\omega_m,\mathbf{x})\right]^{\mathcal{N}-1} \left[a(\mathbf{k},\omega_m,\mathbf{x})\right. \\ &\left. + \mathcal{N} b(\mathbf{k},\omega_m,\mathbf{x})\right] \left[a^*(\mathbf{k},\omega_m,\mathbf{x}) + \mathcal{N} b^*(\mathbf{k},\omega_m,\mathbf{x})\right]. \end{aligned} \tag{56}$$

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Performing the replica limit $\mathcal{N}\to 0$, the respective contributions to the replicated effective potential reduce to

$$V_{\text{eff}}^{(1)} = \lim_{\mathcal{N}\to 0}\frac{V_{\text{eff}}^{(\mathcal{N},1)}}{\mathcal{N}} = \int \mathrm{d}\mathbf{x}\Bigg\{-g\Sigma^2(\mathbf{x}) - \frac{g}{2}\Psi^{*2}(\mathbf{x})\Psi^2(\mathbf{x}) + \Psi^*(\mathbf{x})\left[-\mu - \frac{\hbar^2}{2M}\mathbf{\Delta} + 2g\Sigma(\mathbf{x}) + V(\mathbf{x}) - \frac{D}{2\hbar}Q_0^*(\mathbf{x}) - \frac{D}{2\hbar}Q_0(\mathbf{x})\right]\Psi(\mathbf{x}) + \frac{D}{2\hbar^2\beta}\lim_{\eta\to 0^+}\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\omega_m\eta}\,Q_m(\mathbf{x})Q_{-m}^*(\mathbf{x}) + \frac{D}{2\hbar}\left[Q_0^*(\mathbf{x}) + Q_0(\mathbf{x})\right]\left[q(\mathbf{x}) + \Psi^*(\mathbf{x})\Psi(\mathbf{x})\right]\Bigg\}, \tag{57}$$

and

$$V_{\text{eff}}^{(2)} = \lim_{\mathcal{N}\to 0}\frac{V_{\text{eff}}^{(\mathcal{N},2)}}{\mathcal{N}} = \frac{1}{2\beta}\int \mathrm{d}\mathbf{x}\int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^n}\Bigg\{\lim_{\eta\to 0^+}\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\omega_m\eta}\ln\left[-\mathrm{i}\hbar\omega_m + \epsilon(\mathbf{k}) - \mu + 2g\Sigma(\mathbf{x}) + V(\mathbf{x}) - \frac{D}{\hbar}Q_m(\mathbf{x})\right] + \lim_{\eta\to 0^+}\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\omega_m\eta}\ln\left[\mathrm{i}\hbar\omega_m + \epsilon(\mathbf{k}) - \mu + 2g\Sigma(\mathbf{x}) + V(\mathbf{x}) - \frac{D}{\hbar}Q_m^*(\mathbf{x})\right] - \frac{\beta D\left[q(\mathbf{x}) + \Psi^*(\mathbf{x})\Psi(\mathbf{x})\right]}{\epsilon(\mathbf{k}) - \mu + 2g\Sigma(\mathbf{x}) + V(\mathbf{x}) - \frac{D}{\hbar}Q_0(\mathbf{x})} - \frac{\beta D\left[q(\mathbf{x}) + \Psi^*(\mathbf{x})\Psi(\mathbf{x})\right]}{\epsilon(\mathbf{k}) - \mu + 2g\Sigma(\mathbf{x}) + V(\mathbf{x}) - \frac{D}{\hbar}Q_0^*(\mathbf{x})}\Bigg\}, \tag{58}$$

where we have inserted equations (42), (43) and (56) into equation (55). The remaining $\mathbf{k}$-integrals of the logarithmic functions in equation (58) are UV-divergent in all dimensions, while the $\mathbf{k}$-integrals of the third and the fourth term diverge in two and three dimensions and converge only in one dimension. Thus, we evaluate equation (58) by using, again, the Schwinger integral (46) and the corresponding Schwinger representation of the logarithm:

$$\ln a = -\frac{\partial}{\partial x}\left[\frac{1}{\Gamma(x)}\int_0^\infty ds\; s^{x-1}\,\mathrm{e}^{-as}\right]\Bigg|_{x=0}. \tag{59}$$

With this we obtain:

$$V_{\text{eff}}^{(2)} = -\frac{1}{2\beta}\left(\frac{M}{2\pi\hbar^2}\right)^{n/2}\lim_{\eta\to 0^+}\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\omega_m\eta}\int\mathrm{d}\mathbf{x}\,\Gamma\!\left(-\frac{n}{2}\right)\Bigg\{\left[-\mathrm{i}\hbar\omega_m - \mu + 2g\Sigma(\mathbf{x}) + V(\mathbf{x}) - \frac{D}{\hbar}Q_m(\mathbf{x})\right]^{n/2} + \left[\mathrm{i}\hbar\omega_m - \mu + 2g\Sigma(\mathbf{x}) + V(\mathbf{x}) - \frac{D}{\hbar}Q_m^*(\mathbf{x})\right]^{n/2}\Bigg\} - \frac{D}{2}\int\mathrm{d}\mathbf{x}\left[q(\mathbf{x}) + \Psi^*(\mathbf{x})\Psi(\mathbf{x})\right]\Gamma\!\left(-\frac{n}{2}+1\right)\left(\frac{M}{2\pi\hbar^2}\right)^{n/2} \times\Bigg\{\left[-\mu + 2g\Sigma(\mathbf{x}) + V(\mathbf{x}) - \frac{D}{\hbar}Q_0(\mathbf{x})\right]^{\frac{n}{2}-1} + \left[-\mu + 2g\Sigma(\mathbf{x}) + V(\mathbf{x}) - \frac{D}{\hbar}Q_0^*(\mathbf{x})\right]^{\frac{n}{2}-1}\Bigg\}. \tag{60}$$

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As the extremum of the effective potential yields the thermodynamic potential due to equations (4) and (20), we obtain from equations (57) and (60) the free energy:

$$\Omega = \int \mathrm{d}\mathbf{x} \Bigg\{ - g\Sigma^2(\mathbf{x}) - \frac{g}{2} n_0^2(\mathbf{x}) - \sqrt{n_0(\mathbf{x})} \left[\mu + \frac{\hbar^2}{2M} \boldsymbol{\Delta} - 2g\Sigma(\mathbf{x}) - V(\mathbf{x}) + \frac{D}{\hbar} Q_0(\mathbf{x}) \right] \sqrt{n_0(\mathbf{x})}$$
$$+ \frac{D}{\hbar} Q_0(\mathbf{x}) \left[q(\mathbf{x}) + n_0(\mathbf{x}) \right] + \frac{D}{2\hbar^2 \beta} \lim_{\eta \to 0^+} \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega_m \eta} \, Q_m^2(\mathbf{x})$$
$$- D\Gamma\left(-\frac{n}{2} + 1 \right) \left(\frac{M}{2\pi\hbar^2} \right)^{n/2} \left[q(\mathbf{x}) + n_0(\mathbf{x}) \right] \left[-\mu + 2g\Sigma(\mathbf{x}) + V(\mathbf{x}) - \frac{D}{\hbar} Q_0(\mathbf{x}) \right]^{\frac{n}{2}-1}$$
$$- \frac{1}{\beta} \Gamma\left(-\frac{n}{2} \right) \left(\frac{M}{2\pi\hbar^2} \right)^{n/2} \lim_{\eta \downarrow 0} \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega_m \eta} \left[-\mathrm{i}\hbar\omega_m - \mu + 2g\Sigma(\mathbf{x}) + V(\mathbf{x}) - \frac{D}{\hbar} Q_m(\mathbf{x}) \right]^{n/2} \Bigg\}. \tag{61}$$

Note that the particle density $n(\mathbf{x})$, which is defined from the expression $N = -\frac{\partial \Omega}{\partial \mu} = \int \mathrm{d}\mathbf{x} n(\mathbf{x})$, with the particle number N, turns out to coincide with the mean-field $\Sigma(\mathbf{x})$ due to equations (49)–(51):

$$\Sigma(\mathbf{x}) = n(\mathbf{x}). \tag{62}$$

Furthermore, all self-consistency equations (49)–(52) can be directly obtained by extremising the thermodynamic potential (61) with respect to its variables $Q_{m\neq 0}(\mathbf{x})$, $Q_0(\mathbf{x})$, $q(\mathbf{x})$, and $\sqrt{n_0(\mathbf{x})}$. Indeed the combination of the two extremisations $\frac{\delta \Omega}{\delta Q_{m\neq 0}(\mathbf{x}')} = 0$ and $\frac{\delta \Omega}{\delta q(\mathbf{x}')} = 0$ gives us equation (49), while the extremisations $\frac{\delta \Omega}{\delta Q_0(\mathbf{x}')} = 0$ and $\frac{\delta \Omega}{\delta \sqrt{n_0(\mathbf{x}')}} = 0$ yield equations (49) and (52), respectively.

Now we apply our theory, which is formulated for a general n-dimensional homogeneous system, first to the three-dimensional dirty bosons, since this case turns out to be simpler, and then to the one-dimensional dirty bosons. The two-dimensional case cannot be treated using the actual form of the theory as is discussed in detail below equation (48).

8. Application of Hartree–Fock mean-field theory in 3D

Here we are interested in obtaining the free energy as well as the self-consistency equations of the three-dimensional dirty boson system. To this end, we deduce first the corresponding Matsubara coefficients.

8.1. Matsubara coefficients

In three dimensions ($n = 3$), equations (47) and (48) reduce after performing the replica limit $\mathcal{N} \to 0$ to:

$$Q_m(\mathbf{x}) = -2\sqrt{\pi}\,\hbar \left(\frac{M}{2\pi\hbar^2} \right)^{3/2} \sqrt{ -\mathrm{i}\hbar\omega_m - \mu + 2gn(\mathbf{x}) + V(\mathbf{x}) - \frac{D}{\hbar} Q_m(\mathbf{x}) }\,, \tag{63}$$

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$$q_m(\mathbf{x}) = \sqrt{\pi}\,D\left(\frac{M}{2\pi\hbar^2}\right)^{3/2}\frac{[q_m(\mathbf{x})+\hbar\beta\Psi^*(\mathbf{x})\Psi(\mathbf{x})\delta_{m,0}]}{\sqrt{-\mathrm{i}\hbar\omega_m-\mu+2g\Sigma(\mathbf{x})+V(\mathbf{x})-\frac{D}{\hbar}Q_0(\mathbf{x})}}. \tag{64}$$

Equation (63) represents a quadratic equation for the corresponding Matsubara coefficients $Q_m(\mathbf{x})$, which is solved by:

$$Q_m(\mathbf{x}) = -2\pi\hbar D\left(\frac{M}{2\pi\hbar^2}\right)^{3}\pm 2\sqrt{\pi}\,\hbar\left(\frac{M}{2\pi\hbar^2}\right)^{3/2}\sqrt{-\mathrm{i}\hbar\omega_m-\mu+2gn(\mathbf{x})+V(\mathbf{x})+\pi D^2\left(\frac{M}{2\pi\hbar^2}\right)^{3}}. \tag{65}$$

Now, we treat both cases ($m=0$ and $m\neq 0$) separately.

At first, we consider the case $m=0$ and note that $Q_0(\mathbf{x})$ has to be real according to equation (63). For $m=0$ equation (64) reduces to

$$Q_0(\mathbf{x}) = \begin{cases} -2\sqrt{\pi}\,\hbar\left(\frac{M}{2\pi\hbar^2}\right)^{3/2}\left[\sqrt{\pi}\,D\left(\frac{M}{2\pi\hbar^2}\right)^{3/2}+\sqrt{-\mu_r(\mathbf{x})}\right]; & \mu_r(\mathbf{x})\leqslant 0,\\ -2\sqrt{\pi}\,\hbar\left(\frac{M}{2\pi\hbar^2}\right)^{3/2}\left[\sqrt{\pi}\,D\left(\frac{M}{2\pi\hbar^2}\right)^{3/2}-\sqrt{-\mu_r(\mathbf{x})}\right]; & \mu_r^{(\mathrm{crit})}\leqslant\mu_r(\mathbf{x})\leqslant 0,\end{cases} \tag{66}$$

where we have introduced the renormalized chemical potential:

$$\mu_r(\mathbf{x}) = \mu - V(\mathbf{x}) - 2gn(\mathbf{x}) - \pi D^2\left(\frac{M}{2\pi\hbar^2}\right)^{3}, \tag{67}$$

and the critical chemical potential is defined by $\mu_r^{(\mathrm{crit})} = -\pi D^2\left(\frac{M}{2\pi\hbar^2}\right)^{3}$. Since $\mu_r^{(\mathrm{crit})}\leqslant 0$, we obtain from equation (66) that the condition $\mu_r(\mathbf{x})\leqslant 0$ has to be fulfilled.

Now, we consider the case $m\neq 0$, where equations (64) and (66) are only compatible for the lower sign, i.e. we conclude:

$$Q_m(\mathbf{x}) = -2\pi\hbar D\left(\frac{M}{2\pi\hbar^2}\right)^{3} - 2\sqrt{\pi}\,\hbar\left(\frac{M}{2\pi\hbar^2}\right)^{3/2}\sqrt{-\mathrm{i}\hbar\omega_m-\mu_r(\mathbf{x})},\ m\neq 0. \tag{68}$$

From equation (64) we conclude that $q_0(\mathbf{x}) = \hbar\beta q(\mathbf{x})$ has also to be real, where $q(\mathbf{x})$ satisfies the algebraic equation:

$$q(\mathbf{x}) = \begin{cases} \sqrt{\pi}\,D\left(\frac{M}{2\pi\hbar^2}\right)^{3/2}\frac{n_0(\mathbf{x})}{\sqrt{-\mu_r(\mathbf{x})}}; & \mu_r(\mathbf{x})\leqslant 0,\\ -\sqrt{\pi}\,D\left(\frac{M}{2\pi\hbar^2}\right)^{3/2}\frac{n_0(\mathbf{x})}{\sqrt{-\mu_r(\mathbf{x})}}; & \mu_r^{(\mathrm{crit})}\leqslant\mu_r(\mathbf{x})\leqslant 0,\end{cases} \tag{69}$$

and for $m\neq 0$ we have $q_m(\mathbf{x}) = 0$. At the end of appendix B it is shown that $q(\mathbf{x})$ is a density and this has to be positive, so the negative solution in equation (63) can be rejected. Finally, we obtain

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$$q_m(\mathbf{x}) = \begin{cases} 0; & m \neq 0, \\ \hbar\beta\sqrt{\pi}\,D\left(\dfrac{M}{2\pi\hbar^2}\right)^{3/2} \dfrac{n_0(\mathbf{x})}{\sqrt{-\mu_r(\mathbf{x})}}; & m = 0, \end{cases} \tag{70}$$

Note that, due to the assumed homogeneity in time we had to put $q(\mathbf{x}-\mathbf{x}', \frac{\mathbf{x}+\mathbf{x}'}{2}, \tau-\tau')$ in equation (24) to be time-dependent, but according to equation (70) this quantity turns out to be time-independent.

8.2. Particle density

Taking into account equations (66) and (68), we get from equations (51) and (62) for the particle density:

$$n(\mathbf{x}) = q(\mathbf{x}) + n_0(\mathbf{x}) + \frac{\Delta Q_0(\mathbf{x})}{\hbar\beta}$$
$$-\frac{2\sqrt{\pi}}{\beta}\left(\frac{M}{2\pi\hbar^2}\right)^{3/2} \lim_{\eta\to 0^+} \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega_m\eta}\left[\sqrt{\pi}\,D\left(\frac{M}{2\pi\hbar^2}\right)^{3/2} + \sqrt{-\mathrm{i}\hbar\omega_m - \mu_r(\mathbf{x})}\right], \tag{71}$$

where the following abbreviation has been introduced:

$$\Delta Q_0(\mathbf{x}) = Q_0(\mathbf{x}) - \lim_{m\to 0} Q_m(\mathbf{x}) = \begin{cases} 0\,; & \mu_r(\mathbf{x}) \leqslant 0 \\ 4\sqrt{\pi}\,\hbar\left(\dfrac{M}{2\pi\hbar^2}\right)^{3/2}\sqrt{-\mu_r(\mathbf{x})}\,; & \mu_r^{(\mathrm{crit})} \leqslant \mu_r(\mathbf{x}) \leqslant 0. \end{cases} \tag{72}$$

Then the remaining Matsubara sums (71) are evaluated by using the zeta-function regularization method [54]. The first sum in equation (71) vanishes immediately due to the Poisson formula:

$$\sum_{m=-\infty}^{\infty} \delta(x-m) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{-2\pi\mathrm{i}nx}. \tag{73}$$

In order to calculate the second sum in equation (71), we apply both the Schwinger integral (46) and the Poisson formula (73) to obtain:

$$\lim_{\eta\to 0^+} \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega_m\eta}(-\mathrm{i}\hbar\omega_m + a)^{\nu} = \frac{\zeta_{\nu+1}(\mathrm{e}^{-a\beta})}{\beta^{\nu}\,\Gamma(-\nu)}, \tag{74}$$

with the polylogarithmic function $\zeta_\nu(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\nu}$. Thus, we obtain for the particle density

$$n(\mathbf{x}) = q(\mathbf{x}) + n_0(\mathbf{x}) + \frac{\Delta Q_0(\mathbf{x})}{\hbar\beta} + \left(\frac{M}{2\pi\hbar^2\beta}\right)^{3/2} \zeta_{3/2}(\mathrm{e}^{\beta\mu_r(\mathbf{x})}). \tag{75}$$

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8.3. Free energy

The remaining Matsubara sums in the expression for the thermodynamic potential (61) are evaluated in three dimensions by using, again, the zeta-function regularization method. Taking into account equations (66), (68) and (74) yields

$$\frac{D}{2\hbar^2\beta}\lim_{\eta\to 0^+}\sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega_m\eta}\, Q_m^2(\mathbf{x}) = -2\pi D^2\left(\frac{M}{2\pi\hbar^2}\right)^3\left(\frac{M}{2\pi\hbar^2\beta}\right)^{3/2}\zeta_{3/2}(\mathrm{e}^{\beta\mu_r(\mathbf{x})}) - \frac{2\pi D^2\Delta Q_0(\mathbf{x})}{\hbar\beta}\left(\frac{M}{2\pi\hbar^2}\right)^3 \tag{76}$$

and, correspondingly,

$$-\frac{4\sqrt{\pi}}{3\beta}\left(\frac{M}{2\pi\hbar^2}\right)^{3/2}\lim_{\eta\to 0^+}\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\omega_m\eta}\left[-\mathrm{i}\hbar\omega_m-\mu+2gn(\mathbf{x})+V(\mathbf{x})-\frac{D}{\hbar}Q_m(\mathbf{x})\right]^{3/2} = -\frac{1}{\beta}\left(\frac{M}{2\pi\hbar^2\beta}\right)^{3/2}$$
$$\times\zeta_{5/2}(\mathrm{e}^{\beta\mu_r(\mathbf{x})}) + 2\pi D^2\left(\frac{M}{2\pi\hbar^2}\right)^3\left(\frac{M}{2\pi\hbar^2\beta}\right)^{3/2}\zeta_{3/2}(\mathrm{e}^{\beta\mu_r(\mathbf{x})}) + \frac{2\pi D^2\Delta Q_0(\mathbf{x})}{\hbar\beta}\left(\frac{M}{2\pi\hbar^2}\right)^3 + \frac{\Delta Q_0^3(\mathbf{x})}{24\pi\hbar^3\beta}\left(\frac{M}{2\pi\hbar^2}\right)^{-3}. \tag{77}$$

According to equation (72) we have two solution branches of our mean-field equations for $\mu_r^{(\mathrm{crit})}\leqslant\mu_r(\mathbf{x})\leqslant 0$, one with $\Delta Q_0(\mathbf{x})=0$ and another one with $\Delta Q_0(\mathbf{x})>0$. As the latter solution branch yields a higher thermodynamic potential, we do no longer consider it in the following and restrict ourselves to the case $\Delta Q_0(\mathbf{x})=0$. With this and using the mean-field equation (52), the thermodynamic potential (61) is now given in three dimensions by:

$$\Omega = \int \mathrm{d}\mathbf{x}\Bigg\{-gn^2(\mathbf{x}) + \frac{g}{2}n_0^2(\mathbf{x}) - \frac{1}{\beta}\left(\frac{M}{2\pi\hbar^2\beta}\right)^{3/2}\zeta_{5/2}(\mathrm{e}^{\beta\mu_r(\mathbf{x})}) + \sqrt{n_0(\mathbf{x})}\Bigg\{-gn_0(\mathbf{x})$$
$$+\left[\sqrt{\pi}\,D\left(\frac{M}{2\pi\hbar^2}\right)^{3/2} + \sqrt{-\mu+2gn(\mathbf{x})+V(\mathbf{x})+\pi D^2\left(\frac{M}{2\pi\hbar^2}\right)^3}\,\right]^2 - \frac{\hbar^2}{2M}\boldsymbol{\Delta}\Bigg\}\sqrt{n_0(\mathbf{x})}\Bigg\}. \tag{78}$$

Furthermore, we note that the order parameter $q(\mathbf{x})$ turns out not to explicitly contribute to the thermodynamic potential (78).

8.4. Self-consistency equations

Inserting $\Delta Q_0(\mathbf{x})=0$ in equation (75) we obtain for the particle density $n(\mathbf{x})$ the fundamental decomposition

$$n(\mathbf{x}) = n_0(\mathbf{x}) + q(\mathbf{x}) + n_{\mathrm{th}}(\mathbf{x}). \tag{79}$$

It contains the order parameter of the superfluid $n_0(\mathbf{x})$, which represents the density of the particles in the condensate, the order parameter of the Bose-glass phase $q(\mathbf{x})$, which stands for the density of the particles in the respective minima of the disorder potential and vanishes in absence of disorder, and the thermal component $n_{\mathrm{th}}(\mathbf{x})$ which vanishes in case of zero temperature. Note that both order parameters $n_0(\mathbf{x})$ and $q(\mathbf{x})$ are related to correlation functions, as is elucidated in appendix B. The resulting

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self-consistency equations for $n_0(\mathbf{x})$, $q(\mathbf{x})$, and $n_{\rm th}(\mathbf{x})$ follow from inserting equation (62) into expressions (52), (70) and (75):

$$\left\{-gn_0(\mathbf{x})+\left[\sqrt{-\mu+d^2+2gn(\mathbf{x})+V(\mathbf{x})}+d\right]^2-\frac{\hbar^2}{2M}\Delta\right\}\sqrt{n_0(\mathbf{x})}=0, \tag{80}$$

$$q(\mathbf{x})=\frac{dn_0(\mathbf{x})}{\sqrt{-\mu+d^2+2gn(\mathbf{x})+V(\mathbf{x})}}, \tag{81}$$

$$n_{\rm th}(\mathbf{x})=\left(\frac{M}{2\pi\beta\hbar^2}\right)^{3/2}\varsigma_{3/2}(\mathrm{e}^{\beta\,[\mu-d^2-2gn(\mathbf{x})-V(\mathbf{x})]}), \tag{82}$$

where $d=\sqrt{\pi}\,D(M/2\pi\hbar^2)^{3/2}$characterizes the disorder strength. For physical reasons it is plausible to assume that particles accumulate in the center of the trap. Thus, the differential self-consistency equation (80) has to be solved with the boundary conditions $\frac{\partial n(\mathbf{x})}{\partial \mathbf{x}}|_{\mathbf{x}=0}=0$ and $\frac{\partial n_0(\mathbf{x})}{\partial \mathbf{x}}|_{\mathbf{x}=0}=0$, and the normalization condition

$$N=\int \mathrm{d}\mathbf{x}\, n(\mathbf{x}), \tag{83}$$

In total we have four coupled equations, among them three algebraic equations (79), (81) and (82), and one partial differential equation (80). In the absence of disorder, i.e. $d=0$, the Bose-glass order parameter vanishes and equation (80) reduces to the Hartree–Fock Gross–Pitaevskii equation in the clean case.

Note that those self-consistency equations (79)–(82) can be also obtained in a different way. To this end we rewrite the thermodynamic potential (78) as a function of the chemical potential μ, the condensate density $n_0(\mathbf{x})$, the Bose-glass order parameter $q(\mathbf{x})$ and the thermal density $n_{\rm th}(\mathbf{x})$:

$$\begin{aligned}\Omega=\int \mathrm{d}\mathbf{x}\Bigg\{&-g\,[n_0(\mathbf{x})+q(\mathbf{x})+n_{\rm th}(\mathbf{x})]^2+\frac{g}{2}n_0^2(\mathbf{x})-\frac{1}{\beta}\left(\frac{M}{2\pi\hbar^2\beta}\right)^{3/2}\zeta_{5/2}(\mathrm{e}^{\beta\{\mu-2g[n_0(\mathbf{x})+q(\mathbf{x})+n_{\rm th}(\mathbf{x})]-V(\mathbf{x})+d^2\}})\\&+\sqrt{n_0(\mathbf{x})}\left\{-gn_0(\mathbf{x})+\left[d+\sqrt{-\mu+2g\,[n_0(\mathbf{x})+q(\mathbf{x})+n_{\rm th}(\mathbf{x})]+V(\mathbf{x})-d^2}\right]^2-\frac{\hbar^2}{2M}\mathbf{\Delta}\right\}\sqrt{n_0(\mathbf{x})}\Bigg\}.\end{aligned} \tag{84}$$

Performing a partial derivative with respect to μ and extremising with respect to the condensate density, the Bose-glass order parameter and the thermal density, i.e. $-\frac{\partial\Omega}{\partial\mu}=N, \frac{\delta\Omega}{\delta n_0(\mathbf{x}')}=0, \frac{\delta\Omega}{\delta q(\mathbf{x}')}=0$, and $\frac{\delta\Omega}{\delta n_{\rm th}(\mathbf{x}')}=0$, we reproduce, indeed, equations (79)–(82). Thus, we recognize that in our Hartree–Fock mean-field theory the order parameters can be considered as variational parameters. This allows, in principle, to use a variational solution method based on the principle that, among all possible configurations of a physical system, the one that extremises some specified quantity is realized. This method is used in physics both for theory construction and for calculational purposes (see, for instance, the successful variational perturbation theory worked out in [52, 54–56]).

9. Application of Hartree–Fock mean-field theory in 1D

Now we turn to the one-dimensional case, i.e. $n = 1$, where equations (49)–(52) and (62) reduce to

$$Q_m(x) = \sqrt{\frac{M}{2}}\,\frac{1}{\sqrt{-\mathrm{i}\hbar\omega_m - \mu + 2gn(x) + V(x) - \frac{D}{\hbar}Q_m(x)}}, \tag{85}$$

$$q(x) = \frac{D}{\hbar M}Q_0^3(x)\,\frac{n_0(x)}{1 - \frac{D}{\hbar M}Q_0^3(x)}, \tag{86}$$

the Gross–Pitaevskii equation

$$\left[-\mu + 2gn(x) + V(x) - gn_0(x) - \frac{D}{\hbar}Q_0(x) - \frac{\hbar^2}{2M}\frac{\partial^2}{\partial x^2}\right]\sqrt{n_0(x)} = 0, \tag{87}$$

and the particle density equation

$$n(x) = q(x) + n_0(x) + \lim_{\eta \to 0^+}\sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega_m \eta}\,\frac{Q_m(x)}{\hbar\beta}. \tag{88}$$

Equation (85) represents a cubic equation with respect to $Q_m(x)$:

$$-\frac{D}{\hbar}Q_m^3(x) + \left[-\mathrm{i}\hbar\omega_m - \mu + 2gn(x) + V(x)\right]Q_m^2(x) - \frac{M}{2} = 0, \tag{89}$$

whose solution should be inserted into equations (85)–(89). To this end we have to use the Cardan method [57], which is characterized by a discriminant. For the sake of simplicity, we restrict ourselves to the zero temperature case, where only the $m = 0$ term contributes. In this case the discriminant has a real value and the Cardan method can be applied. According to the sign of the discriminant δ_0 we get the following real solutions for $Q_0(x)$:

$$Q_0(x) = \begin{cases} \sqrt[3]{\dfrac{-p+\sqrt{\delta_0}}{2}} + \sqrt[3]{\dfrac{-p-\sqrt{\delta_0}}{2}} + \dfrac{\hbar}{3D}\left[-\mu + 2gn(x) + V(x)\right]; & \delta_0 > 0, \\ \sqrt[3]{\dfrac{-p+\mathrm{i}\sqrt{-\delta_0}}{2}} + \sqrt[3]{\dfrac{-p-\mathrm{i}\sqrt{-\delta_0}}{2}} + \dfrac{\hbar}{3D}\left[-\mu + 2gn(x) + V(x)\right]; & \delta_0 \leqslant 0, \\ \mathrm{e}^{\pm\frac{2\mathrm{i}\pi}{3}}\sqrt[3]{\dfrac{-p+\mathrm{i}\sqrt{-\delta_0}}{2}} + \mathrm{e}^{\mp\frac{2\mathrm{i}\pi}{3}}\sqrt[3]{\dfrac{-p-\mathrm{i}\sqrt{-\delta_0}}{2}} + \dfrac{\hbar}{3D}\left[-\mu + 2gn(x) + V(x)\right]; & \delta_0 \leqslant 0, \end{cases} \tag{90}$$

with the abbreviation $p = -\frac{2\hbar^3}{27D^3}[-\mu + 2gn(x) + V(x)]^3 + \frac{\hbar M}{2D}$. The correct solution of $Q_0(x)$ has, according to equation (85), to be positive and can only be selected after choosing the form of the trap and by ensuring a minimal free energy.

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At zero temperature equations (85) and (87) remain the same, but equation (88) reduces to:

$$n(x) = q(x) + n_0(x), \tag{91}$$

and the free energy (61) specializes, with (85), to:

$$\Omega = \int dx \Bigg\{ -g\,[n_0(x)+q(x)]^2 - \frac{g}{2} n_0^2(x) - \sqrt{n_0(x)} \left\{ \mu + \frac{\hbar^2}{2M}\frac{\partial^2}{\partial x^2} - 2g\,[n_0(x)+q(x)] - V(x) + \frac{D}{\hbar} Q_0(x) \right\} \sqrt{n_0(x)} \Bigg\}. \tag{92}$$

After inserting equation (91) into equation (90) and then inserting the result into the free energy expression (92), the three self-consistency equations (85), (87) and (91) can be directly obtained by extremising the free energy with respect to its variables $q(x)$, $n_0(x)$ and μ, i.e. $-\frac{\partial\Omega}{\partial\mu} = N$, $\frac{\delta\Omega}{\delta n_0(x')} = 0$ and $\frac{\delta\Omega}{\delta q(x')} = 0$, respectively. So also in one dimension our Hartree–Fock mean-field theory can be based on identifying the order parameters as variational parameters.

10. Conclusions and outlook

In this paper, we developed in detail a Hartree–Fock mean-field theory on the basis of the replica method for a trapped delta-correlated weakly interacting Bose gas in n dimensions at finite temperature. This allowed us to get the free energy as well as the underlying self-consistency equations for the respective components of the particle density. In the end, we applied this theory to one-dimensional and three-dimensional dirty bosons.

On the basis of these self-consistency relations the possible emergence of a Bose-glass region in trapped quasi-1D Bose–Einstein condensed systems in the presence of delta-correlated disorder is analyzed in [58]. Analytical calculations based on the present Hartree–Fock mean-field theory as well as detailed numerical simulations show unambiguously the existence of a Bose-glass region, whose spatial distribution turns out to change with the disorder strength. For small disorder strengths the Bose-glass region emerges at the edge of the atomic cloud, while in the intermediate disorder regime it is located in the trap center. But no quantum phase transition from the superfluid to the Bose-glass phase could be detected neither in the weak nor in the intermediate disorder regime.

The case of tree-dimensional trapped dirty bosons is investigated within the Hartree–Fock mean-field theory in [59], where the existence of a first-order quantum phase transition from the superfluid to the Bose-glass at zero temperature for a harmonically trapped delta-correlated dirty boson is detected at a critical disorder strength, which qualitatively agrees with findings in the literature. At finite temperature the impact of both temperature and disorder fluctuations on the respective components of the density as well as their Thomas–Fermi radii are studied. In particular, we found that a superfluid region, a Bose-glass region, and a thermal region coexist. Furthermore, depending on the respective system parameters, three phase transitions are detected, namely, one from the superfluid to the Bose-glass phase, another one from the Bose-glass to the thermal phase, and, finally, one directly from the superfluid to the thermal phase.

We expect that the seminal results obtained in [58, 59] which follow from the theory worked out in this paper, are useful for a quantitative analysis of ongoing experiments for dirty bosons in quasi one- and three-dimensional harmonic traps. Furthermore, we expect that the UV-divergency encountered in our two-dimensional theory according to section 6 can be eliminated within a proper renormalization program. The resulting self-consistency equations in two dimensions would then be suitable, for instance, to analyze the localization properties of dirty photons in a microcavity [60]. This seems to be insofar a quite challenging research problem as the superfluid to Bose-glass transition could (not) be found in 3D (1D) on the basis of the theory of this paper [58, 59]. Thus, in view of the existence of the Bose-glass phase, the case of trapped dirty photons is marginal.

It should be noted that the replica symmetry can break [61]. For instance, the so-called replica-symmetric solution of the Sherrington–Kirkpatrick was shown to break down below a critical temperature [62, 63]. Therefore, Parisi introduced the scheme of replica-symmetry breaking (RSB) [64–67]. It turns out to yield a stable solution for the Sherrington–Kirkpatrick model for all temperatures. The physical origin of RSB is the existence of many local minima of the complicated free energy, which are separated by high barriers. Practically one has to compare the free energies associated with the RS and RSB solutions and verify whether the free energy of the RSB solution is smaller. If this is the case this proofs that RS is broken. In the case of dirty bosons it still has to be shown whether RSB lowers the free energy or not.

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Appendix A. Disorder potential

Here we introduce the statistical properties of the considered disorder potential $U(\mathbf{x})$ which fluctuates at each space point $\mathbf{x}$ from realization to realization (see figure A1). Such a frozen disorder potential serves, for instance, for modeling superfluid helium in porous media [2–5], where the pores can be modeled by statistically distributed local scatterers. In the following we assume for the disorder potential that it is homogeneous after the disorder ensemble average, i.e. after having performed the average $\overline{\bullet}$ over all possible realizations. Thus, the expectation value of the disorder potential vanishes without loss of generality

$$\overline{U(\mathbf{x})} = 0. \tag{A.1}$$

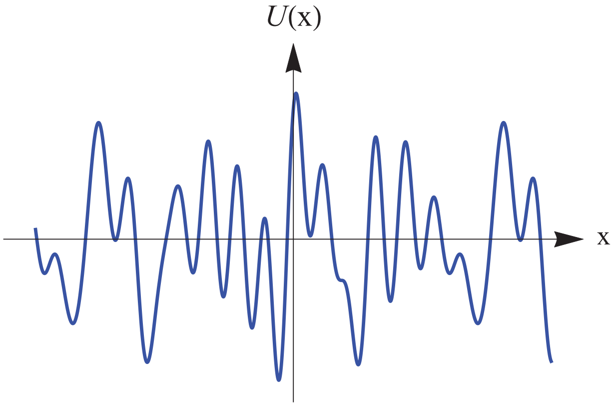

Figure A1. Example for a realization of a frozen disorder potential $U(\mathbf{x})$ with vanishing expectation value (A.1).

Indeed, due to the homogeneity, the disorder ensemble average $\overline{U(\mathbf{x})}$ represents a constant, which can be absorbed without loss of generality into the chemical potential within a grand-canonical description. Furthermore, a homogeneous disorder potential has a correlation function which depends on the difference of the space points:

$$\overline{U(\mathbf{x}_1)U(\mathbf{x}_2)} = D(\mathbf{x}_1 - \mathbf{x}_2). \tag{A.2}$$

In case of a Gaussian correlated disorder in n spatial dimensions we have

$$D(\mathbf{x}_1 - \mathbf{x}_2) = D\,\frac{\mathrm{e}^{-(\mathbf{x}_1-\mathbf{x}_2)^2/2\xi^2}}{(2\pi\xi^2)^{n/2}}, \tag{A.3}$$

where its coherence length ξ can be identified with the average extension of the pores [24]. If one is not interested in a quantitative model for interpreting experimental measurements, one can neglect this spatial extension of the pores. In the limit of a vanishing coherence length ξ we obtain a qualitative model for disordered bosons with a delta correlation:

$$D(\mathbf{x}_1 - \mathbf{x}_2) = D\,\delta(\mathbf{x}_1 - \mathbf{x}_2). \tag{A.4}$$

Here the parameter D is proportional to the density of pores and represents a measure for the disorder strength.

As a next step we consider the probability distribution $P[U]$, which is a functional of the disorder potential $U(\mathbf{x})$. To this end we define expectation values such as (A.1) and (A.2) by the functional integral:

$$\overline{\bullet} = \int \mathcal{D}U \; \bullet \; P[U]. \tag{A.5}$$

Here the functional integral stands for an infinite product of ordinary integrals with respect to all possible values of the disorder potential $U(\mathbf{x})$ at all space points $\mathbf{x}$ [68]:

$$\int \mathcal{D}U = \prod_{\mathbf{x}} \int_{-\infty}^{\infty} \mathrm{d}\,U(\mathbf{x}). \tag{A.6}$$

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The functional measure has to be chosen according to

$$\int \mathcal{D}U\, P[U] = 1, \tag{A.7}$$

so that the probability distribution is normalized: $\langle 1 \rangle = 1$.

Provided that $P[U]$ is Gaussian distributed, it is uniquely fixed by both expectation values (A.1) and (A.2) according to

$$P[U] = \exp\left\{ -\frac{1}{2} \int d\mathbf{x} \int d\mathbf{x}'\, D^{-1}(\mathbf{x}-\mathbf{x}')\, U(\mathbf{x})\, U(\mathbf{x}') \right\}, \tag{A.8}$$

where the integral kernel $D^{-1}(\mathbf{x}-\mathbf{x}')$ represents the functional inverse of the correlation function (A.3):

$$\int d\mathbf{x}\, D^{-1}(\mathbf{x}_1-\mathbf{x}) D(\mathbf{x}-\mathbf{x}_2) = \delta(\mathbf{x}_1-\mathbf{x}_2). \tag{A.9}$$

For instance, we obtain for the δ-correlation (A.4) from (A.9) the integral kernel:

$$D^{-1}(\mathbf{x}_1-\mathbf{x}_2) = \frac{1}{D}\, \delta(\mathbf{x}_1-\mathbf{x}_2). \tag{A.10}$$

We are interested in calculating higher moments of the probability distribution (A.8). To this end we consider the following generating functional

$$I[j] = \overline{\exp\left\{ \int d\mathbf{x}\, j(\mathbf{x})\, U(\mathbf{x}) \right\}}, \tag{A.11}$$

with the auxiliary current field $j(\mathbf{x})$ which represents according to (A.5) and (A.8) a Gaussian functional integral with the result [68]

$$I[j] = \exp\left\{ \frac{1}{2} \int d\mathbf{x} \int d\mathbf{x}'\, D(\mathbf{x}-\mathbf{x}') j(\mathbf{x}) j(\mathbf{x}') \right\}. \tag{A.12}$$

The respective moments of the probability distribution (A.8) follow from successive functional derivatives of the generating functional (A.11) with respect to the auxiliary current field $j(\mathbf{x})$. Indeed, we obtain for the first two moments:

$$\overline{U(\mathbf{x}_1)} = \left. \frac{\delta I[j]}{\delta j(\mathbf{x}_1)} \right|_{j(\mathbf{x})=0}, \tag{A.13}$$

$$\overline{U(\mathbf{x}_1)\, U(\mathbf{x}_2)} = \left. \frac{\delta^2 I[j]}{\delta j(\mathbf{x}_1) \delta j(\mathbf{x}_2)} \right|_{j(\mathbf{x})=0}. \tag{A.14}$$

Inserting (A.12) into (A.13) and (A.14) leads then, indeed, to (A.1) and (A.2). In a similar way also higher correlation functions are evaluated. Whereas the expectation values of all odd products of disorder potentials vanish, those with an even product are evaluated according to the Wick rule. So we obtain, for instance:

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$$\overline{U(\mathbf{x}_1)U(\mathbf{x}_2)U(\mathbf{x}_3)U(\mathbf{x}_4)} = D(\mathbf{x}_1-\mathbf{x}_2)D(\mathbf{x}_3-\mathbf{x}_4)+D(\mathbf{x}_1-\mathbf{x}_3)D(\mathbf{x}_2-\mathbf{x}_4)+D(\mathbf{x}_1-\mathbf{x}_4)D(\mathbf{x}_2-\mathbf{x}_3). \tag{A.15}$$

In the case that the probability distribution $P[U]$ is not Gaussian, its generating functional (A.11) contains more than the second cumulant [69], so we have as a straight-forward generalization of (A.12):

$$I[j] = \exp\left\{\sum_{i=2}^{\infty}\frac{(-1)^{i-1}}{i!}\int \mathrm{d}\mathbf{x}_1\cdots\int \mathrm{d}\mathbf{x}_i\, D^{(i)}(\mathbf{x}_1,\ldots,\mathbf{x}_i)j(\mathbf{x}_1)\cdots j(\mathbf{x}_i)\right\}, \tag{A.16}$$

where $D^{(i)}(\mathbf{x}_1,\ldots,\mathbf{x}_i)$ denotes the ith cumulant. Indeed, equation (A.16) reduces with $D^{(2)}(\mathbf{x}_1,\mathbf{x}_2)=D(\mathbf{x}_1,\mathbf{x}_2)$ and $D^{(i)}(\mathbf{x}_1,\ldots,\mathbf{x}_i)=0$ for $i\geqslant 3$ to equation (A.12).

Appendix B. Correlation functions and order parameters

In the following we fix the physical interpretation of the two order parameters $n_0(\mathbf{x})$ and $q(\mathbf{x})$ that our mean-field theory contains. To this end we follow the notion of classical and quantum spin-glass theory [65, 70, 71] and investigate how these quantities are related to correlation functions.

We start with considering the grand-canonical average of the Bose field:

$$\langle\psi(\mathbf{x},\tau)\rangle = \frac{1}{\mathcal{Z}}\oint\mathcal{D}\psi^*\oint\mathcal{D}\psi\,\psi(\mathbf{x},\tau)\,\mathrm{e}^{-\mathcal{A}[\psi^*,\psi]/\hbar}, \tag{B.1}$$

which represents a functional of the disorder potential $U(\mathbf{x})$ due to the action (2). In order to evaluate its disorder expectation value we apply again the replica method. To this end we identify $\psi(\mathbf{x},\tau)$ with $\psi_\alpha(\mathbf{x},\tau)$ and add further $\mathcal{N}-1$ Bose fields according to:

$$\langle\psi(\mathbf{x},\tau)\rangle = \frac{1}{\mathcal{Z}^{\mathcal{N}}}\left\{\prod_{\alpha'=1}^{\mathcal{N}}\oint\mathcal{D}\psi^*_{\alpha'}\oint\mathcal{D}\psi_{\alpha'}\right\}\psi_\alpha(\mathbf{x},\tau)\,\exp\left\{-\frac{1}{\hbar}\sum_{\alpha'=1}^{\mathcal{N}}\mathcal{A}[\psi^*_{\alpha'},\psi_{\alpha'}]\right\}. \tag{B.2}$$

As the right-hand side is independent of the replica index α, we obtain in the replica limit $\mathcal{N}\to 0$:

$$\langle\psi(\mathbf{x},\tau)\rangle = \lim_{\mathcal{N}\to 0}\frac{1}{\mathcal{N}}\sum_{\alpha=1}^{\mathcal{N}}\left\{\prod_{\alpha'=1}^{\mathcal{N}}\oint\mathcal{D}\psi^*_{\alpha'}\oint\mathcal{D}\psi_{\alpha'}\right\}\psi_\alpha(\mathbf{x},\tau)\,\exp\left\{-\frac{1}{\hbar}\sum_{\alpha'=1}^{\mathcal{N}}\mathcal{A}[\psi^*_{\alpha'},\psi_{\alpha'}]\right\}. \tag{B.3}$$

Now we are in a position to perform the averaging with respect to the disorder potential $U(\mathbf{x})$ by applying again the generating functional (A.16) with the auxiliary current field (6). Thus we obtain the following replica representation of the grand-canonical average of the Bose field:

$$\overline{\langle\psi(\mathbf{x},\tau)\rangle} = \lim_{\mathcal{N}\to 0}\frac{1}{\mathcal{N}}\sum_{\alpha=1}^{\mathcal{N}}\left\{\prod_{\alpha'=1}^{\mathcal{N}}\oint\mathcal{D}\psi^*_{\alpha'}\oint\mathcal{D}\psi_{\alpha'}\right\}\psi_\alpha(\mathbf{x},\tau)\,\mathrm{e}^{-\mathcal{A}^{(\mathcal{N})}[\psi^*,\psi]/\hbar} \tag{B.4}$$

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with the replica action (9) as we restrict ourselves also here to the second cumulant. In a similar way we yield for the two-point function:

$$\overline{\langle\psi(\mathbf{x},\tau)\psi^*(\mathbf{x}',\tau')\rangle} = \lim_{\mathcal{N}\to 0}\frac{1}{\mathcal{N}}\sum_{\alpha=1}^{\mathcal{N}}\left\{\prod_{\alpha'=1}^{\mathcal{N}}\oint\mathcal{D}\psi_{\alpha'}^*\oint\mathcal{D}\psi_{\alpha'}\right\}\psi_\alpha(\mathbf{x},\tau)\,\psi_\alpha^*(\mathbf{x}',\tau')\,\mathrm{e}^{-\mathcal{A}^{(\mathcal{N})}[\psi^*,\psi]/\hbar}. \tag{B.5}$$

In order to further evaluate n-point functions of the form (B.4) and (B.5), we introduce the generating functional:

$$\overline{\mathcal{Z}[j^*,j]} = \left\{\prod_{\alpha=1}^{\mathcal{N}}\oint\mathcal{D}\psi_\alpha^*\oint\mathcal{D}\psi_\alpha\right\}\mathrm{e}^{-\mathcal{A}^{(\mathcal{N})}[\psi^*,\psi;j^*,j]/\hbar}, \tag{B.6}$$

where each Bose field $\psi_\alpha^*(\mathbf{x},\tau),\psi_\alpha(\mathbf{x},\tau)$ is coupled to its own current field $j_\alpha(\mathbf{x},\tau), j_\alpha^*(\mathbf{x},\tau)$ via the action:

$$\mathcal{A}^{(\mathcal{N})}[\psi^*,\psi;j^*,j] = \mathcal{A}^{(\mathcal{N})}[\psi^*,\psi] - \int_0^{\hbar\beta}\mathrm{d}\tau\int\mathrm{d}\mathbf{x}\sum_{\alpha=1}^{\mathcal{N}}\{j_\alpha^*(\mathbf{x},\tau)\psi_\alpha(\mathbf{x},\tau)+\psi_\alpha^*(\mathbf{x},\tau)j_\alpha(\mathbf{x},\tau)\}. \tag{B.7}$$

Indeed, performing successive functional derivatives with respect to the current fields $j_\alpha(\mathbf{x},\tau), j_\alpha^*(\mathbf{x},\tau)$, we obtain the 1- and 2-point function (B.4) and (B.5) from the generating functional (B.6) and (B.7) according to:

$$\overline{\langle\psi(\mathbf{x},\tau)\rangle} = \lim_{\mathcal{N}\to 0}\frac{\hbar}{\mathcal{N}}\sum_{\alpha=1}^{\mathcal{N}}\left.\frac{\delta\,\overline{\mathcal{Z}[j^*,j]}}{\delta j_\alpha^*(\mathbf{x},\tau)}\right|_{j(\mathbf{x},\tau)=0}^{j^*(\mathbf{x},\tau)=0}, \tag{B.8}$$

$$\overline{\langle\psi(\mathbf{x},\tau)\psi^*(\mathbf{x}',\tau')\rangle} = \lim_{\mathcal{N}\to 0}\frac{\hbar^2}{\mathcal{N}}\sum_{\alpha=1}^{\mathcal{N}}\left.\frac{\delta^2\,\overline{\mathcal{Z}[j^*,j]}}{\delta j_\alpha^*(\mathbf{x},\tau)\delta j_\alpha(\mathbf{x}',\tau')}\right|_{j(\mathbf{x},\tau)=0}^{j^*(\mathbf{x},\tau)=0}. \tag{B.9}$$

Thus, it remains to calculate the generating functional $\overline{\mathcal{Z}[j^*,j]}$ within our Hartree–Fock mean-field theory. To this end we perform the background expansions (10) and assume again that the background fields have the replica symmetry form (23), so we have:

$$\overline{\mathcal{Z}[j^*,j]} = \exp\left\{-\beta V_{\mathrm{eff}}^{(N)} + \frac{1}{\hbar}\int_0^{\hbar\beta}\mathrm{d}\tau\int\mathrm{d}\mathbf{x}\sum_{\alpha=1}^{N}[j_\alpha^*(\mathbf{x},\tau)\,\Psi(\mathbf{x})+\Psi^*(\mathbf{x})\,j_\alpha(\mathbf{x},\tau)]\right.$$
$$\left. + \frac{1}{\hbar^2}\int_0^{\hbar\beta}\mathrm{d}\tau\int_0^{\hbar\beta}\mathrm{d}\tau'\int\mathrm{d}\mathbf{x}\int\mathrm{d}\mathbf{x}'\sum_{\alpha=1}^{N}\sum_{\alpha'=1}^{N}j_\alpha^*(\mathbf{x},\tau)\,\langle\delta\psi_\alpha(\mathbf{x},\tau)\delta\psi_{\alpha'}^*(\mathbf{x}',\tau')\rangle j_{\alpha'}(\mathbf{x}',\tau')\right\}. \tag{B.10}$$

Inserting (B.10) into (B.8) and (B.9), yields

$$\overline{\langle\psi(\mathbf{x},\tau)\rangle} = \sqrt{n_0(\mathbf{x})} \tag{B.11}$$

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and, by taking into account (35):

$$\overline{\langle\psi(\mathbf{x},\tau)\psi^*(\mathbf{x}',\tau')\rangle} = \sqrt{n_0(\mathbf{x})n_0(\mathbf{x}')} + g_1\left(\mathbf{x}-\mathbf{x}', \frac{\mathbf{x}+\mathbf{x}'}{2};\tau-\tau'\right) + g_2\left(\mathbf{x}-\mathbf{x}', \frac{\mathbf{x}+\mathbf{x}'}{2};\tau-\tau'\right). \tag{B.12}$$

Now we need just to evaluate the functions $g_1\left(\mathbf{x}-\mathbf{x}', \frac{\mathbf{x}+\mathbf{x}'}{2};\tau-\tau'\right)$ and $g_2\left(\mathbf{x}-\mathbf{x}', \frac{\mathbf{x}+\mathbf{x}'}{2};\tau-\tau'\right)$, respectively. Inserting (42), (43) into (36), (37) and using the Schwinger integral (46), [53, 3.471.9], and [53, 8.469.3], as well as performing the replica limit $\mathcal{N}\to 0$ yields:

$$\begin{aligned} g_1\left(\mathbf{x}-\mathbf{x}', \frac{\mathbf{x}+\mathbf{x}'}{2};\tau-\tau'\right) &= \frac{\sqrt{\pi}}{\beta}\left(\frac{M}{2\pi\hbar^2}\right)^{n/2}\left(\frac{2\hbar^2}{M(\mathbf{x}-\mathbf{x}')^2}\right)^{\frac{n-1}{4}} \\ &\times \sum_{m=-\infty}^{\infty}\frac{1}{\left[-\mathrm{i}\hbar\omega_m + V\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right) - \mu + 2g\Sigma\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right) - \frac{D}{\hbar}Q_m\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right)\right]^{\frac{3-n}{4}}} \\ &\times \exp\left\{-\mathrm{i}\omega_m(\tau-\tau') - \sqrt{\frac{2M}{\hbar^2}\left[-\mathrm{i}\hbar\omega_m + V\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right) - \mu + 2g\Sigma\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right) - \frac{D}{\hbar}Q_m\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right)\right]}\;|\mathbf{x}-\mathbf{x}'|\right\} \end{aligned} \tag{B.13}$$

and

$$\begin{aligned} g_2\left(\mathbf{x}-\mathbf{x}', \frac{\mathbf{x}+\mathbf{x}'}{2};\tau-\tau'\right) &= \sqrt{\pi}\,D\left(\frac{M}{2\pi\hbar^2}\right)^{n/2}\left(\frac{2\hbar^2}{M(\mathbf{x}-\mathbf{x}')^2}\right)^{\frac{n-1}{4}}\left[q\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right) + \Psi^*\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right)\Psi\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right)\right] \\ &\times \frac{\sqrt{\frac{M}{2\hbar^2}\left[V\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right) - \mu + 2g\Sigma\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right) - \frac{D}{\hbar}Q_0\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right)\right]}\;|\mathbf{x}-\mathbf{x}'| + \frac{3-n}{4}}{\left[V\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right) - \mu + 2g\Sigma\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right) - \frac{D}{\hbar}Q_0\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right)\right]^{\frac{7-n}{4}}} \\ &\times \exp\left\{-\sqrt{\frac{2M}{\hbar^2}\left[V\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right) - \mu + 2g\Sigma\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right) - \frac{D}{\hbar}Q_0\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right)\right]}\;|\mathbf{x}-\mathbf{x}'|\right\}, \end{aligned} \tag{B.14}$$

respectively. Note that the function $g_2\left(\mathbf{x}-\mathbf{x}', \frac{\mathbf{x}+\mathbf{x}'}{2};\tau-\tau'\right)$ turns out not to depend on $\tau-\tau'$ at all.

Correspondingly, we determine the disorder average of the 4-point function $|\langle\psi(\mathbf{x},\tau)\psi^*(\mathbf{x}',\tau')\rangle|^2 = \langle\psi(\mathbf{x},\tau)\psi^*(\mathbf{x}',\tau')\rangle\,\langle\psi^*(\mathbf{x},\tau)\psi(\mathbf{x}',\tau')\rangle$, which has the replica representation:

$$\overline{|\langle\psi(\mathbf{x},\tau)\psi^*(\mathbf{x}',\tau')\rangle|^2} = \lim_{N\to 0}\frac{\hbar^4}{\mathcal{N}(\mathcal{N}-1)}\sum_{\alpha\neq\alpha'}\left.\frac{\delta^4\overline{\mathcal{Z}[j^*,j]}}{\delta j^*_\alpha(\mathbf{x},\tau)j_\alpha(\mathbf{x}',\tau')\delta j^*_{\alpha'}(\mathbf{x}',\tau')j_{\alpha'}(\mathbf{x},\tau)}\right|_{j(\mathbf{x},\tau)=0}^{j^*(\mathbf{x},\tau)=0}. \tag{B.15}$$

Inserting the generating functional (B.10) into (B.15) leads to:

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$$\overline{|\langle\psi(\mathbf{x},\tau)\psi^*(\mathbf{x}',\tau')\rangle|^2} = |\overline{\langle\psi(\mathbf{x},\tau)\psi^*(\mathbf{x}',\tau')\rangle}|^2 + n_0(\mathbf{x})\, g_2(\mathbf{0},\mathbf{x}';0) + n_0(\mathbf{x}')\, g_2(\mathbf{0},\mathbf{x};0) + g_2(\mathbf{0},\mathbf{x};0) g_2(\mathbf{0},\mathbf{x}';0). \tag{B.16}$$

Now we are in the position to investigate the 2- and the 4-point function (B.12) and (B.16) for special values of their spatio-temporal arguments. At first, we set $\tau=\tau'$ and study their behavior in the long-range limit $|\mathbf{x}-\mathbf{x}'|\to\infty$. From (B.12) with (B.13) and (B.14) we obtain for the 2-point function:

$$\lim_{|\mathbf{x}-\mathbf{x}'|\to\infty} \overline{\langle\psi(\mathbf{x},\tau)\psi^*(\mathbf{x}',\tau)\rangle} = \sqrt{n_0(\mathbf{x})n_0(\mathbf{x}')}\,. \tag{B.17}$$

We read off from (49) and (B.14) that $q(\mathbf{x}) = g_2(0,\mathbf{x};0)$, so that the 4-point function (B.16) leads to:

$$\lim_{|\mathbf{x}-\mathbf{x}'|\to\infty} \overline{|\langle\psi(\mathbf{x},\tau)\psi^*(\mathbf{x}',\tau)\rangle|^2} = [n_0(\mathbf{x}) + q(\mathbf{x})]\;[n_0(\mathbf{x}') + q(\mathbf{x}')]. \tag{B.18}$$

Following the notion of classical spin-glass theory [65, 70], this result justifies to consider the quantities $n_0(\mathbf{x})$ and $q(\mathbf{x})$ as the order parameters of the condensate and the Bose-glass phase, respectively. However, in analogy to quantum spin-glass theory [71], the Bose-glass order parameter $q(\mathbf{x})$, which has been introduced in [41] in close analogy to the Edward-Anderson order parameter of spin-glasses [71], should also be related to the long-time limit $|\tau-\tau'|\to\infty$ of the 2-point function (B.12) at $T=0$. At $T=0$ the term (B.13) vanishes, whereas (B.14) remains valid as it is temperature independent. By setting $\mathbf{x}=\mathbf{x}'$, we consider the behavior of the 2-point function (B.12) in the long-time limit $|\tau-\tau'|\to\infty$ and read off from (49), (B.12)–(B.14):

$$\lim_{|\tau-\tau'|\to\infty} \overline{\langle\psi(\mathbf{x},\tau)\psi^*(\mathbf{x},\tau')\rangle} = n_0(\mathbf{x}) + q(\mathbf{x}). \tag{B.19}$$

Note, furthermore, that the localization of the Bose-glass states can be inferred from the spatial exponential fall-off of the correlation function $g_2(\mathbf{x}-\mathbf{x}', \frac{\mathbf{x}+\mathbf{x}'}{2}; \tau-\tau')$ describing correlations of the locally condensed component. In the Bose-glass phase equation (49) yields $-\mu + 2g\Sigma(\mathbf{x}) + V(\mathbf{x}) - \frac{D}{\hbar}Q_0(\mathbf{x}) = \left[D\Gamma\left(2-\frac{n}{2}\right)\left(\frac{M}{2\pi\hbar^2}\right)^{n/2}\right]^{\frac{2}{4-n}}$. Inserting this result into the exponential part of function (B.14) allows us to extract for the zero Matsubara mode $m=0$ the temperature-independent Larkin length $\mathcal{L} = \frac{\hbar}{\sqrt{2M}}\left[D\Gamma\left(2-\frac{n}{2}\right)\left(\frac{M}{2\pi\hbar^2}\right)^{n/2}\right]^{\frac{1}{n-4}}$, which is also found in [39–41, 72]. Note that this Larkin length is independent of both the densities and the interaction strength g, since the Hartree–Fock approximation is an effective free-particle theory.

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