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Transport and scattering in inhomogeneous quantum wires

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We consider scattering and transport in interacting quantum wires that are connected to leads. Such a setup can be represented by a minimal model of interacting fermions with sudden changes in interaction strength and/or velocity. The inhomogeneities generally cause relevant backscattering, so it is *a priori* unclear if perfect ballistic transport is possible in the low-temperature limit. We demonstrate that a conducting fixed point surprisingly exists even for large abrupt changes, which in the considered model corresponds to a velocity-matching condition. The general position-dependent Green's function is calculated in the presence of a sudden change, and is confirmed numerically with high accuracy. The exact form of the interference pattern in the form of density oscillations around inhomogeneities can be used to estimate the effective strength of local backscattering sources, offering a route to design experiments where the effects of the contacts are minimized.

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The description of transport through quantum wires has become a very well-studied research area, as it directly ties together conductivity experiments[1–4] with central theoretical models in one dimension, such as the Landauer formalism[5] and Luttinger liquids.[6] Landauer showed that the conductance of a single spinless noninteracting quantum channel is always finite and given by $G = (1 - R^2)e^2/h$, where R is the backscattering amplitude.[5] However, interaction effects change this picture dramatically as R becomes temperature dependent through renormalization. In the pioneering work of Kane and Fisher it was found that a local perturbation in a repulsive Luttinger liquid is relevant, which results in a characteristic power-law dependence $R \propto T^{g-1}$ where $g < 1$ is the interaction parameter.[7] A number of impurity models in one dimension have since been analyzed in detail[8–16]and confirm that generically a local perturbation cuts the transport at low temperatures, unless the relevant operator is forbidden by symmetry.

Nearly perfect connections between leads and wires can be achieved for different types of quantum wires,[1–4] which show quantized conductance at moderate temperatures. At lower temperatures, however, scattering in the connections to the leads plays an increasingly important role. The higher-dimensional contacts are effectively weakly interacting, so there has been great interest in describing the transport through a quantum wire attached to noninteracting leads.[16–28] The minimal model for this setup is a single channel of interacting spinless fermions, where the interaction parameter changes along the wire. Under the assumption that it is possible to use a hydrodynamic Luttinger liquid description for this model,[17] it would be expected that backscattering follows a nontrivial renormalization behavior which is position dependent.[18] Even if the connections are effectively free from imperfections with a homogeneous lattice structure,[4] there have to be small regions of the wire, the junctions, where the interaction changes, which does not necessarily occur adiabatically and will induce intrinsic backscattering. This immediately invites the question if it is ever possible to create a perfect connection in the low-temperature limit. Indeed it is in general unclear how large such intrinsic backscattering is for generic junctions, and if it is even justified to use an inhomogeneous Luttinger-liquid description in the first place for this setup, since the usual bosonization procedure assumes a translational and scale-invariant theory.

In this Rapid Communication we consider the intrinsic backscattering in an inhomogeneous wire and show that a perfectly conducting fixed point can indeed be found by adjusting parameters such as the velocities on both sides. At half filling it is also possible to estimate the bare strength of the backscattering from the change in the local velocity field. The full correlation function of an inhomogeneous Luttinger liquid is calculated and agrees with numerical simulations to high accuracy. This confirms the validity of the inhomogeneous theory and proves that a low-energy conducting fixed point can exist even in the presence of large abrupt jumps.

A hydrodynamic description of interacting fermions with a changing interaction parameter g_x leads to a generalized inhomogeneous Luttinger-liquid action[17,18]

$$S_0 = \int_0^{1/T} d\tau \int dx \frac{u_x}{2g_x}\left[\frac{(\partial_\tau\phi)^2}{u_x^2} + (\partial_x\phi)^2\right], \qquad (1)$$

where ϕ is a canonical bosonic field. Here the effective velocity u_x generally also changes with the interaction strength. We will derive the corresponding correlation function for an abrupt jump below. Additionally, however, backscattering must be considered. In particular, it is known that the Hamiltonian density contains a leading relevant oscillating operator given by[7]

$$e^{-i2k_Fx}\psi_+^\dagger\psi_- \propto e^{-i2k_Fx}e^{-i\sqrt{4\pi}\phi}, \qquad (2)$$

where $\psi_\pm$ are left- and right-moving fermion fields and k_F is the Fermi wavenumber. Normally this operator can be neglected under the integral, but this is no longer the case for inhomogeneous systems. In fact, it is exactly this operator as a local perturbation which causes the renormalization of defects in wires[7] and spin chains.[8] The oscillating part of the interaction results in the same bosonic operator,[8] so that even a change in interaction alone will induce backscattering.

In principle, the conductance depends on the connections of the wire to both of the leads. We will concentrate here on the backscattering in the case where the length of the wire is larger than the coherence length, which is proportional to u/T, so that it suffices to consider each junction separately.[18] A single junction can be characterized by changes taking place in a small region around $x = 0$. The field ϕ_x is assumed to be slowly varying on the scale of the Fermi wavelength so that it is possible to use an expansion of ϕ_x in Eq. (2) and a partial integration to derive an effectively local perturbation

$$H' \approx \lambda e^{-i\sqrt{4\pi}\phi_{x=0}} + \text{H.c.}, \tag{3}$$

while the oscillating operator cancels everywhere in the uniform regions. So far we have kept only the leading relevant operator, but there are higher-order terms which will be discussed later. In general λ is complex, except for particle-hole-symmetric situations. Even relatively smooth junctions become effectively sharper and sharper under renormalization, so that the relevant backscattering is nonzero unless the amplitude λ is adjusted to zero, which requires the fine tuning of two parameters. As we will see later, it is indeed possible to identify such conducting fixed points in a lattice model by an appropriate choice of parameters on both sides of the junction. The existence of a conducting fixed point is also of relevance for the discussion about possible charge fractionalization in Luttinger liquids.[3,29,30] If the Luttinger liquid supports only charges $e(1 \pm g_x)/2$ related to the chiral eigenstates of the Hamiltonian on each side of the wire, then it seems to be impossible that backscattering can be tuned to zero. Our results thus support the analysis in Ref. 30 that at such a junction an *arbitrary* charge can be injected into a Luttinger liquid.

In order to perform a renormalization group analysis we have to consider the full partition function

$$\mathcal{Z} = \int D\phi e^{-S_0 - \int_0^{1/T} d\tau H'} \tag{4}$$

with the quadratic action in Eq. (1). For an abrupt change in the interaction constant at $x = 0$ from $g_{x<0} = g_\ell$ to $g_{x\geqslant 0} = g_r$, and similarly for the velocity u_x the general bosonic Green's function $\tilde{G}(x,x';\tau) = \langle \phi_{x,\tau}\phi_{x',0}\rangle_0$ is found by matching the left and right parts. In particular, it is possible to solve

$$\tilde{G}(x,x';\tau) = T\sum_m e^{i\omega_m\tau}\tilde{G}_m(x,x') \tag{5}$$

with

$$\left[\frac{\omega_m^2}{2g_x u_x} - \frac{\partial}{\partial x}\left(\frac{u_x}{2g_x}\frac{\partial}{\partial x}\right)\right]\tilde{G}_m(x,x') = \delta(x-x') \tag{6}$$

by allowing a discontinuity in the derivative of $\tilde{G}_m$ at $x = x'$.[17] We determine the general Green's function to be

$$\tilde{G}(x,x';\tau) = -\frac{\bar{g}}{\pi}\ln\left|\sinh\left[\pi T\left(\frac{|x|}{u_x} + \frac{|x'|}{u_{x'}} - i\tau\right)\right]\right| + \frac{\mathcal{L}[x,x']g_x}{\pi}\ln\left|\frac{\sinh\left[\pi T\left(\frac{|x|}{u_x} + \frac{|x'|}{u_{x'}} - i\tau\right)\right]}{\sinh\left[\pi T\left(\frac{|x-x'|}{u_x} - i\tau\right)\right]}\right|, \tag{7}$$

where we have defined a new effective interaction parameter $\bar{g} = 2(\frac{1}{g_\ell} + \frac{1}{g_r})^{-1}$. Here $\mathcal{L}[x,x']$ is 1 when x and x' are in the same region, and 0 when they are not. The renormalization of a local perturbation in Eq. (3) can be determined with the help of the Green's function by integrating out the Fourier components above a cutoff Λ,[6,18] which gives

$$\frac{1}{\lambda}\frac{d\lambda}{d\ln\Lambda} = 1 - \bar{g}. \tag{8}$$

We therefore expect that the effective backscattering renormalizes as a power law in the temperature, $R \propto T^{\bar{g}-1}$.

As a concrete lattice model we can consider spinless fermions at half filling,

$$H = \sum_x\left[-t_x(\psi_x^\dagger\psi_{x+1} + \text{H.c.}) + U_x\left(n_x - \frac{1}{2}\right)\left(n_{x+1} - \frac{1}{2}\right)\right], \tag{9}$$

where $n_x = \psi_x^\dagger\psi_x$. The corresponding interaction parameter g and the renormalized velocity u in Eq. (1) are functions of U and t, which are known from the Bethe ansatz.[6] For small jumps and interactions we have estimated the size of λ perturbatively; it turns out to be proportional to the corresponding renormalized velocity field u_x,

$$\lambda \propto \sum_x e^{-i2k_F x}(u_{x+1} - u_x), \tag{10}$$

with $2k_F = \pi$, so to lowest order it does not matter if the velocity change occurs due to inhomogeneous interactions or hopping amplitudes, which may even compensate each other. In particular, Eq. (10) suggests that a conducting fixed point can be achieved for a sharp discontinuity by a velocity matching $u_\ell = u_r$, but the derivation above is valid only for small interactions. The numerical simulations show that this condition holds even in a strongly interacting model. Equation (10) also implies that backscattering can be made arbitrarily small by very slow "adiabatic" changes.

In order to calculate the backscattering amplitude R numerically, we use quantum Monte Carlo simulations[31] on systems with an abrupt junction at $x = 0$, i.e., $U_{x<0} = U_\ell$ and $U_{x\geqslant 0} = U_r$, and analogously for the hopping amplitude t_x. For long system sizes $L \gtrsim 40t_x/T$ the boundary condition at $\pm L/2$ becomes irrelevant. The backscattering R induces a $2k_F$ interference pattern in the density, the so-called Friedel oscillations,[13,16,32] which can be calculated directly in the simulations and also give additional information about the correlation functions. In particular, we consider the local oscillating density in a half-filled lattice in response to changing the chemical potential,

$$\chi_x = \frac{\partial}{\partial\mu}\langle n_x\rangle\Big|_{\mu=0}, \tag{11}$$

where the density is bosonized as

$$n_x = n_0 - \frac{1}{\sqrt{\pi}}\partial_x\phi_x + \text{const}\times\frac{(-1)^x}{\pi}\sin[\sqrt{4\pi}\phi_x], \tag{12}$$

analogously to the local magnetization and susceptibility in spin chains.[8,13,32–35]

The results for the Friedel oscillations $\chi = \chi_0 + (-1)^x\chi_{\text{alt}}$ near an abrupt junction at $x = 0$ are shown in Fig. 1 for

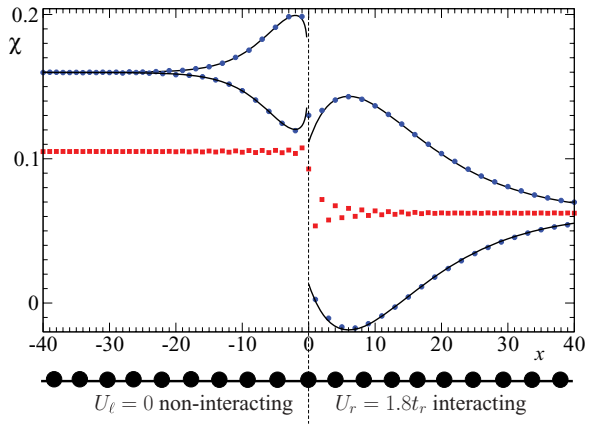

FIG. 1. (Color online) The local response in the density χ for a jump in interaction from $U_\ell = 0$ to $U_r = 1.8t_r$ at $T = 0.1t_r$. Circles (blue): No discontinuity in hopping, $t_\ell = t_r$. Squares (red): The hopping on the left is adjusted to $t_\ell \approx 1.518t_r$ in order to match the velocities on both sides. Solid lines (black) are fits to the predicted behavior in Eq. (14) for even and odd sites separately.

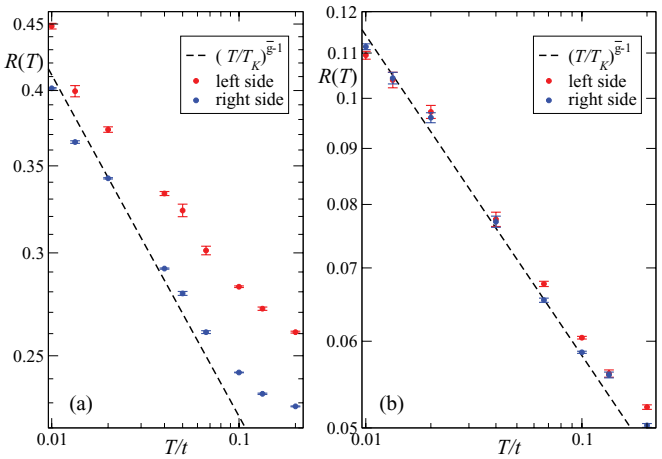

FIG. 2. (Color online) Effective backscattering amplitude $R(T)$ on a logarithmic scale extracted from the amplitude of the density oscillations in Eq. (14) for a jump from (a) $U_\ell = 0$ to $U_r = 1.8t$ ($T_K = 3.4 \times 10^{-4}t$), and (b) $U_\ell = 1t$ to $U_r = 1.4t$ with $t = t_\ell = t_r$ ($T_K = 6 \times 10^{-6}t$). The amplitudes are extracted from fitting the local response for $x < 0$ ("left side") and $x > 0$ ("right side") separately.

a change from $U_\ell = 0$ to $U_r = 1.8t_r$ at finite temperatures $T = 0.1t_r$ for two cases: In the case that the interaction strength changes but the hopping is equal $t_\ell = t_r$, we have $u_\ell < u_r$ and strong alternating $2k_F = \pi$ oscillations are observed (circles). In the second case, the hopping t_ℓ on the left was also increased in order to exactly match the velocities $2t_\ell = u_\ell = u_r \approx 3.036t_r$ (squares). Clearly, the backscattering oscillation is strongly suppressed with a different position dependence, but it is not zero. As the numerical data will show, it turns out that the *relevant* backscattering term is exactly zero in this case. The uniform parts of χ_x on the two sides are constant and approximately given by the zero-temperature result $\chi_0 = \frac{g_x}{\pi u_x}$, which are not equal in either case. We have also tested other cases with more complicated changes in hopping and interactions over three sites in order to tune $\lambda = 0$ in Eq. (10), and in all cases backscattering is strongly suppressed.

Let us first analyze the position dependence of the Friedel amplitude in the case of unequal velocities, which shows a characteristic maximum in Fig. 1 reminiscent of the local susceptibility near ends in spin chains.[32,35] However, we will show that the behavior is *not* exactly of the same form as for scattering from open ends as was conjectured before.[13] In order to calculate the alternating response in the presence of the perturbation (3) with small $|\lambda| \propto |u_r - u_\ell|$ we use the bosonized form of the density (12),

$$\chi_{\rm alt} = \lambda \int_0^{1/T} d\tau \frac{\partial}{\partial \mu} \langle \sin \sqrt{4\pi}\phi_{x,\tau} \cos \sqrt{4\pi}\phi_{0,0}\rangle_0, \qquad (13)$$

where the dependence on μ is given by a shift of the field $\partial_x\phi$ by $\frac{\mu g_x}{\sqrt{\pi}u_x}$ which turns the sine dependence into a cosine dependence with an additional factor of x. Using the Green's function in Eq. (7), we arrive at integrals of the form $\int_0^1 d\tau |\sinh(X - i\pi\tau)|^{-2g} = 2^g (\sinh 2X)^{-g} P_{-g}(\coth 2X)$ where $P_l(z)$ is the Legendre function. Therefore we find

$$\chi_{\rm alt} \propto \lambda \frac{g_x x}{u_x} \int_0^{1/T} d\tau \langle \cos \sqrt{4\pi}\phi_{x,\tau} \cos \sqrt{4\pi}\phi_{0,0}\rangle_0$$
$$\propto \lambda T^{\bar{g}-1} x \left(\frac{u_x}{T} \sinh\left[\frac{2\pi T x}{u_x}\right]\right)^{-g_x} P_{-\bar{g}}(z), \qquad (14)$$

where $z = \coth[2\pi T x/u_x]$. Compared to the scattering from an open boundary[13,32] there are two notable differences: First of all there is an additional factor in the form of the Legendre function, which changes the shape significantly near the scatterer, but quickly approaches unity for $x \gtrsim u_x/\pi T$. Second, there is an additional temperature-dependent factor proportional to $T^{\bar{g}-1}$ which is in agreement with the renormalization behavior predicted in Eq. (8). As can be seen by the fit in Fig. 1, the behavior (14) describes the numerical data perfectly, where x is taken to be the position from an effective scattering center. We have confirmed the position dependence for many different temperatures and discontinuities. This unambiguously shows that the action in Eq. (1) and the Green's function (7) leading up to Eq. (14) are a reliable description of the problem.

An interesting detail in Fig. 1 is the maximum in the noninteracting region where $U_\ell = 0$ and $g_\ell = 1$, which is absent for an open end but now arises due to the Legendre function with $\bar{g} < 1$. Physically this can be understood as a proximity effect, where the behavior in a range on the noninteracting side is influenced by the collective excitations on the other side.

The temperature-dependent amplitudes of the fits to the position-dependent part in Eq. (14) directly give the backscattering $R(T)$ in Fig. 2, where we have used the amplitude from open ends corresponding to $R = 1$, which is known exactly, as the normalization.[35,36] The data confirm the predicted power-law behavior at low temperatures. Note that the independent fits on both sides give roughly the same $R(T)$. For the larger jump in the interaction $U = 1.8t_r$ there are deviations at higher

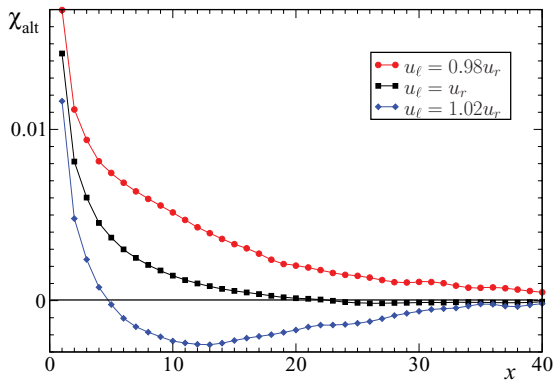

FIG. 3. (Color online) The alternating part of the density oscillations at $T = 0.1t_r$ on the interacting side $U = 1.8t_r$, where the hopping on the noninteracting side is adjusted so that $u_\ell = 0.98u_r$, $u_\ell = u_r$, and $u_\ell = 1.02u_r$, respectively.

temperatures coming from higher-order operators (left panel). Below a characteristic temperature T_K we expect the power law to break down as the stable fixed point $R(0) = 1$ is approached, but this energy scale could not be reached in the simulations.

In order to find a conducting scenario in the low-temperature limit, it is interesting to analyze in more detail the case of equal velocities $u_\ell = u_r$ with strong discontinuities in both hopping and interaction. As shown in Fig. 1 density oscillations are still observed in this case, but we are interested only in the contribution of the leading *relevant* operator in Eq. (3), which will grow while the temperature is lowered and must change sign as a function of velocity at the conducting fixed point. In Fig. 3 we show the alternating part of the oscillations for a jump from $U_\ell = 0$ to $U_r = 1.8t_r$, where the hopping on the left has been adjusted to three different cases: $u_\ell = 0.98u_r$, $u_\ell = u_r$, and $u_\ell = 1.02u_r$. It is quite apparent that a sign change takes place exactly at $u_\ell = u_r$, which means that the relevant backscattering vanishes. The remaining oscillations visible in Fig. 1 for $u_\ell = u_r$ are caused by higher-order local operators in Eq. (3). In particular, the next-leading terms are given by $\partial_x e^{i\sqrt{4\pi}\phi}$ and $e^{i\sqrt{16\pi}\phi}$ with scaling dimension $\bar{g} + 1$ and $4\bar{g}$, respectively, which are irrelevant unless $4\bar{g} < 1$. Away from half filling the marginal operator $\partial_x \phi_{x=0}$ will also be present,[12] but does not affect the scattering to first order. The velocity-matching rule for a conducting fixed point is surprisingly simple, considering that it is not linked to any special symmetry in this scenario. Perfect conduction in quantum wires with impurities has been described before in cases where a renormalization to the periodic boundary condition fixed point occurs.[7] This, however, is not the case here. We find a conducting fixed point described by Eq. (7) which does not correspond to a standard boundary conformally invariant theory.

To conclude, we have studied the intrinsic backscattering present when connecting a quantum wire to a lead. All inhomogeneities and impurities in the junction are in general relevant for repulsive Luttinger liquids, leading to a conductance which scales as $G(T) \propto \frac{e^2}{h}[1 - (T/T_K)^{2\bar{g}-2}]$, with an unusual power-law exponent given by $\bar{g} = 2(\frac{1}{g_\ell} + \frac{1}{g_r})^{-1}$ in terms of the interaction parameters on both sides of the junction g_ℓ, g_r. We have shown that it is possible to achieve a perfect connection, i.e., an absence of relevant backscattering, in inhomogeneous wires by tuning other parameters such as the velocity in the lead and in the wire. The general Green's function was calculated in the presence of an abrupt jump along the wire based on an inhomogeneous Luttinger-liquid action, which was in excellent agreement with numerical simulations. The results suggest that the observation of Friedel oscillations in the density along the wire, which can be attempted by scanning probe microscopy, can be used to analyze local scattering centers. A systematic study of inhomogeneities and thus an experimental test of our results seems feasible in semiconductor heterostructures. One could start from a device where a quantum wire adiabatically broadens into a two-dimensional electron gas.[4] By applying a sharp potential barrier at the interface a local backscattering center could then be realized, allowing it to continuously tune the backscattering amplitude λ.

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