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Entanglement of Hard-Core Bosons on the Honeycomb Lattice

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The entanglement of hard-core bosons in square and honeycomb lattices with nearest-neighbor interactions is estimated by means of quantum Monte Carlo (QMC) simulations and spin-wave (SW) analysis. The particular $U(1)$-invariant form of the concurrence is used to establish a connection with observables, such as density and superfluid density. For specific regimes, the concurrence is expressed as a combination of boson density and superfluid density.

1. Introduction

The past few years have seen a large explosion of interest in the studies of the interfaces between quantum information and many-body systems. Among the subjects of interest can be cited quantum information processing in ultracold atomic gases.[1] This subject was initiated by the first proposal of using ultracold atoms on optical lattices for quantum information.[2] The physics of quantum ultracold gases in optical lattices has rapidly grown in interest.[3] Theoretical studies suggest that ultracold gases may be used for the experimental realization of the phenomenon of a supersolid.[4–11]

In another register, entanglement is an important element in quantum information. It is used in quantum computation[12] and is also a valuable resource in quantum thermodynamics.[13,14] Meanwhile it can characterize quantum phase transitions,[15–23] which are of central interest in a variety of many-body quantum systems, such as models with spins, bosons, or fermions on frustrated lattices,[10,23–27] quasi one-dimensional ladders[28,29] and chains,[16–18,30] and unfrustrated geometries in 2D or higher dimensions.[9,19,20,31–33] In this paper, we will take a closer look at the pairwise entanglement between two sites for the example of hard-core bosons on the square lattice and the honeycomb lattice, for a fundamental understanding to make this measure more useful for a deeper analysis of quantum many body systems. Entanglement is maximal close to the critical points and its derivatives can signal precisely the presence of a quantum phase transition at the critical points.[16–23] In pioneering works, Syljuåsen showed how entanglement can be measured by quantum Monte Carlo (QMC) methods,[34] which has also been applied to the XXZ-model on the square lattice.[35] The purpose of this paper is to show how the entanglement changes on the honeycomb lattice with less connectivity and also provide an independent approximate estimate based on spin wave (SW) theory.

There are various quantities that can be used to extract information on the entanglement, such as the concurrence,[36,37] the entropy of entanglement,[38] and the negativity[39] are mentioned. Quantum discord is a related measure of non-classical correlations.[40,41] While the entropy of entanglement and negativity are bipartite measures, concurrence is a pairwise measure of entanglement which can be conveniently determined by means of QMC simulations.[34] Here we focus our attention on the hard-core boson model with nearest-neighbor interactions in two spatial dimensions, which can be mapped onto a two dimensional spin-1/2 XXZ model.[42,43] By means of QMC simulations and SW analysis, we estimate the entanglement by using concurrence. For a specific region of the phase diagram of the XXZ model, the concurrence takes a very simple $U(1)$-invariant form.[34] This particular $U(1)$-invariant form is used to establish a connection with observables, such as boson density and superfluid density of the hard-core boson system. Concurrence can henceforth be expressed as a combination of boson density and superfluid density.

The outline of the paper is as follows. In section 2, we present the hard-core boson model and its mapping onto the XXZ spin model. In section 3, we recall the elements of information theory which leads to the $U(1)$-invariant form of the concurrence. In section 4, the QMC and SW approaches used to derive the quantum correlations are presented. In section 5, we provide the results of QMC simulations and SW analysis. In section 6, the connection between concurrence and observables is established. In section 7, we conclude and discuss on potential outlooks.

2. The Model

We will consider a two dimensional system of hard-core bosons on the square lattice and the honeycomb lattice. The Hamiltonian is given by

$$H = -t\sum_{\langle ij\rangle}\left(a_i^\dagger a_j + a_j^\dagger a_i\right) - \mu\sum_i \hat{n}_i + V\sum_{\langle ij\rangle}\hat{n}_i\hat{n}_j \tag{1}$$

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DOI: 10.1002/pssb.201800639

where $\langle ij \rangle$ denotes nearest neighbor bonds, a_i $\left(a_i^\dagger\right)$ destroys (creates) a hard-core boson on site i, and μ is the chemical potential. The hopping parameter is denoted by t and the interaction between nearest neighbors is introduced by V.

To enforce the hard-core constraint in a simple way, the Hamiltonian (1) is mapped onto the two dimensional XXZ model with external magnetic field. The exact mapping is performed by $a_i^\dagger \leftrightarrow S_i^+$, $a_i \leftrightarrow S_i^-$, and $\widehat{n}_i \leftrightarrow S_i^z + 1/2$.[42–44] For the particular case with $V/2t = \Delta$ and $\mu = \lambda V$, where λ is half the coordination number z ($z = 3$ for a honeycomb lattice and $z = 4$ for a square lattice), and in units of $t/2$ the Hamiltonian reduces to the familiar XXZ spin model

$$H_{XXZ} = \sum_{\langle i,j \rangle} \left[-\left(\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y \right) + \Delta \sigma_i^z \sigma_j^z \right] + \kappa_\Delta \tag{2}$$

where the sum is taken over all nearest neighbor sites on a lattice which is bipartite. The operators σ^a with $a = x, y, z$ are Pauli matrices and σ^0 is the unit matrix. The constant κ_Δ is equal to $-\Delta z N/8$ and simply arises from the mapping between the spins and bosons operators. N is the number of sublattice spins of the bipartite honeycomb lattice. We keep explicitly the constant κ_Δ present in the Hamiltonian (2) because it will be important for the computation of nearest-neighbor spin–spin correlation functions from the precise value of the bond energy using the Hellmann–Feynman theorem[47,48] in section 4. The Hamiltonian is real and invariant under $U(1)$ rotation about the spin z axis. This continuous symmetry can only be spontaneously broken in dimensions higher than one and for $|\Delta| < 1$. A global Z_2 symmetry about the spin x (or y) is also present.

At the critical point $\Delta_c = 1$ the XXZ spin lattice undergoes a quantum phase transition between an XY phase for $-1 < \Delta < 1$ and an Ising antiferromagnetic phase for $\Delta > 1$. For $\Delta < -1$, the XXZ system is in a ferromagnetic phase.

In order to extract information about entanglement in the system by means of concurrence, we need to build the joint state of two spin sites. The two-site density matrix provides such requirement.

3. Concurrence

The information on the joint state is contained in the two-site density matrix ρ_{ij}, which is derived from the following operator expansion[21]

$$\rho_{ij} = \mathrm{Tr}_{\bar{ij}}[\rho] = \frac{1}{4} \sum_{\alpha,\beta=0}^{3} \Theta_{\alpha\beta} \sigma_i^\alpha \otimes \sigma_j^\beta \tag{3}$$

where the trace is taken over the whole system excluding the sites i and j. The coefficients $\Theta_{\alpha\beta}$ of the expansion are related to the spin–spin correlation functions through the relation

$$\Theta_{\alpha\beta} = \mathrm{Tr}\left[\sigma_i^\alpha \sigma_j^\beta \rho_{ij} \right] = \langle \sigma_i^\alpha \sigma_j^\beta \rangle \tag{4}$$

Owing to the symmetry of the Hamiltonian most of the coefficients $\Theta_{\alpha\beta}$ are equal to zero. Translation invariance requires that the density matrix ρ_{ij} is only a function of distance $r = i - j$ independent of the position i. The reflection symmetry leads to $\rho_{ij} = \rho_{ji}$, the Hamiltonian being real the density matrix verifies $\rho_{ij}^* = \rho_{ij}$. Combining all symmetry constraints the density matrix expressed in the natural basis $\{|\downarrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\uparrow\uparrow\rangle\}$ reduces to

$$\rho_{ij} = \begin{pmatrix} u & g & g & y \\ g & w & x & g \\ g & x & w & g \\ y & g & g & u \end{pmatrix} \tag{5}$$

where the matrix elements are given by $u = \frac{1}{4} + \frac{\langle \sigma_i^z \sigma_j^z \rangle}{4}$, $w = \frac{1 - \langle \sigma_i^z \sigma_j^z \rangle}{4}$, $x = \frac{\langle \sigma_i^x \sigma_j^x \rangle + \langle \sigma_i^y \sigma_j^y \rangle}{4}$, $g = \frac{\langle \sigma^x \rangle}{4}$, and $y = \frac{\langle \sigma_i^x \sigma_j^x \rangle - \langle \sigma_i^y \sigma_j^y \rangle}{4}$. Therefore, information on entanglement of the system can be extracted easily from the two-point correlators.

As mentioned before, a good indicator of entanglement is provided by the concurrence $\mathcal{C}$. The concurrence of two spins may be computed from the joint state ρ_{ij} through the formula $\mathcal{C} = \max\{0, \gamma_1 - \gamma_2 - \gamma_3 - \gamma_4\}$, where γ_i are the eigenvalues in decreasing order of the matrix $R = \sqrt{\rho_{ij} \widetilde{\rho_{ij}}}$,[36,37] where $\widetilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)$. The square root of the eigenvalues of the matrix R are given by

$$\Gamma_\pm = \frac{1}{4} \left| \sqrt{\left(1 + \langle \sigma_i^x \sigma_j^x \rangle\right)^2 - 4\langle \sigma^x \rangle} \pm \left| \langle \sigma_i^y \sigma_j^y \rangle - \langle \sigma_i^z \sigma_j^z \rangle \right| \right|$$

$$\Theta_\pm = \frac{1}{4} \left| 1 - \langle \sigma_i^x \sigma_j^x \rangle \pm \left(\langle \sigma_i^y \sigma_j^y \rangle + \langle \sigma_i^z \sigma_j^z \rangle \right) \right| \tag{6}$$

For two-space dimensions and for $|\Delta| < 1$ the average $\langle \sigma^x \rangle$ takes spontaneous non-zero values, the $U(1)$ symmetry of the XXZ model is spontaneously broken. However, as was argued by Syljuåsen[34] for bi-partite lattices the concurrence in the symmetry-broken state (5) still takes on a $U(1)$ invariant form

$$\mathcal{C} = \frac{1}{2} \left(\langle \sigma_i^x \sigma_j^x \rangle + \langle \sigma_i^y \sigma_j^y \rangle - \langle \sigma_i^z \sigma_j^z \rangle - 1 \right) \tag{7}$$

if and only if the spin–spin correlation functions verify $\langle \sigma_i^y \sigma_j^y \rangle + \langle \sigma_i^z \sigma_j^z \rangle > \langle \sigma_i^x \sigma_j^x \rangle - 1$ and $\langle \sigma_i^y \sigma_j^y \rangle > \langle \sigma_i^z \sigma_j^z \rangle$. While this expression is valid of any two spins, we will restrict our discussion in the following to nearest neighbors only.

The $U(1)$-invariant form of the concurrence is very convenient for the computation by means of QMC simulations. The corresponding correlations can also be estimated by SW theory as shown below. Moreover, the expression in Equation (7) helps to establish a connection between entanglement and observables such as boson density and superfluid density. In particular, we show below that the concurrence can be expressed as a linear combination of boson density and superfluid density.

To work out the correlation functions in the following we will apply both an analytical approach using SW theory and a numerical approach with QMC methods.

4. SW Analysis and QMC Simulations

SW analysis provides a good analytical approach as was shown for the hard-core bosons problem on the square lattice[43] and the honeycomb lattice.[8] A SW analysis can be performed in the regime $-1 < \Delta < 1$ for which the ground state corresponds to spins aligned in any direction within the XY plane. In order to grab fully the particular symmetry of the XY interactions a Haldane mapping is performed[45] and a semi-classical approach can be applied[46] to compute the spin–spin correlation function.

We first recall the transformations applied to the Hamiltonian (2) that lead to a diagonalized Hamiltonian as demonstrated in ref. [46]. The following demonstration is in some steps very similar to the derivation of the non-linear sigma model.[45] We are going to work out the spin–spin correlations function by means of a semi-classical version of the Hamiltonian (2).

First the spin operators are expressed by means of the Haldane mapping. In terms of the in-plane angular coordinate ϕ_i and the spin projection σ_i^z in the z-direction, the spin operators read

$$\boldsymbol{\sigma}_i = \left(\sqrt{1-\sigma_i^{z2}}\cos\phi_i, \sqrt{1-\sigma_i^{z2}}\sin\phi_i, \sigma_i^z\right) \tag{8}$$

With this mapping the Hamiltonian (2) becomes

$$H = \sum_{\langle i,j\rangle}\left(-\sqrt{\left(1-\sigma_i^{z2}\right)\left(1-\sigma_j^{z2}\right)}\cos\left(\phi_i-\phi_j\right) + \Delta\sigma_i^z\sigma_j^z\right) + \kappa_\Delta \tag{9}$$

At zero temperature we can assume a dilute SW boson gas. In this case, the Hamiltonian can reasonably be expanded for small σ^z and ϕ. The expansion of the Hamiltonian up to quadratic terms and to second order in σ^z and ϕ reads

$$H^{(2)} = \sum_{\langle i,j\rangle}\left(-1+\frac{1}{2}\left(\sigma_i^{z2}+\sigma_j^{z2}\right)+\frac{1}{2}\left(\phi_i-\phi_j\right)^2+\Delta\sigma_i^z\sigma_j^z\right)+\kappa_\Delta \tag{10}$$

Higher orders of the expansion are not explicitly considered. Later in the derivation of the spin–spin correlation functions we introduce corrections arising from those neglected higher order terms. After Fourier transformation the Hamiltonian $H^{(2)}$ becomes

$$H^{(2)} = \sum_k\left(\left(1-|\gamma_k|\cos\varphi_k\right)\phi_k\phi_{-k} + \left(1+\Delta|\gamma_k|\cos\varphi_k\right)\sigma_k^z\sigma_{-k}^z\right) + \kappa_\Delta + \kappa_0 \tag{11}$$

where $\kappa_0 = -zN$ collects the constant parts of $H^{(2)}$. We also introduced the structure factor $\gamma_k = \frac{1}{z}\sum_d e^{i\mathbf{k}\cdot\mathbf{r}_d} = |\gamma_k|e^{i\varphi_k}$ which is a complex number for the honeycomb lattice and where the sum runs over nearest neighbors sites. The amplitude of the structure factor for the honeycomb lattice reads

$$|\gamma_k| = \frac{1}{3}\left[3+4\cos\frac{3k_x}{2}\cos\frac{\sqrt{3}k_y}{2}+2\cos\left(\sqrt{3}k_y\right)\right]^{1/2} \tag{12}$$

For square lattices the phase φ_k equals zero and the structure factor is given by $\gamma_k = (\cos k_x + \cos k_y)/2$. We then use the canonical transformation $\phi_k = \alpha_k(b_k^\dagger + b_{-k})$ and $\sigma_k^z = i\beta_k(b_k^\dagger - b_{-k})$ where b_k's are bosons and

$$\alpha_k = \left[\frac{1+\Delta|\gamma_k|\cos\varphi_k}{1-|\gamma_k|\cos\varphi_k}\right]^{1/4},\qquad \beta_k = \left[\frac{1-|\gamma_k|\cos\varphi_k}{1+\Delta|\gamma_k|\cos\varphi_k}\right]^{1/4} \tag{13}$$

Finally, the Hamiltonian takes the diagonalized form

$$H^{(2)} = \kappa_0 + \kappa_\Delta + \sum_k \omega_k\left(b_k^\dagger b_k + 1/2\right) \tag{14}$$

where $\omega_k = 4z\left[\left(1-|\gamma_k|\cos\varphi_k\right)\left(1+\Delta|\gamma_k|\cos\varphi_k\right)\right]^{1/2}$.

In order to compute the nearest-neighbor spin–spin correlations functions, we can employ the Hellmann–Feynman theorem which relates the correlations functions to the bond-energy of the system.[47,48] The bond-energy $e(\Delta)$ is defined as the average of the Hamiltonian H divided by the number of bonds, $e(\Delta) = \langle H\rangle/zN$. The spin–spin correlations functions are easily given by $\langle\sigma_i^z\sigma_j^z\rangle = \frac{\partial e(\Delta)}{\partial\Delta}$ and $\langle\sigma_i^x\sigma_j^x + \sigma_i^y\sigma_j^y\rangle = -e(\Delta) + \Delta\frac{\partial e(\Delta)}{\partial\Delta}$. Taking the average of the second order Hamiltonian $H^{(2)}$ over the ground state leads to the approximated ground state energy $e^{(2)} = \langle H^{(2)}\rangle/zN$. The spin–spin correlations functions can then be expressed in terms of the approximated bond energies

$$\langle\sigma_i^z\sigma_j^z\rangle \simeq \frac{\partial e^{(2)}(\Delta)}{\partial\Delta} + \kappa_{zz} \tag{15}$$

$$\langle\sigma_i^x\sigma_j^x + \sigma_i^y\sigma_j^y\rangle \simeq -e^{(2)}(\Delta) + \Delta\frac{\partial e^{(2)}\Delta}{\partial\Delta} + \kappa_{xy} \tag{16}$$

Non-negligible corrections to the spin-spin correlation functions (15) and (16) may be taken into account from higher order terms of the expansion of the Hamiltonian (9). These corrections, κ_{zz} and κ_{xy}, are functions of the anisotropic parameter Δ. In the region $-1 < \Delta < 1$, we approximate κ_{zz} and κ_{xy} by constants. For the honeycomb lattice we find $\kappa_{zz} \simeq 0.15$ and $\kappa_{xy} \simeq 1.65$, while for the square lattice we obtain $\kappa_{zz} \simeq 0.5$ and $\kappa_{xy} \simeq 2$ from fits to the Monte Carlo simulation results at $\Delta = -1$. Despite the aggressive approximations we have applied here we will see that a reasonable description of the system is obtained.

The QMC simulations used in the present work are based on the stochastic series expansion algorithm.[49–51] The numerical results are obtained for lattices of size $L \times L$ ($L = 12$, 18, and 24 for honeycomb lattices, and $L = 16$, 20, and 24 for square lattices) with periodic boundary conditions and a finite inverse temperature of $\beta t = 50$.

5. Results

Figure 1(a) and (b) provides a comparison between the spin–spin correlation functions derived from QMC simulations and the

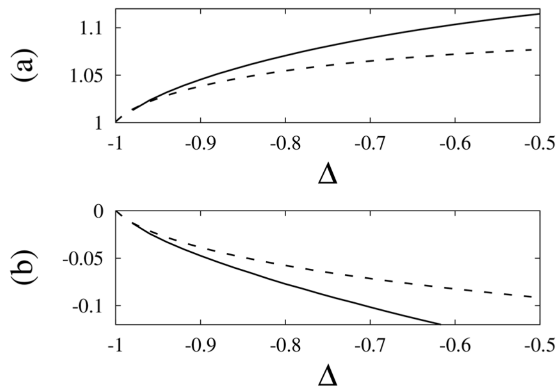

Figure 1. Spin–spin correlation functions for nearest neighbors on the honeycomb lattice, with a) $\langle \sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y \rangle$ and b) $\langle \sigma_i^z \sigma_j^z \rangle$ obtained from Monte Carlo simulation (full line) and SW analysis (dashed line).

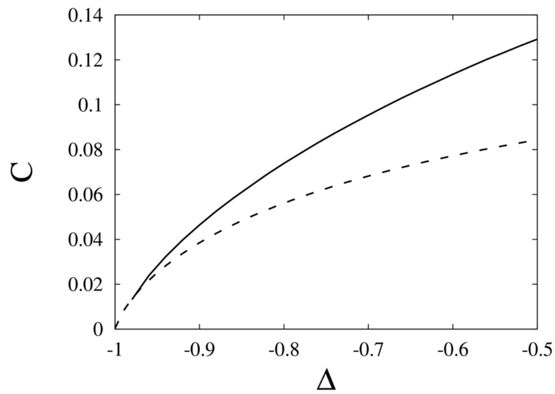

Figure 2. Nearest neighbor concurrence derived from Monte Carlo simulation (full line) and SW analysis (dashed line) on the honeycomb lattice.

SW approach on the honeycomb lattice. For the region $\Delta < 0$, SW theory provides a good approximation of the correlation functions. For larger anisotropic parameter Δ, the deviations increase slowly, since the interaction between SWs become more and more relevant. Note that the correlation function $\langle \sigma^x \sigma^x + \sigma^y \sigma^y \rangle$ is larger than one for $\Delta > -1$. This may surprise at a first glance, however, considering the two inequalities $-1 \leq \langle \sigma^x \sigma^x \rangle \leq 1$ and $-1 \leq \langle \sigma^y \sigma^y \rangle \leq 1$ we expect that $\langle \sigma^x \sigma^x + \sigma^y \sigma^y \rangle$ belongs to the range $[-2, 2]$. In particular, for the spin singlet and triplet $|T_\pm\rangle = (|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle)/\sqrt{2}$ the XY-correlation function reads $\langle T_\pm | \sigma^x \sigma^x + \sigma^y \sigma^y | T_\pm \rangle = \pm 2$. This supports the fact that the absolute value of the XY-correlation function can take values larger than one.

Figure 2 depicts the concurrence obtained from QMC on honeycomb lattices. For $\Delta < -1$, the spin systems is in a ferromagnetic phase. The state of the system can be expressed as a product of separate states, the concurrence is equal to zero and the system is separable. For $\Delta \geq 1$, the system is in an antiferromagnetic phase. For $\Delta = 1$, the concurrence is maximal. Increasing the parameter Δ from the ferromagnetic to the antiferromagnetic phase increases the entanglement of the spin system (or equivalently the hard-core boson system).

The concurrence obtained from SW approach on honeycomb lattices (dashed line in Figure 2) agree with QMC predictions close to the ferromagnetic phase transition, for $\Delta \to -1$. For larger values of Δ, the gas of SW excitations becomes denser. Hence the approximation of a dilute gas no longer holds and our present SW analysis is no longer valid.

Similar results are obtained for hard-core bosons on square lattices as depicted in **Figure 3** and 4, in full agreement with previous calculations.[35]

It is expected that the SW results underestimate the concurrence in both cases, since higher order correlations are ignored. However, interestingly the deviation is stronger for the honeycomb lattice, which also shows a larger concurrence compared to the square lattice. The lower coordination number appears to increase quantum correlations in this case, since the pairwise entanglement has to be divided over a fewer number of neighbors. We find that the SW theory works better for larger coordination number and for smaller values of Δ close to -1.

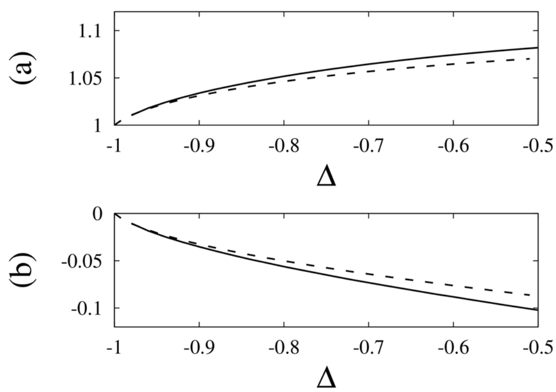

Figure 3. Spin–spin correlation functions for nearest neighbors on the square lattice, a) $\langle \sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y \rangle$ and b) $\langle \sigma_i^z \sigma_j^z \rangle$ obtained from Monte Carlo simulation (full line) and spin-wave analysis (dashed line).

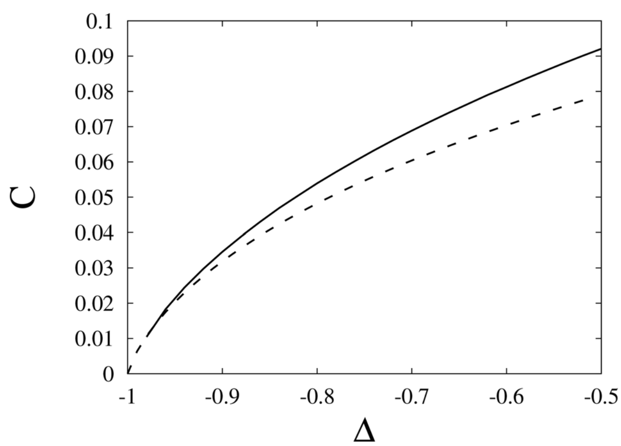

Figure 4. Nearest neighbor concurrence derived from Monte Carlo simulation (full line) and SW analysis (dashed line) on the square lattice.

6. Concurrence, Boson Density, and Superfluidity

According to the mapping relating the hard-core bosons and spins, the $U(1)$-invariant form of the concurrence in Equation (7) can be expressed in terms of the boson density and superfluid density of the hard-core boson system.

Indeed the spin–spin correlation function in the z direction can be easily expressed in terms of the boson density by making use of the exact mapping $\hat{n}_i \leftrightarrow S_i^z + 1/2$ which leads to $\langle S_i^z S_j^z \rangle = 1/4 - \langle \hat{n} \rangle + \langle \hat{n}_i \hat{n}_j \rangle$.

The superfluid density is related to the energy cost to introduce a twist v between pairs of nearest neighbors spins. The superfluid density is given by the second derivative of the energy of the spin system with respect to the twist v, $\rho_s = d^2 \langle H(v) \rangle / d^2 v$.[52,53] The Hamiltonian $H(v)$ is derived from the XXZ model by application of a local rotation at site i by an angle v_i around the z axis, $S_i^+ \rightarrow S_i^+ e^{iv_i}$, $S_i^- \rightarrow S_i^- e^{-iv_i}$, and $S_i^z \rightarrow S_i^z$. Expanding the Hamiltonian around $v_{ij} = v_i - v_j = 0$ leads to $H(v) = H + \sum_{\langle i,j \rangle} (v_{ij} J_{ij}^s + v_{ij}^2 T_{ij}/2)$, where $J_{ij}^s = (i/2) t (S_i^+ S_j^- - S_j^+ S_i^-)$ is the spin current in the z direction and $T_{ij} = (1/2) t (S_i^+ S_j^- + S_j^+ S_i^-)$ is the spin-kinetic energy.[52–54] In first-order perturbation theory, the spin stiffness is given by $\rho_s = \frac{1}{2N} \frac{\partial^2}{\partial v^2} \sum_{ij} v_{ij}^2 \langle T_{ij} \rangle$. Second order perturbation leads to a term integrating the current–current correlator, with respect to J_s, that is neglected in our SW approach. The spin stiffness for a uniform twist v and a given direction leads to $\langle S_i^x S_j^x + S_j^y S_i^y \rangle \simeq 2\rho_s / t$.

Replacing the spin–spin correlation functions by their linear expressions with respect to the boson density and superfluid density the concurrence reads

$$\mathcal{C} \simeq \max\{0, K_0 + K_b \rho_b + K_s \rho_s + K_{cor,b} \langle \rho_b \rho_b \rangle\} \qquad (17)$$

where $K_0 = -1$, $K_b = K_{cor,b} = 2$, and $K_s = 4/t$ are constants. This expression provides a direct way to approximately measure the entanglement between two hard-core bosons experimentally.

7. Conclusion

In conclusion, we proposed a SW analysis as well as numerical approach to estimate the entanglement in a hard-core boson model on the honeycomb lattice and the square lattice. By means of the particular $U(1)$-invariant form that the concurrence takes we compare SW theory and QMC measures of entanglement. Moreover, we also show the existence of an approximate linear relation between concurrence, boson density, and superfluid density. This relation may be used in experimental measurements for direct evaluation of the entanglement in hard-core boson system.

Our results show that the numerical data generally shows a larger increase of entanglement with anisotropy parameter Δ than would be predicted from the SW approximation. Compared to the square lattice this increase is even larger for the honeycomb lattice, which has the lowest possible connectivity in two dimensions, and therefore stronger pairwise entanglement.

It has to be mentioned that the present demonstration is generic and may be easily applied to different lattice symmetries. The $U(1)$-form of the concurrence is sensitive to the lattice symmetry only through the spin-spin correlation functions. The linear relation of the concurrence with observables should also remain valid.

Acknowledgements

This work was supported by the DFG via the Research Center Transregio 49. R.D. would like to thank A. Metavitsiadis for the helpful discussions on the present subject.

Conflict of Interest

The authors declare no conflict of interest.

Keywords

boson density, concurrence, entanglement, hard-core bosons, quantum observables, spin wave theory, superfluid density

Received: October 31, 2018
Revised: June 11, 2019
Published online: July 31, 2019

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