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Bogoliubov theory of dipolar Bose gas in a weak random potential

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We consider a dilute homogeneous Bose gas with both an isotropic short-range contact interaction and an anisotropic long-range dipole-dipole interaction in a weak random potential at low temperature in three dimensions. Within the realm of Bogoliubov theory, we analyze how both condensate and superfluid are depleted due to quantum and thermal fluctuations as well as disorder fluctuations.

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I. INTRODUCTION

Since the first observation of Bose-Einstein condensation (BEC) in 1995 for alkali atomic vapors of ^{87}Rb and ^{23}Na atoms, both theoretical and experimental research have accelerated to study this newly discovered macroscopic quantum phenomenon, where a fraction of these bosons occupies the same quantum mechanical ground state. Achieving BEC in experiment for alkali atoms was only possible due to the discovery of efficient cooling and trapping techniques which became available by the end of the 20th century [1–3]. With more cooling advances, research interest has not only increased in the field of ultracold quantum gases with isotropic short-range contact interaction, which represents the effective interaction between atoms and is dominated by the s-wave scattering length. Also, the anisotropic long-range dipole-dipole interaction has been made accessible to detailed study by the formation of a BEC in a dipolar quantum gas of ^{52}Cr atoms [4,5]. Further atomic BECs with magnetic dipole-dipole interaction followed soon with ^{164}Dy [6] and ^{168}Er [7] atoms. Strong electric dipole-dipole interactions were realized by a stimulated Raman adiabatic passage (STIRAP) experiment, which allowed the creation of ^{40}K ^{87}Rb molecules in the rovibrational ground state [8,9].

The research field of ultracold quantum gases allows us today to analyze many condensed matter problems in the spirit of Feynman's quantum simulator [10], as all ingredients of the quantum many-body Hamiltonian can be experimentally tuned with an unprecedented precision. The kinetic energy can be controlled with spin-orbit coupling [11,12], the potential energy can be harmonic [13], boxlike [14], or even anharmonic [15], and also the strength of contact as well as dipolar interaction are tunable [16–18]. In addition, in order to make quantum gas simulators even more realistic, various experimental and theoretical methods for controlling the effect of disorder have been designed. Using superfluid helium in vycor glass, which represents some kind of porous media, the random distribution of pores represents a random environment [19]. Laser speckles are produced by focusing the laser beam on a glass plate, where the resulting random interference pattern is reflected to a BEC cell [20–23]. Wire traps represent magnetic traps on atomic chips where the roughness and the imperfection of the wire surface generates a disorder potential [24,25]. Another possibility to create a random potential is to trap one species of atoms randomly in a deep optical lattice, which serves as frozen scatterers for a second atomic species [26,27]. In addition also, incommensurable lattices provide a useful random environment [28–30].

In order to study the properties of interacting bosons in such a random potential, Huang and Meng proposed a Bogoliubov theory, which was applied to the case of superfluid helium in porous media [31], and extended later by others [32–35]. For a delta-correlated disorder, it was found that both a condensate and a superfluid depletion occur due to the localization of bosons in the respective minima of the external random potential which is present even at zero temperature. A generalization to the corresponding situation, where the disorder correlation function falls off with a characteristic correlation length as, for instance, a Gauss function [36,37], laser speckles [38], or a Lorentzian [39] is straightforward.

In this paper, we extend the Bogoliubov theory of Huang and Meng [31] to a dipolar Bose gas at finite temperature. With this we go beyond the zero-temperature mean-field approach where the Gross-Pitaevskii equation is solved perturbativly with respect to the random potential [37,39]. This extension to the finite-temperature regime allows us to study in detail how the anisotropy of superfluidity can be tuned, a phenomenon which also occurs at zero temperature [37,39], but turns out to be enhanced by the thermal fluctuations. We observe in addition that contact and dipolar interaction have different effects upon quantum, thermal, and disorder depletion of condensate and superfluid.

The paper is organized as follows. In Sec. II, we revisit the Bogoliubov theory of the homogeneous dirty boson problem. With this, we determine for a general two-particle interaction and a general disorder correlation for different observables the beyond mean-field corrections which stem from quantum, thermal, and disorder fluctuations. In Sec. III, we specialize our treatment for a dipolar Bose gas and a delta-correlated disorder in the zero temperature and the thermodynamic limit. In particular, we investigate the particle and condensate density as well as the inner energy. In Sec. IV, we extend the Bogoliubov theory in order to derive the superfluid depletion which turns out to be only due to the external random potential and the thermal excitation. In Sec. V, we consider how finite-temperature effects on the condensate depletion as well as the normal fluid component depend on the respective strength of contact and dipolar interaction.

II. BOGOLIUBOV THEORY

A three-dimensional ultracold dipolar Bose gas in a weak random potential is modeled by the grand-canonical Hamiltonian

$$\hat{\mathcal{K}} = \int d^3\mathbf{x}\,\hat{\psi}^\dagger(\mathbf{x})\left[\frac{-\hbar^2\nabla^2}{2m} - \mu + U(\mathbf{x})\right]\hat{\psi}(\mathbf{x}) + \frac{1}{2}\int d^3\mathbf{x}\int d^3\mathbf{x}'\,\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{x}')V(\mathbf{x},\mathbf{x}')\hat{\psi}(\mathbf{x}')\hat{\psi}(\mathbf{x}), \quad (1)$$

where $\hat{\psi}(\mathbf{x})$ and $\hat{\psi}^\dagger(\mathbf{x})$ are the usual field operators for Bose particles of mass m, which satisfy the following commutation relations:

$$[\hat{\psi}(\mathbf{x}),\hat{\psi}^\dagger(\mathbf{x}')] = \delta(\mathbf{x}-\mathbf{x}'),$$
$$[\hat{\psi}(\mathbf{x}),\hat{\psi}(\mathbf{x}')] = [\hat{\psi}^\dagger(\mathbf{x}),\hat{\psi}^\dagger(\mathbf{x}')] = 0. \quad (2)$$

Here, μ denotes the chemical potential and $U(\mathbf{x})$ represents the random potential. Irrespective of the physical origin of the disorder potential, we assume that the average over the random potential vanishes and that it has some kind of correlation, i.e., $\langle U(\mathbf{x})\rangle = 0$ and $\langle U(\mathbf{x})U(\mathbf{x}')\rangle = R(\mathbf{x}-\mathbf{x}')$. Furthermore, the two-particle interaction potential for a dipolar Bose gas consists of two different parts: $V(\mathbf{x},\mathbf{x}') = V_\delta(\mathbf{x}-\mathbf{x}') + V_{\rm dd}(\mathbf{x}-\mathbf{x}')$. On the one hand, the short-range isotropic contact interaction is given by $V_\delta(\mathbf{x}) = g\delta(\mathbf{x})$, where $g = 4\pi a\hbar^2/m$ represents its strength with the s-wave scattering length a. From now on, we assume a positive a, i.e., a repulsive contact interaction, which depletes the particles from the ground state due to quantum and thermal fluctuations and also forbids them from being localized in the minima of the external random potential $U(\mathbf{x})$. Thus, unlike the case of free fermions, where Pauli blocking prevents the particles from localizing in a single orbital, bosons require a repulsive interaction to prevent them from collapsing into the respective minima of the external random potential. On the other hand, the long-range anisotropic polarized dipolar part is written in real space for dipoles aligned along the z-axis direction according to

$$V_{\rm dd}(\mathbf{x}) = \frac{C_{\rm dd}}{4\pi}\,\frac{(x^2+y^2+z^2)-3z^2}{(x^2+y^2+z^2)^{5/2}}, \quad (3)$$

where $C_{\rm dd}$ represents the dipolar interaction strength due to magnetic or electric dipole moments. In the first case, we have $C_{\rm dd} = \mu_0 d_{\rm m}^2$, with the magnetic dipole moment $d_{\rm m}$ and the magnetic permeability of vacuum μ_0, and in the latter case we have $C_{\rm dd} = d_{\rm e}^2/\epsilon_0$, with the electric dipole moment $d_{\rm e}$ and the vacuum dielectric constant ϵ_0. Note that, due to the anisotropic character of the dipolar interaction, many static and dynamic properties of a dipolar Bose gas become tunable [4,5,40,41], consequences for a random environment have only recently been explored [37,39].

With the help of a Fourier transformation we can rewrite the Hamiltonian (1) in momentum space according to

$$\hat{\mathcal{K}} = \sum_{\mathbf{k}}\left(\frac{\hbar^2\mathbf{k}^2}{2m} - \mu\right)\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \frac{1}{v}\sum_{\mathbf{p},\mathbf{k}} U_{\mathbf{p}-\mathbf{k}}\,\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{k}} + \frac{1}{2v}\sum_{\mathbf{p},\mathbf{k},\mathbf{q}} V_{\mathbf{q}}\,\hat{a}^\dagger_{\mathbf{k}+\mathbf{q}}\hat{a}^\dagger_{\mathbf{p}-\mathbf{q}}\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{k}}, \quad (4)$$

where v denotes the volume and the interaction potential in momentum space is given by [17]

$$V_{\mathbf{q}} = g + \frac{C_{\rm dd}}{3}(3\cos^2\theta - 1). \quad (5)$$

Here, θ denotes the angle between the polarization direction, which is here along the z axis, and the wave vector $\mathbf{q}$. Note that (5) is not continuous at $\mathbf{q} = \mathbf{0}$, as the limit $\mathbf{q}\to\mathbf{0}$ is direction dependent. This is the origin for various anisotropic properties, which are characteristic for dipolar Bose gases [37,39,42,43]. The operators $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$ are the annihilation and creation operators in Fourier space, respectively, which turn out to satisfy the bosonic commutation relations

$$[\hat{a}_{\mathbf{k}},\hat{a}^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k}\mathbf{k}'}, \quad [\hat{a}_{\mathbf{k}},\hat{a}_{\mathbf{k}'}] = [\hat{a}^\dagger_{\mathbf{k}},\hat{a}^\dagger_{\mathbf{k}'}] = 0. \quad (6)$$

Near absolute zero temperature, the number of the particles $N_{\mathbf{0}}$ in the ground state $|\Phi_{\mathbf{0}}\rangle$ become macroscopically large. In this case, we have due to (6) $\hat{a}_{\mathbf{0}}|\Phi_{\mathbf{0}}\rangle \approx \sqrt{N_{\mathbf{0}}}|\Phi_{\mathbf{0}}\rangle$ and $\hat{a}^\dagger_{\mathbf{0}}|\Phi_{\mathbf{0}}\rangle \approx \sqrt{N_{\mathbf{0}}}|\Phi_{\mathbf{0}}\rangle$, so the operators $\hat{a}_{\mathbf{0}}$ and $\hat{a}^\dagger_{\mathbf{0}}$ approximately commute with each other. Thus, we can apply the Bogoliubov prescription [44] and replace the creation and annihilation operators of the ground state $\mathbf{k} = \mathbf{0}$ by a c-number, i.e., $\hat{a}_{\mathbf{0}} = \hat{a}^\dagger_{\mathbf{0}} = \sqrt{N_{\mathbf{0}}}$. As a consequence, we have to decompose the respective momentum summations in the Hamiltonian (4) into their ground state $\mathbf{k} = \mathbf{0}$ and excited states $\mathbf{k} \neq \mathbf{0}$ contributions. By doing so, we perform the following two physical approximations. At first, we ignore terms which contain creation and annihilation operators of the excited states $\mathbf{k}\neq\mathbf{0}$, which are of third and fourth order, as they represent higher-order interactions of the particles out of the condensate. Such an approximation is justified in the case of weakly interacting systems. Second, we assume for weak enough disorder that disorder fluctuations decouple in lowest order. Therefore, we ignore the terms $U_{\mathbf{p}-\mathbf{k}}\,\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{k}}$ with both $\mathbf{k}\neq\mathbf{0}$ and $\mathbf{p}\neq\mathbf{0}$. Note that these two physical approximations imply that the excited states are only rarely occupied, i.e., $N - N_{\mathbf{0}} \ll N$. With this we get the simplified Hamiltonian

$$\begin{aligned}\hat{\mathcal{K}}' = {}& \left(-\mu + \frac{1}{v}U_0\right)N_{\mathbf{0}} + \frac{1}{2v}V_{\mathbf{0}}N_{\mathbf{0}}^2 \\ &+ \frac{1}{2}{\sum_{\mathbf{k}}}'\left(\frac{\hbar^2\mathbf{k}^2}{2m} - \mu\right)(\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{k}}) \\ &+ \frac{1}{v}\sqrt{N_{\mathbf{0}}}{\sum_{\mathbf{k}}}' U_{\mathbf{k},\mathbf{0}}\,(\hat{a}^\dagger_{\mathbf{k}} + \hat{a}_{-\mathbf{k}}) \\ &+ \frac{1}{2v}N_{\mathbf{0}}{\sum_{\mathbf{k}}}'(V_{\mathbf{0}} + V_{\mathbf{k}})\,(\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{k}}) \\ &+ \frac{1}{2v}N_{\mathbf{0}}{\sum_{\mathbf{k}}}' V_{\mathbf{k}}\,(\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} + \hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}}).\end{aligned} \quad (7)$$

Here, the prime over the summation symbol indicates that the ground state $\mathbf{k} = \mathbf{0}$ is excluded. The appearance of off-diagonal terms $\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}$ with $\mathbf{k}\neq\mathbf{0}$ in Eq. (7) indicates that the state $\mathbf{k} = \mathbf{0}$ is no longer the ground state of the interacting system. In order to determine this new mean-field ground state, we have to diagonalize the Hamiltonian (7). To this end, we follow Ref. [31] and use the inhomogeneous Bogoliubov transformation

$$\hat{a}_{\mathbf{k}} = u_{\mathbf{k}}\hat{\alpha}_{\mathbf{k}} - v_{\mathbf{k}}\hat{\alpha}^{\dagger}_{-\mathbf{k}} - z_{\mathbf{k}}, \quad \hat{a}^{\dagger}_{\mathbf{k}} = u_{\mathbf{k}}\hat{\alpha}^{\dagger}_{\mathbf{k}} - v_{\mathbf{k}}\hat{\alpha}_{-\mathbf{k}} - z^{*}_{\mathbf{k}}. \tag{8}$$

Note that the Bogoliubov amplitudes $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ can be chosen to be real without loss of generality. Furthermore, we impose upon the new operators $\hat{\alpha}_{\mathbf{k}}$, $\hat{\alpha}^{\dagger}_{\mathbf{k}}$ that they also satisfy bosonic commutation relations

$$[\hat{\alpha}_{\mathbf{k}},\hat{\alpha}^{\dagger}_{\mathbf{k}'}] = \delta_{\mathbf{k}\mathbf{k}'}, \quad [\hat{\alpha}_{\mathbf{k}},\hat{\alpha}_{\mathbf{k}'}] = [\hat{\alpha}^{\dagger}_{\mathbf{k}},\hat{\alpha}^{\dagger}_{\mathbf{k}'}] = 0. \tag{9}$$

Inserting (8) in the simplified Hamiltonian (7) we obtain via diagonalization the following results for the Bogoliubov parameters:

$$u^2_{\mathbf{k}} = \frac{1}{2}\left[\frac{\frac{\hbar^2\mathbf{k}^2}{2m} - \mu + n_0(V_0 + V_{\mathbf{k}})}{E_{\mathbf{k}}} + 1\right],$$

$$v^2_{\mathbf{k}} = \frac{1}{2}\left[\frac{\frac{\hbar^2\mathbf{k}^2}{2m} - \mu + n_0(V_0 + V_{\mathbf{k}})}{E_{\mathbf{k}}} - 1\right], \tag{10}$$

and the translations

$$z_{\mathbf{k}} = \frac{\frac{1}{v}\sqrt{N_0}U_{\mathbf{k}}\left(\frac{\hbar^2\mathbf{k}^2}{2m} - \mu + n_0 V_0\right)}{E^2_{\mathbf{k}}}. \tag{11}$$

Here, we have introduced for brevity the condensate density $n_0 = N_0/v$ and the quasiparticle dispersion

$$E_{\mathbf{k}} = \sqrt{\left[\frac{\hbar^2\mathbf{k}^2}{2m} - \mu + n_0\,(V_0 + V_{\mathbf{k}})\right]^2 - (n_0 V_{\mathbf{k}})^2}. \tag{12}$$

Note that the diagonalized Hamiltonian changes with each realization of the disorder potential $U(\mathbf{x})$. Therefore, we get the final Hamiltonian of the dirty dipolar Bose system by performing the disorder ensemble average

$$\begin{aligned}\langle\hat{\mathcal{K}}'\rangle = {}& v\left(-\mu n_0 + \frac{1}{2}V_0 n_0^2\right)\\ &+ \frac{1}{2}\sum_{\mathbf{k}}{}'\left\{E_{\mathbf{k}} - \left[\frac{\hbar^2\mathbf{k}^2}{2m} - \mu + n_0(V_0 + V_{\mathbf{k}})\right]\right\}\\ &+ \frac{1}{2}\sum_{\mathbf{k}}{}' E_{\mathbf{k}}(\hat{\alpha}^{\dagger}_{\mathbf{k}}\hat{\alpha}_{\mathbf{k}} + \hat{\alpha}^{\dagger}_{-\mathbf{k}}\hat{\alpha}_{-\mathbf{k}})\\ &- \sum_{\mathbf{k}}{}'\frac{n_0 R_{\mathbf{k}}}{E^2_{\mathbf{k}}}\left(\frac{\hbar^2\mathbf{k}^2}{2m} - \mu + n_0 V_0\right).\end{aligned} \tag{13}$$

With this it is straightforward to determine the corresponding grand-canonical potential $\Omega_{\text{eff}} = -\beta^{-1}\ln\mathcal{Z}_G$, where $\mathcal{Z}_G = \operatorname{Tr} e^{-\beta\langle\hat{\mathcal{K}}'\rangle}$ denotes the grand-canonical partition function and $\beta = 1/(k_B T)$ is the reciprocal temperature:

$$\begin{aligned}\Omega_{\text{eff}} = {}& v\left(-\mu n_0 + \frac{1}{2}V_0 n_0^2\right)\\ &+ \frac{1}{2}\sum_{\mathbf{k}}{}'\left\{E_{\mathbf{k}} - \left[\frac{\hbar^2\mathbf{k}^2}{2m} - \mu + n_0(V_0 + V_{\mathbf{k}})\right]\right\}\\ &+ \sum_{\mathbf{k}}{}'\frac{1}{\beta}\ln(1 - e^{-\beta E_{\mathbf{k}}})\\ &- \sum_{\mathbf{k}}{}'\frac{n_0 R_{\mathbf{k}}}{E^2_{\mathbf{k}}}\left(\frac{\hbar^2\mathbf{k}^2}{2m} - \mu + n_0 V_0\right).\end{aligned} \tag{14}$$

Extremizing Eq. (14) with respect to the condensate density n_0 for fixed chemical potential μ we find up to first order in quantum, thermal, and disorder fluctuations

$$\begin{aligned}n_0 = {}& \frac{\mu}{V_0} - \frac{1}{2v}\sum_{\mathbf{k}}{}'\left\{\frac{\left(\frac{\hbar^2\mathbf{k}^2}{2m} - \mu\right)(V_0 + V_{\mathbf{k}})}{V_0 E_{\mathbf{k}}}\right.\\ &\left. + \frac{\frac{\mu}{V_0}\left(V_0^2 + 2V_0 V_{\mathbf{k}}\right) - E_{\mathbf{k}}(V_0 + V_{\mathbf{k}})}{V_0 E_{\mathbf{k}}}\right\}\\ &- \frac{1}{v}\sum_{\mathbf{k}}{}'\frac{\left(\frac{\hbar^2\mathbf{k}^2}{2m} - \mu\right)(V_0 + V_{\mathbf{k}}) + \frac{\mu}{V_0}\left(V_0^2 + 2V_0 V_{\mathbf{k}}\right)}{(e^{\beta E_{\mathbf{k}}} - 1)V_0 E_{\mathbf{k}}}\\ &+ \frac{1}{v}\sum_{\mathbf{k}}{}'\frac{R_{\mathbf{k}}}{E^4_{\mathbf{k}}}\left\{\frac{1}{V_0}\left(\frac{\hbar^2\mathbf{k}^2}{2m}\right)^3 - \frac{\mu}{V_0}\left(\frac{\hbar^2\mathbf{k}^2}{2m}\right)^2\right\},\end{aligned} \tag{15}$$

where the quasiparticle dispersion (12) now reads as

$$E_{\mathbf{k}} = \sqrt{\left(\frac{\hbar^2\mathbf{k}^2}{2m}\right)^2 + \mu\frac{\hbar^2\mathbf{k}^2}{m}\frac{V_{\mathbf{k}}}{V_0}}. \tag{16}$$

Note that the zeroth order $n_0 = \mu/V_0$ in (15) represents the mean-field result and corresponds to the Hugenholtz-Pines relation [45]. It makes the quasiparticle dispersion (12) according to (16) linear and gapless (massless) in the long-wavelength limit $\mathbf{k} \to \mathbf{0}$, in accordance with the Nambu-Goldstone theorem [46,47].

Inserting (15) in (14), the grand-canonical potential Ω_{eff} reduces up to first order in all fluctuations to the grand-canonical free energy

$$\begin{aligned}\mathcal{F} = {}& -\frac{v\mu^2}{2V_0} + \frac{1}{2}\sum_{\mathbf{k}}{}'\left[E_{\mathbf{k}} - \left(\frac{\hbar^2\mathbf{k}^2}{2m} + \mu\frac{V_{\mathbf{k}}}{V_0}\right)\right]\\ &+ \sum_{\mathbf{k}}{}'\frac{1}{\beta}\ln(1 - e^{-\beta E_{\mathbf{k}}}) - \sum_{\mathbf{k}}{}'\frac{R_{\mathbf{k}}}{E^2_{\mathbf{k}}}\frac{\hbar^2\mathbf{k}^2}{2m}\frac{\mu}{V_0}.\end{aligned} \tag{17}$$

The particle-number density follows then from Eq. (17) via the thermodynamic relation $n = -\frac{1}{v}\frac{\partial\mathcal{F}}{\partial\mu}$:

$$\begin{aligned}n = {}& \frac{\mu}{V_0} - \frac{1}{2v}\sum_{\mathbf{k}}{}'\left(\frac{\frac{\hbar^2\mathbf{k}^2}{2m}\frac{V_{\mathbf{k}}}{V_0}}{E_{\mathbf{k}}} - \frac{V_{\mathbf{k}}}{V_0}\right)\\ &- \frac{1}{v}\sum_{\mathbf{k}}{}'\frac{\frac{\hbar^2\mathbf{k}^2}{2m}\frac{V_{\mathbf{k}}}{V_0}}{(e^{\beta E_{\mathbf{k}}} - 1)E_{\mathbf{k}}}\\ &+ \frac{1}{v}\sum_{\mathbf{k}}{}'\frac{R_{\mathbf{k}}}{E^4_{\mathbf{k}}}\frac{1}{V_0}\left(\frac{\hbar^2\mathbf{k}^2}{2m}\right)^3.\end{aligned} \tag{18}$$

Eliminating from (15) and (18) the chemical potential μ allows us to determine the condensate depletion up to first order in the respective fluctuations:

$$n - n_0 = n' + n_{\text{th}} + n_{\text{R}}. \tag{19}$$

Thus, the total particle density consists of four parts. Within the Bogoliubov theory, the main contribution is due to the condensate density n_0, whereas the occupation of the excited states above the ground state consists of three distinct terms: the condensate depletion due to interaction and quantum fluctuations is denoted by [48]

$$n' = \frac{1}{2v} \sum_{\mathbf{k}}{}' \left(\frac{\frac{\hbar^2\mathbf{k}^2}{2m} + nV_{\mathbf{k}}}{E_{\mathbf{k}}} - 1 \right). \tag{20}$$

Note that we neglected here any impact of the external random potential upon the quantum fluctuations, an interesting case which was only recently studied in detail in Refs. [49,50].

The condensate depletion due to the thermal fluctuations is given by

$$n_{\rm th} = \frac{1}{v} \sum_{\mathbf{k}}{}' \frac{\frac{\hbar^2\mathbf{k}^2}{2m} + nV_{\mathbf{k}}}{E_{\mathbf{k}}} \frac{1}{e^{\beta E_{\mathbf{k}}} - 1}. \tag{21}$$

And, finally, the condensate depletion due to the external random potential results in

$$n_{\rm R} = \frac{1}{v} \sum_{\mathbf{k}}{}' \frac{nR_{\mathbf{k}}}{E_{\mathbf{k}}^4} \left(\frac{\hbar^2\mathbf{k}^2}{2m} \right)^2. \tag{22}$$

Note that the validity of our approximation depends on the condition $n - n_0 \ll n$, i.e., the depletion is small, so that condensate density n_0 and total density n are approximately the same. With this dispersion (16) in all fluctuation terms, (20)–(22) reduce to

$$E_{\mathbf{k}} = \sqrt{\left(\frac{\hbar^2\mathbf{k}^2}{2m} \right)^2 + nV_{\mathbf{k}} \frac{\hbar^2\mathbf{k}^2}{m}}, \tag{23}$$

which represents the celebrated Bogoliubov spectrum for the collective excitations. Note that in the Popov approximation, the particle density n in (23) is substituted by the condensate density n_0, so that this Bogoliubov dispersion acquires fluctuation corrections [51].

The grand-canonical ground-state energy $E - \mu N$ follows directly from Eq. (17), by using the thermodynamic relation $E - \mu N = -T^2(\frac{\partial}{\partial T} \frac{\mathcal{F}}{T})_{v,\mu}$:

$$E - \mu N = -\frac{v\mu^2}{2V_0} + E' + E_{\rm th} + E_{\rm R}. \tag{24}$$

The first term on the right-hand side of (24) represents the ground-state mean-field energy for the homogeneous Bose-Einstein condensate, whereas the other three terms denote energy contributions which are due to the partial occupation of the excited states. The energy shift due to interaction and quantum fluctuations is denoted by

$$E' = \frac{1}{2} \sum_{\mathbf{k}}{}' \left[E_{\mathbf{k}} - \left(\frac{\hbar^2\mathbf{k}^2}{2m} + nV_{\mathbf{k}} \right) \right]. \tag{25}$$

The energy shift due to the thermal fluctuations is given by

$$E_{\rm th} = \sum_{\mathbf{k}}{}' E_{\mathbf{k}} \frac{1}{e^{\beta E_{\mathbf{k}}} - 1}. \tag{26}$$

And, finally, the energy shift due to the external random potential reads as

$$E_{\rm R} = -\sum_{\mathbf{k}}{}' \frac{R_{\mathbf{k}}}{E_{\mathbf{k}}^2} \frac{\hbar^2\mathbf{k}^2}{2m} n. \tag{27}$$

Within the next section, these general results are further evaluated and discussed for a dipolar Bose gas.

III. ZERO-TEMPERATURE RESULTS

In this section, we specialize our results to the zero-temperature limit for a delta-correlated random potential $R(\mathbf{x} - \mathbf{x}') = R_0 \delta(\mathbf{x} - \mathbf{x}')$, where R_0 denotes the disorder strength. In the thermodynamic limit, when both the particle number N and the volume v diverge, i.e., $N \to \infty$ and $v \to \infty$, but their ratio $n = \frac{N}{v}$ remains constant, the respective wave-vector sums converge towards integrals: $\frac{1}{v} \sum_{\mathbf{k}}{}' \to \int \frac{d^3\mathbf{k}}{(2\pi)^3}$. With this we obtain for the quantum depletion (20) the concrete result [42,43]

$$n' = n_{\rm B} Q_{\frac{3}{2}}(\epsilon_{\rm dd}), \tag{28}$$

with the original Bogoliubov expression [44]

$$n_{\rm B} = \frac{8}{3\sqrt{\pi}} (na)^{\frac{3}{2}}, \tag{29}$$

whereas the disorder depletion (22) specializes to [37,39]

$$n_{\rm R} = n_{\rm HM} Q_{-\frac{1}{2}}(\epsilon_{\rm dd}). \tag{30}$$

Here, $\epsilon_{\rm dd} = C_{\rm dd}/3g$ denotes the relative dipolar interaction strength, and

$$n_{\rm HM} = \frac{m^2 R_0}{8\hbar^4 \pi^{\frac{3}{2}}} \sqrt{\frac{n}{a}} \tag{31}$$

represents the Huang-Meng result for the isotropic contact interaction [31]. In addition, the functions

$$Q_\alpha(\epsilon_{\rm dd}) = \int_0^1 [1 + \epsilon_{\rm dd}(3x^2 - 1)]^\alpha dx, \tag{32}$$

which describe the dipolar effect, can be expressed analytically [52]

$$Q_\alpha(\epsilon_{\rm dd}) = (1 - \epsilon_{\rm dd})^\alpha \, {}_2F_1\left(-\alpha, \frac{1}{2}; \frac{3}{2}; \frac{-3\epsilon_{\rm dd}}{1 - \epsilon_{\rm dd}} \right) \tag{33}$$

in terms of the hypergeometric function ${}_2F_1$. The dipolar function $Q_\alpha(\epsilon_{\rm dd})$ is depicted in Fig. 1, and increases for increasing magnitude of α for both positive and negative signs. Note that, in order to avoid any instability, we restricted $\epsilon_{\rm dd}$ in Fig. 1 to the maximum value one, so that the radicand in the Bogoliubov spectrum (23) remains positive when $\mathbf{k} \to 0$ [42,43].

We conclude that Eqs. (28) and (30) reproduce the contact interaction results (29) and (31) for $\epsilon_{\rm dd} = 0$ due to the property $Q_\alpha(0) = 1$. Note that increasing the repulsive contact interaction has opposite effects on the zero-temperature condensate depletion. On the one hand, more and more bosons are then scattered from the ground state to some excited states so that the quantum depletion (29) increases. On the other hand, the localization of bosons in the respective minima of the

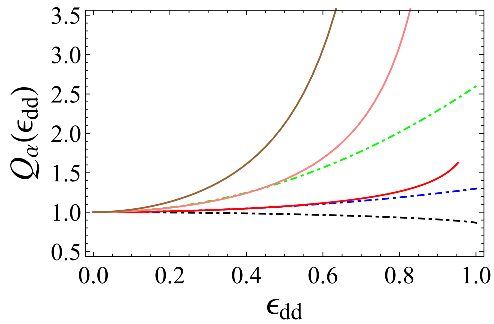

FIG. 1. (Color online) Dipolar enhancement function $Q_\alpha(\epsilon_{dd})$ versus relative dipolar interaction strength ϵ_{dd} for different values of α: $-\frac{5}{2}$ (brown, upper solid line), $-\frac{3}{2}$ (pink, middle solid line), $-\frac{1}{2}$ (red, lower solid line), $\frac{1}{2}$ (black, lower dotted-dashed line), $\frac{3}{2}$ (blue, middle dotted-dashed line), $\frac{5}{2}$ (green, upper dotted-dashed line).

random potential is hampered, so that disorder depletion (31) decreases. In contrast to that, the long-range dipolar interaction turns out to enhance both the quantum deletion (28) and the disorder depletion (30) (see Fig. 1). As the dipolar interaction potential (3) has two repulsive and only one attractive direction in real space, it yields a net expulsion of bosons from the ground state which is described by $Q_{\frac{3}{2}}(\epsilon_{dd})$ in (28). In contrast to that, the dipolar interaction supports the localization of bosons in the random environment [37,39] according to $Q_{-\frac{1}{2}}(\epsilon_{dd})$ in (30). This is due to the fact that the BEC droplets in the respective minima of the random potential can minimize their energy by deformation along the polarization axis [5]. The dipolar enhancement in both the quantum depletion n' and the external random potential depletion n_R are approximately the same with respect to ϵ_{dd} up to $\epsilon_{dd} = 0.6$, but then the dipolar enhancement in n_R starts to increase faster than in n' (see Fig. 1).

Furthermore, we read off from (25) that the energy shift due to the interaction and the quantum fluctuations are ultraviolet divergent. Thus, following Refs. [43,53], we can evaluate (25) by introducing an ultraviolet cutoff, where the interaction has to be renormalized by inserting the term $\frac{m(nV_{\mathbf{k}})^2}{\hbar^2\mathbf{k}^2}$, so that the divergent part is removed

$$\frac{E'}{N} = \frac{1}{2n}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[E_{\mathbf{k}} - \left(\frac{\hbar^2\mathbf{k}^2}{2m} + nV_{\mathbf{k}}\right) + \frac{m(nV_{\mathbf{k}})^2}{\hbar^2\mathbf{k}^2}\right], \tag{34}$$

evaluations in the thermodynamic limit yielding

$$\frac{E'}{N} = \frac{2\pi a\hbar^2 n}{m}\frac{128}{15}\left(\frac{na^3}{\pi}\right)^{\frac{1}{2}} Q_{\frac{5}{2}}(\epsilon_{dd}). \tag{35}$$

Another method for (25) relies on Schwinger trick [54] which leads again to (35). The energy shift due to the external random potential (27) is also ultraviolet divergent:

$$\frac{E_R}{N} = -\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{R_0}{\frac{\hbar^2\mathbf{k}^2}{2m} + 2nV_{\mathbf{k}}}. \tag{36}$$

Thus, it is calculated either with an ultraviolet cutoff by subtracting the divergent term $\frac{2mR_0}{v\hbar^2\mathbf{k}^2}$, or with the help of the Schwinger trick, leading to

$$\frac{E_R}{N} = \frac{2mR_0}{\hbar^2}\sqrt{\frac{na}{\pi}}\,Q_{\frac{1}{2}}(\epsilon_{dd}). \tag{37}$$

Note that energy shift E' due to interaction and quantum fluctuations in Eq. (35) and the external random potential energy shift E_R in Eq. (37) reproduce the well-known contact interaction results when $\epsilon_{dd} = 0$, and that E' increases quicker than E_R with respect to ϵ_{dd}. Finally, note that both energy shifts (35) and (37) could be used to study the collective properties of a trapped dipolar Bose gas within the local density approximation [55,56].

IV. SUPERFLUIDITY

Now, we extend the Bogoliubov theory in order to study the transport phenomenon of superfluidity within the framework of linear-response theory. With this we calculate the superfluid depletions due to both the external random potential and the thermal fluctuations.

A. Bogoliubov theory revisited

Following Refs. [34,57] we calculate the superfluid component n_s of the Bose gas, which moves with the superfluid velocity $\mathbf{v}_s$, via a linear-response approach. To this end, we perform a Galilean boost for our system with a boost velocity $\mathbf{u}$ so that the normal fluid component n_n is dragged along this boost. Inserting the Galilean transformations $\mathbf{x}' = \mathbf{x} + \mathbf{u}t$, $t = t'$, and rewriting the field operator in Heisenberg picture according to $\hat{\psi}'(\mathbf{x}',t') = \hat{\psi}(\mathbf{x},t)e^{\frac{i}{\hbar}m\mathbf{v}_s\mathbf{x}}$, the Hamiltonian in momentum space reads as, instead of (4),

$$\begin{aligned}\hat{\mathcal{K}} = {}& \frac{1}{2}\sum_{\mathbf{k}}\left[\frac{\hbar^2\mathbf{k}^2}{2m} - \mu_{\text{eff}}\right](\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{k}}) \\ &+ \frac{1}{2}\sum_{\mathbf{k}}\hbar\mathbf{k}(\mathbf{u} - \mathbf{v}_s)(\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} - \hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{k}}) \\ &+ \frac{1}{2v}\sum_{\mathbf{p},\mathbf{k}} U_{\mathbf{p}-\mathbf{k}}(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{k}} + \hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{p}}) \\ &+ \frac{1}{2v}\sum_{\mathbf{p},\mathbf{k},\mathbf{q}} V_{\mathbf{q}}\,\hat{a}^\dagger_{\mathbf{k}+\mathbf{q}}\hat{a}^\dagger_{\mathbf{p}-\mathbf{q}}\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{k}},\end{aligned} \tag{38}$$

with the effective chemical potential $\mu_{\text{eff}} = \mu - \frac{1}{2}m\mathbf{v}_s^2 + m\mathbf{u}\mathbf{v}_s$. Applying the Bogoliubov theory along similar lines as in Sec. II and after performing the disorder ensemble average of the grand-canonical Hamiltonian, we obtain instead of (13)

$$\begin{aligned}\langle\hat{\mathcal{K}}'\rangle = {}& v\left(-\mu_{\text{eff}}n_{\mathbf{0}} + \frac{1}{2}V_0 n_{\mathbf{0}}^2\right) \\ &+ \frac{1}{2}\sum_{\mathbf{k}}{}'\left\{E_{\mathbf{k}} - \left[\frac{\hbar^2\mathbf{k}^2}{2m} - \mu_{\text{eff}} + n_{\mathbf{0}}(V_0 + V_{\mathbf{k}})\right]\right\} \\ &+ \sum_{\mathbf{k}}{}'[E_{\mathbf{k}} + \hbar\mathbf{k}(\mathbf{u} - \mathbf{v}_s)]\hat{\alpha}^\dagger_{\mathbf{k}}\hat{\alpha}_{\mathbf{k}} \\ &- \sum_{\mathbf{k}}{}'\frac{n_{\mathbf{0}}R_{\mathbf{k}}\left(\frac{\hbar^2\mathbf{k}^2}{2m} - \mu_{\text{eff}} + n_{\mathbf{0}}V_0\right)}{\left\{E_{\mathbf{k}}^2 - [\hbar\mathbf{k}(\mathbf{u} - \mathbf{v}_s)]^2\right\}},\end{aligned} \tag{39}$$

where $E_{\mathbf{k}}$ stands for the quasiparticle dispersion (12) with μ substituted by $\mu_{\rm eff}$. Now, it is straightforward to find the corresponding grand-canonical effective potential, where we get instead of (14)

$$\Omega_{\rm eff} = v\left(-\mu_{\rm eff}n_{\mathbf{0}} + \frac{1}{2}V_{\mathbf{0}}n_{\mathbf{0}}^2\right) + \frac{1}{2}\sum_{\mathbf{k}}{}' \left\{E_{\mathbf{k}} - \left[\frac{\hbar^2\mathbf{k}^2}{2m} - \mu_{\rm eff} + n_{\mathbf{0}}(V_{\mathbf{0}}+V_{\mathbf{k}})\right]\right\} + \sum_{\mathbf{k}}{}' \frac{1}{\beta}\ln\{1 - e^{-\beta[E_{\mathbf{k}}+\hbar\mathbf{k}(\mathbf{u}-\mathbf{v}_{\rm s})]}\} - \sum_{\mathbf{k}}{}' \frac{n_{\mathbf{0}}R_{\mathbf{k}}\left(\frac{\hbar^2\mathbf{k}^2}{2m} - \mu_{\rm eff} + n_{\mathbf{0}}V_{\mathbf{0}}\right)}{\{E_{\mathbf{k}}^2 - [\hbar\mathbf{k}(\mathbf{u}-\mathbf{v}_{\rm s})]^2\}}. \tag{40}$$

Within a linear-response approach, we are only interested in small velocities $\mathbf{v}_{\rm s}$ and $\mathbf{u}$, so we expand Eq. (40) up to second order in $\hbar\mathbf{k}(\mathbf{u}-\mathbf{v}_{\rm s})$, and we get

$$\Omega_{\rm eff} = v\left(-\mu_{\rm eff}n_{\mathbf{0}} + \frac{1}{2}V_{\mathbf{0}}n_{\mathbf{0}}^2\right) + \frac{1}{2}\sum_{\mathbf{k}}{}' \left\{E_{\mathbf{k}} - \left[\frac{\hbar^2\mathbf{k}^2}{2m} - \mu_{\rm eff} + n_{\mathbf{0}}(V_{\mathbf{0}}+V_{\mathbf{k}})\right]\right\} + \sum_{\mathbf{k}}{}' \frac{1}{\beta}\ln(1-e^{-\beta E_{\mathbf{k}}}) - \sum_{\mathbf{k}}{}' \frac{\beta e^{\beta E_{\mathbf{k}}}[\hbar\mathbf{k}(\mathbf{u}-\mathbf{v}_{\rm s})]^2}{2(e^{\beta E_{\mathbf{k}}}-1)^2} - \sum_{\mathbf{k}}{}' \frac{n_{\mathbf{0}}R_{\mathbf{k}}}{E_{\mathbf{k}}^2}\left[\frac{\hbar^2\mathbf{k}^2}{2m} - \mu_{\rm eff} + n_{\mathbf{0}}V_{\mathbf{0}}\right] - \sum_{\mathbf{k}}{}' \frac{n_{\mathbf{0}}R_{\mathbf{k}}}{E_{\mathbf{k}}^4}\left[\frac{\hbar^2\mathbf{k}^2}{2m} - \mu_{\rm eff} + n_{\mathbf{0}}V_{\mathbf{0}}\right][\hbar\mathbf{k}(\mathbf{u}-\mathbf{v}_{\rm s})]^2. \tag{41}$$

Note that the terms linear in $\hbar\mathbf{k}(\mathbf{u}-\mathbf{v}_{\rm s})$ vanish due to symmetry reasons. Extremizing the grand-canonical effective potential (41) with respect the the condensate density $n_{\mathbf{0}}$ yields in zeroth order the mean-field result $\mu_{\rm eff} = n_{\mathbf{0}}V_{\mathbf{0}}$. With this we are ready to determine the momentum of the system $\mathbf{P} = (-\frac{\partial\Omega_{\rm eff}}{\partial\mathbf{u}})_{v,T,\mu}$ in the small-velocity limit, which turns out to be of the following form:

$$\mathbf{P} = mv\,(n\mathbf{v}_s + n_{\rm n}\mathbf{v}_{\rm n}), \tag{42}$$

where the total density $n = n_{\rm s} + n_{\rm n}$ decomposes into the superfluid and normal components $n_{\rm s}$ and $n_{\rm n}$, respectively, and $\mathbf{v}_{\rm n} = \mathbf{u} - \mathbf{v}_{\rm s}$ denotes the normal fluid velocity. The normal fluid density turns out to decompose according to $n_{\rm n} = n_{\rm R} + n_{\rm th}$, where the contribution due to the external random potential, which only exists in the zero-temperature limit, reads as [37]

$$n_{\mathrm{R}ij} = \frac{1}{v}\sum_{\mathbf{k}}{}' \frac{2nR_{\mathbf{k}}\hbar^2k_ik_j}{m\left(\frac{\hbar^2\mathbf{k}^2}{2m}\right)\left(\frac{\hbar^2\mathbf{k}^2}{2m} + 2nV_{\mathbf{k}}\right)^2}, \tag{43}$$

while the normal fluid density due to the thermal fluctuations is given by

$$n_{\mathrm{th}ij} = \frac{1}{v}\sum_{\mathbf{k}}{}' \frac{\beta}{m}\hbar^2k_ik_j\frac{e^{\beta E_{\mathbf{k}}}}{(e^{\beta E_{\mathbf{k}}}-1)^2}, \tag{44}$$

where $E_{\mathbf{k}}$ stands for the Bogoliubov energy spectrum (23). Note that there is no superfluid depletion in (42) which is due to quantum fluctuations, in contrast to the condensate depletion which has a quantum fluctuation component determined by Eq. (20).

B. Zero-temperature superfluid depletion

Again, we specialize at first to the zero-temperature limit and to a delta-correlated random potential and evaluate the corresponding superfluid depletions in the thermodynamic limit. Depending on the boost direction, we have two different superfluid depletions in the directions parallel or perpendicular to the dipole polarization. In the first case, the superfluid depletion reads as

$$n_{\mathrm{R}\|} = 4n_{\rm HM}\,J_{-\frac{1}{2}}(\epsilon_{\rm dd}), \tag{45}$$

whereas in the second case the superfluid depletion turns out to be

$$n_{\mathrm{R}\perp} = 2n_{\rm HM}\left[Q_{-\frac{1}{2}}(\epsilon_{\rm dd}) - J_{-\frac{1}{2}}(\epsilon_{\rm dd})\right]. \tag{46}$$

Here, we have introduced the new function

$$J_\alpha(\epsilon_{\rm dd}) = \int_0^1 x^2\,[1+\epsilon_{\rm dd}(3x^2-1)]^\alpha dx, \tag{47}$$

which can also be expressed analytically with the help of the integral table [52] according to

$$J_\alpha(\epsilon_{\rm dd}) = \frac{1}{3}(1-\epsilon_{\rm dd})^\alpha\,{}_2F_1\left(-\alpha,\frac{3}{2};\frac{5}{2};\frac{-3\epsilon_{\rm dd}}{1-\epsilon_{\rm dd}}\right). \tag{48}$$

Note that the function $J_\alpha(\epsilon_{\rm dd})$ shown in Fig. 2 has the property $J_\alpha(0) = \frac{1}{3}$ and decreases slowly with increasing $\epsilon_{\rm dd}$ for $\alpha = -\frac{1}{2}$, while it behaves nonmonotonically with increasing $\epsilon_{\rm dd}$ for $\alpha = -\frac{5}{2}$.

We observe that, due to $Q_\alpha(0) = 1$ and $J_\alpha(0) = \frac{1}{3}$, both depletions $n_{\mathrm{R}\|}$ and $n_{\mathrm{R}\perp}$ coincide for vanishing dipolar interaction and equate $\frac{4}{3}n_{\rm HM}$, which reproduces the Huang-Meng result for the isotropic contact interaction [31]. Due to the localization of bosons in the respective minima of the random potential the superfluid is hampered, so that the superfluid depletion is larger than the condensate depletion. Switching on the dipolar interaction has a significant effect on both depletions $n_{\mathrm{R}\|}$ and $n_{\mathrm{R}\perp}$. Their ratio with the condensate depletion $n_{\rm R}$ turns out to

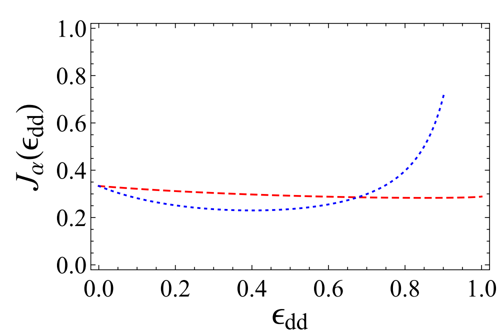

FIG. 2. (Color online) Function $J_\alpha(\epsilon_{\rm dd})$ versus relative dipolar interaction strength $\epsilon_{\rm dd}$ for different values of α: $-\frac{1}{2}$ (red, dashed line), $-\frac{5}{2}$ (blue, dotted line).

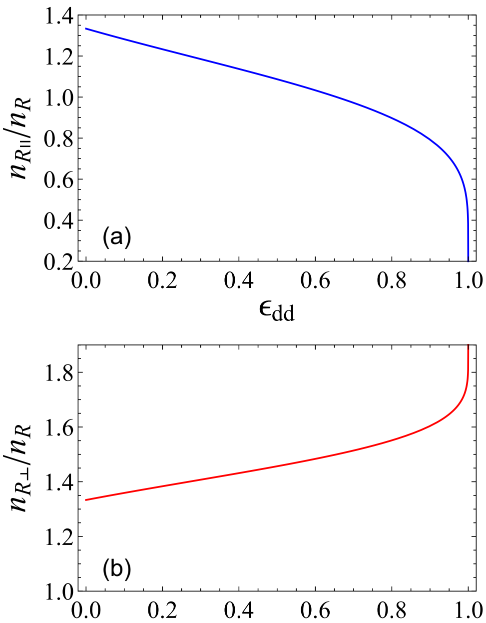

FIG. 3. (Color online) Ratios of superfluid depletions $n_{R\parallel}$ and $n_{R\perp}$ and condensate depletion n_R versus relative dipolar interaction strength ϵ_{dd}.

be monotonically decreasing (increasing) with respect to the relative dipolar interaction strength ϵ_{dd} according to Fig. 3. This is explained qualitatively by considering the dipolar interaction in Fourier space (5) as an effective contact interaction [37,39]. Therefore, for sufficiently large values of ϵ_{dd}, it turns out that $n_{R\parallel}$ becomes even smaller than the condensate depletion. This surprising finding suggests that the localization of the bosons in the minima of the random potential only occurs for a finite period of time [58]. As a consequence, the ratio $n_{R\parallel}/n_{R\perp}$ decreases monotonically for increasing values of the relative dipolar interaction strength ϵ_{dd}.

V. FINITE-TEMPERATURE EFFECTS

In the previous sections, we restricted the application of the Bogoliubov theory of dirty bosons to the zero-temperature limit. Nevertheless, any experiment is performed at finite temperature, where thermal fluctuations play an important role besides quantum and disorder fluctuations. Therefore, we calculate in this section for a dipolar BEC the contributions to both the condensate and the normal fluid component which are due to thermal excitations.

A. Condensate depletion

At zero temperature, the condensate depletion consists of two contributions, one due to the quantum fluctuation (20) and another one due to the disorder potential (22), which are evaluated in (28) and (30) for a dipolar BEC, respectively. But, for increasing temperature also the third contribution (21) of the depletion (19) will play an important role and should be taken into account. This additional term represents the condensate depletion due to the thermal excitations, which

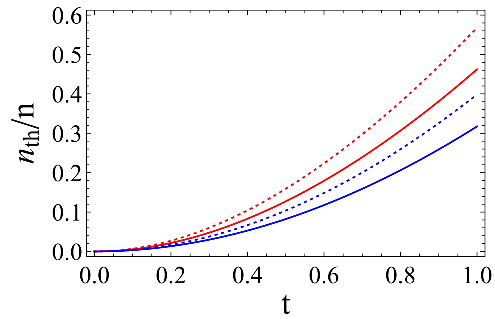

FIG. 4. (Color online) Thermal fractional depletion n_{th}/n versus relative temperature $t = T/T_c^0$ for different values of relative dipolar interaction strength $\epsilon_{dd} = 0$ (solid line), $\epsilon_{dd} = 0.8$ (dotted line), and gas parameter $\gamma = 0.01$ (red, upper two curves), $\gamma = 0.20$ (blue line).

reads as, in the thermodynamics limit,

$$n_{th} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\frac{\hbar^2\mathbf{k}^2}{2m} + nV_{\mathbf{k}}}{E_{\mathbf{k}}} \frac{1}{e^{\beta E_{\mathbf{k}}} - 1}, \tag{49}$$

where $E_{\mathbf{k}}$ denotes the Bogoliubov spectrum (23). The integral (49) can be recast in dimensionless form

$$\frac{n_{th}}{n} = \frac{\gamma^{-\frac{1}{6}} t^2}{2\pi^{\frac{1}{2}}\left[\zeta\left(\frac{3}{2}\right)\right]^{\frac{4}{3}}} I(\gamma,\epsilon_{dd},t), \tag{50}$$

where we have introduced the gas parameter $\gamma = na^3$, and the relative temperature $t = T/T_c^0$, with $T_c^0 = \frac{2\pi\hbar^2 n^{\frac{2}{3}}}{[\zeta(\frac{3}{2})]^{\frac{2}{3}} m k_B}$ being the critical temperature for the noninteracting Bose gas, and $\zeta(x)$ denotes the Riemann zeta function with the value $\zeta(\frac{3}{2}) = 2.612\,38\ldots$. The remaining integral $I(\gamma,\epsilon_{dd},t)$ reads as

$$I(\gamma,\epsilon_{dd},t) = \int_0^\infty dx \int_0^\pi d\theta \frac{x \sin\theta\left(1 + \frac{\alpha x^2}{8\Theta^2}\right)}{\sqrt{\Theta + \frac{\alpha x^2}{16\Theta}}\left(e^{\sqrt{x^2 + \frac{\alpha x^4}{16\Theta^2}}} - 1\right)}, \tag{51}$$

with the abbreviations $\alpha = [\frac{t}{\gamma^{\frac{1}{3}}[\zeta(\frac{3}{2})]^{\frac{2}{3}}}]^2$ and $\Theta = 1 + \epsilon_{dd}(3\cos^2\theta - 1)$. Figure 4 shows how the condensate thermal fractional depletion (50) increases with the relative temperature t for different values of the relative dipolar interaction strength ϵ_{dd} and the gas parameter γ. For increasing contact interaction, it turns out that n_{th} decreases as collisions between the thermal particles yield partially a scattering into the macroscopic ground state. In contrast to that we observe that the dipolar interaction has the opposite effect: increasing ϵ_{dd} enhances the thermal depletion n_{th}. This can be understood from considering the consequences of the dipolar interaction as an effective contact interaction [37,39], yielding two attractive and one repulsive direction in Fourier space according to (5).

In order to investigate this qualitative finding more quantitatively in the vicinity of zero temperature but far below the critical temperature, we approximate the condensate depletion (50) analytically like

$$\frac{n_{th}}{n} = \frac{\pi^{\frac{3}{2}}\gamma^{-\frac{1}{6}}t^2}{6\left[\zeta\left(\frac{3}{2}\right)\right]^{\frac{4}{3}}} Q_{-\frac{1}{2}}(\epsilon_{dd}) - \frac{\pi^{\frac{7}{2}}\gamma^{-\frac{5}{6}}t^4}{480\left[\zeta\left(\frac{3}{2}\right)\right]^{\frac{8}{3}}} Q_{-\frac{5}{2}}(\epsilon_{dd}) + \cdots . \tag{52}$$

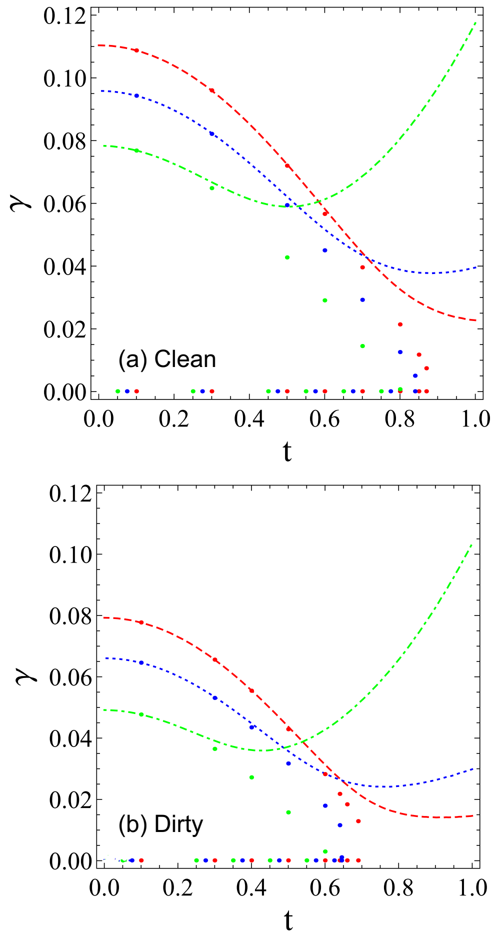

FIG. 5. (Color online) Validity range of Bogoliubov theory in the t-γ plane for (a) clean case, i.e., $R_0 = 0$, and (b) dirty case with $R_0 = \frac{2\hbar^4\pi^{\frac{3}{2}}n^{\frac{1}{3}}}{5m^2}$ for different values of relative dipolar interaction strength $\epsilon_{\rm dd} = 0$ (red, dashed line), $\epsilon_{\rm dd} = 0.5$ (blue, dotted line), $\epsilon_{\rm dd} = 0.8$ (green, dotted-dashed line). Condensate depletion (53) equals $n/2$ for curves with $n_{\rm th}/n$ from (52). Dots represent the case upon inserting $n_{\rm th}/n$ from (50) and (51).

This reproduces the isotropic contact interaction case for vanishing dipolar interaction due to $Q_\alpha(0) = 1$ [59]. Note that (52) turns out to approximate (51) quite well irrespective of γ and $\epsilon_{\rm dd}$ in the temperature range $0 \leqslant t \leqslant 0.3$. Here also each term in (52) decreases for increasing γ, while it is enhanced for increasing $\epsilon_{\rm dd}$.

Now, we investigate how the validity range of the Bogoliubov theory depends on the gas parameter γ and the relative temperature t for different values of the relative dipolar interaction strength $\epsilon_{\rm dd}$ and the disorder strength R_0. To this end, we restrict the condensate depletion $\Delta n = n - n_{\bf 0}$ in Eq. (19) to be limited maximally by half of the particle-number density n. Inserting the critical temperature for the noninteracting Bose gas T_c^0 and the gas parameter γ in the condensate depletion due to quantum fluctuation (28) and random potential (30), the fractional depletion reads as

$$\frac{\Delta n}{n} = \frac{8}{3\sqrt{\pi}}\gamma^{\frac{1}{2}} Q_{\frac{3}{2}}(\epsilon_{\rm dd}) + \frac{n_{\rm th}}{n} + \frac{m^2 R_0}{8\hbar^4\pi^{\frac{3}{2}} n^{\frac{1}{3}}}\gamma^{-\frac{1}{6}} Q_{-\frac{1}{2}}(\epsilon_{\rm dd}). \tag{53}$$

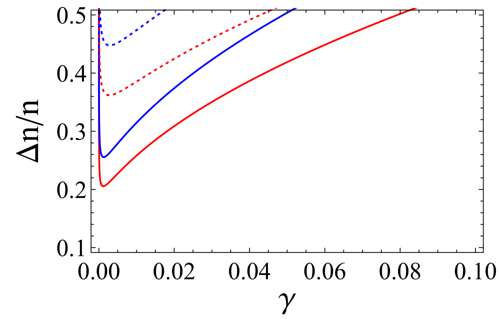

FIG. 6. (Color online) Fractional depletion $\Delta n/n$ versus gas parameter $\gamma = na^3$ for different values of relative dipolar interaction strength $\epsilon_{\rm dd} = 0$ (red line), $\epsilon_{\rm dd} = 0.8$ (blue line) and relative temperature $t = 0$ (solid line), $t = 0.5$ (dotted line) with the disorder strength $R_0 = \frac{2\hbar^4\pi^{\frac{3}{2}}n^{\frac{1}{3}}}{5m^2}$.

Here, the thermal fractional condensate depletion (50) in (53) is determined numerically from (51) and from (52) analytically in the vicinity of zero temperature for comparison. Figure 5 depicts the resulting validity range of the Bogoliubov theory in the clean case, i.e., the disorder strength $R_0 = 0$, and in the dirty case with $R_0 = \frac{2\hbar^4\pi^{\frac{3}{2}}n^{\frac{1}{3}}}{5m^2}$. It is represented by the area below the curves when the condensate depletion is equal to $n/2$. Outside this area, the Bogoliubov approximation is not valid. Note that the validity range (50), (53) is well approximated by (52) near zero temperature. From (50) and (53), we read off that the Bogoliubov theory is not applicable near the critical temperature. Furthermore, the validity range decreases for increasing values of the relative dipolar interaction strength $\epsilon_{\rm dd}$, and it is also smaller in the dirty case compared to the clean one.

Finally, we plot in Fig. 6 the total fractional condensate depletion $\frac{\Delta n}{n}$ versus the gas parameter γ with the thermal fractional condensate depletion $\frac{n_{\rm th}}{n}$ in (53) inserted from (50) for the disorder strength $R_0 = \frac{2\hbar^4\pi^{\frac{3}{2}}n^{\frac{1}{3}}}{5m^2}$. Note that the sharp increase for small gas parameter is unphysical due to the fact that the Bogoliubov theory is no longer valid (see Fig. 5). We conclude that increasing the parameters t and $\epsilon_{\rm dd}$ yields an increasing fractional depletion.

B. Superfluid depletion

Now, we calculate the superfluid depletion for the dipolar Bose gas which is due to thermal fluctuations. In the thermodynamic limit of Eq. (44), the components parallel and perpendicular to the dipole polarization direction have to be evaluated separately. Parallel to the dipoles the integral reads as in dimensionless form

$$\frac{n_{\rm th\|}}{n} = \frac{\gamma^{-\frac{5}{6}} t^4}{8\pi^{\frac{1}{2}}\left[\zeta\left(\frac{3}{2}\right)\right]^{\frac{8}{3}}} I_\|(\gamma,\epsilon_{\rm dd},t), \tag{54}$$

where the remaining integral $I_\|(\gamma,\epsilon_{\rm dd},t)$ is written explicitly as

$$I_\|(\gamma,\epsilon_{\rm dd},t) = \int_0^\infty dx \int_0^\pi d\theta \frac{x^4 \sin\theta \cos^2\theta\, e^{\sqrt{x^2+\frac{\alpha x^4}{16\Theta^2}}}}{\Theta^{\frac{5}{2}}\left(e^{\sqrt{x^2+\frac{\alpha x^4}{16\Theta^2}}} - 1\right)^2}. \tag{55}$$

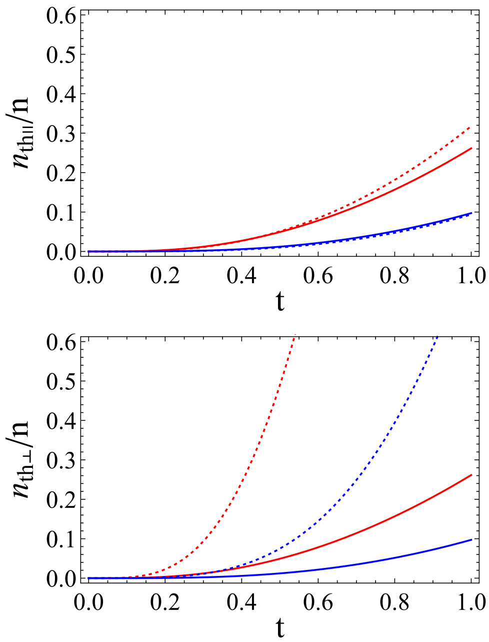

FIG. 7. (Color online) Superfluid thermal fractional depletions $n_{\text{th}\parallel}/n$ and $n_{\text{th}\perp}/n$ versus relative temperature $t = T/T_c^0$ for different values of relative dipolar interaction strength $\epsilon_{\text{dd}} = 0$ (solid), $\epsilon_{\text{dd}} = 0.6$ (dotted) and gas parameter $\gamma = 0.01$ (red), $\gamma = 0.20$ (blue).

In the direction perpendicular to the dipoles, the superfluid depletion reads as in dimensionless form

$$\frac{n_{\text{th}\perp}}{n} = \frac{\gamma^{-\frac{5}{6}} t^4}{8\pi^{\frac{1}{2}}\left[\zeta\left(\frac{3}{2}\right)\right]^{\frac{8}{3}}} I_\perp(\gamma,\epsilon_{\text{dd}},t), \tag{56}$$

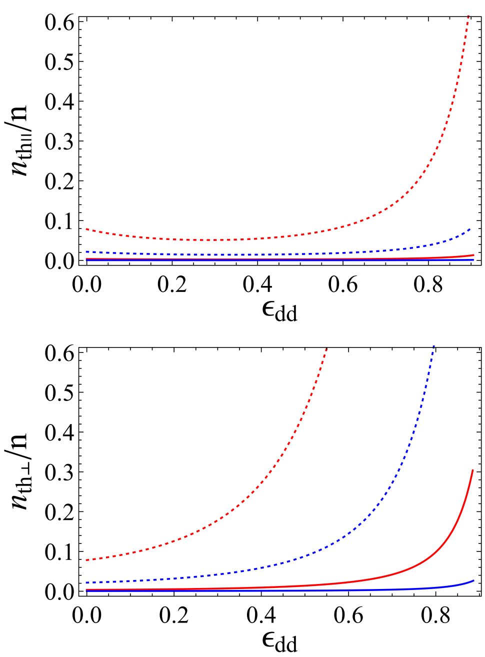

FIG. 8. (Color online) Superfluid thermal fractional depletions $n_{\text{th}\parallel}/n$ and $n_{\text{th}\perp}/n$ versus relative dipolar interaction strength ϵ_{dd} for different values of relative temperature $t = 0.2$ (solid line), $t = 0.6$ (dotted line) and gas parameter $\gamma = 0.01$ (red line), $\gamma = 0.20$ (blue line).

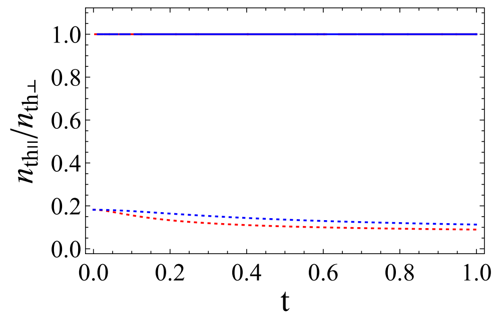

FIG. 9. (Color online) Ratios of thermal superfluid depletions $n_{\text{th}\parallel}/n_{\text{th}\perp}$ versus relative temperature $t = T/T_c^0$ for different values of relative dipolar strength $\epsilon_{\text{dd}} = 0$ (solid line), $\epsilon_{\text{dd}} = 0.6$ (dotted line) and gas parameter $\gamma = 0.01$ (red line), $\gamma = 0.20$ (blue line).

where the remaining integral $I_\perp(\gamma,\epsilon_{\text{dd}},t)$ is written explicitly as

$$I_\perp(\gamma,\epsilon_{\text{dd}},t) = \int_0^\infty dx \int_0^\pi d\theta \frac{x^4 \sin^3\theta \, e^{\sqrt{x^2+\frac{\alpha x^4}{16\Theta^2}}}}{2\Theta^{\frac{5}{2}}\left(e^{\sqrt{x^2+\frac{\alpha x^4}{16\Theta^2}}}-1\right)^2}. \tag{57}$$

Figure 7 shows how the superfluid thermal fractional depletions parallel and perpendicular to the polarized dipoles depend on the respective values of the parameters γ, ϵ_{dd}, and t. The perpendicular superfluid depletion $n_{\text{th}\perp}$ behaves similar to the condensate thermal depletion n_{th}, i.e., $n_{\text{th}\perp}$ is enhanced for increasing t and ϵ_{dd}, whereas it is reduced for increasing γ. Also, for $n_{\text{th}\parallel}$ we observe that it depends on t and γ, like n_{th} and $n_{\text{th}\perp}$, i.e., it increases with t but decreases with γ. But, surprisingly, it turns out that $n_{\text{th}\parallel}$ has a nontrivial dependence on ϵ_{dd}, where we have first a decrease and then an increase for increasing ϵ_{dd}. Figure 8 highlights in more detail how the superfluid thermal fractional depletions change versus relative dipolar interaction strength ϵ_{dd}.

Note that both $n_{\text{th}\perp}$ and $n_{\text{th}\parallel}$ coincide for vanishing dipolar interaction as follows from Fig. 9 which shows the thermal superfluid depletion ratio $n_{\text{th}\parallel}/n_{\text{th}\perp}$ versus the relative temperature t. When the dipolar interaction is present, we notice that the ratio $n_{\text{th}\parallel}/n_{\text{th}\perp}$ reveals only a tiny t and γ dependence which is decreasing with t and increasing with γ, respectively. In Fig. 10, we plot the thermal superfluid depletion ratio $n_{\text{th}\parallel}/n_{\text{th}\perp}$

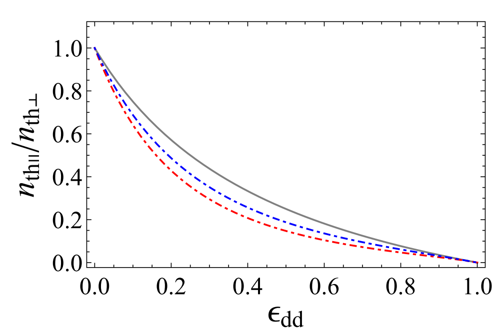

FIG. 10. (Color online) Ratios of thermal superfluid depletions $n_{\text{th}\parallel}/n_{\text{th}\perp}$ versus relative dipolar interaction strength ϵ_{dd} for different values of the gas parameter $\gamma = 0.01$ (red line), $\gamma = 0.20$ (blue line) and relative temperature $t = 0$ (solid gray line), $t = 0.5$ (dotted-dashed line) with $t = 0$ limit from (58) and (59).

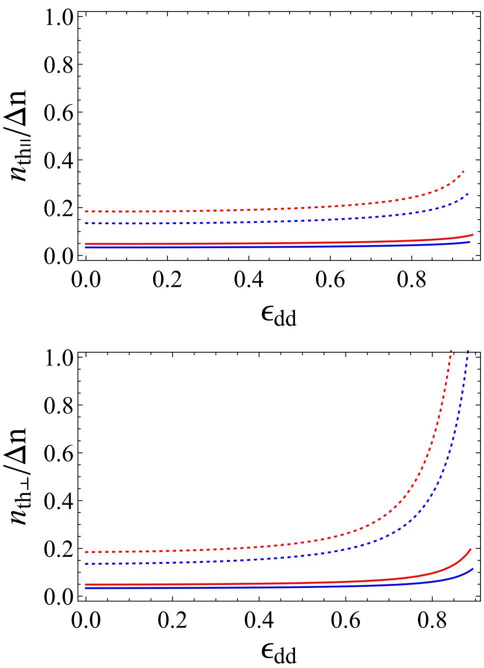

FIG. 11. (Color online) Ratios of superfluid and condensate depletions $n_{\text{th}\parallel}/\Delta n$ (upper) and $n_{\text{th}\perp}/\Delta n$ (lower) versus relative dipolar strength ϵ_{dd} for different values of relative temperature $t = 0.3$ (solid line), $t = 0.6$ (dotted line) and gas parameter $\gamma = 0.01$ (red line), $\gamma = 0.11$ (blue line) in the clean case $R_0 = 0$.

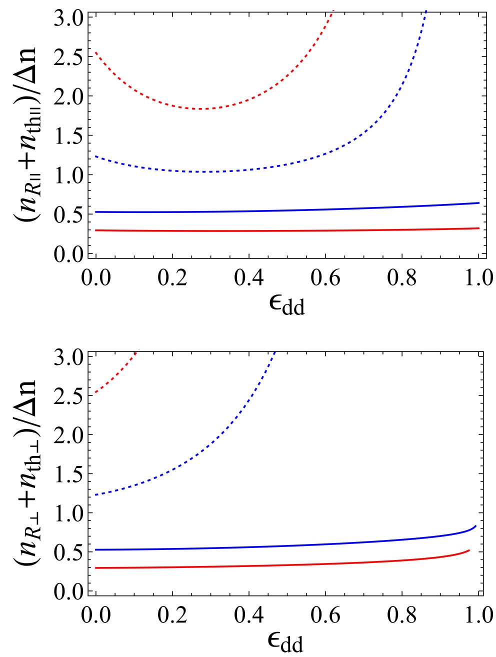

FIG. 12. (Color online) Ratios of superfluid depletions $(n_{\text{th}\parallel} + n_{\text{R}\parallel})/\Delta n$ (upper) and $(n_{\text{th}\perp} + n_{\text{R}\perp})/\Delta n$ (lower) versus relative dipolar strength ϵ_{dd} for different values of relative temperature $t = 0$ (solid line), $t = 0.5$ (dotted line) and gas parameter $\gamma = 0.01$ (red line), $\gamma = 0.08$ (blue line) in the dirty case $R_0 = \frac{2\hbar^4\pi^{\frac{3}{2}}n^{\frac{1}{3}}}{5m^2}$.

versus the relative dipolar strength ϵ_{dd}. It shows that both $n_{\text{th}\perp}$ and $n_{\text{th}\parallel}$ coincide for vanishing dipolar interaction, and the ratio $n_{\text{th}\parallel}/n_{\text{th}\perp}$ decreases for increasing values of the relative dipolar strength ϵ_{dd}.

In the vicinity of the zero temperature but far below critical temperature, we can approximate the superfluid depletion in (54) analytically. We find that the superfluid depletion parallel to the dipoles reads as

$$\frac{n_{\text{th}\parallel}}{n} = \frac{\pi^{\frac{7}{2}}\gamma^{-\frac{5}{6}}t^4}{15\left[\zeta\left(\frac{3}{2}\right)\right]^{\frac{8}{3}}} J_{-\frac{5}{2}}(\epsilon_{\text{dd}}) + \cdots, \tag{58}$$

which has a nonmonotonic behavior versus ϵ_{dd} due to the function $J_{-\frac{5}{2}}(\epsilon_{\text{dd}})$, which is plotted in Fig. 2 where we have a decrease at first and then an increase for increasing ϵ_{dd}. Furthermore, the superfluid depletion perpendicular to the dipoles reads as

$$\frac{n_{\text{th}\perp}}{n} = \frac{\pi^{\frac{7}{2}}\gamma^{-\frac{5}{6}}t^4}{30\left[\zeta\left(\frac{3}{2}\right)\right]^{\frac{8}{3}}}\left[Q_{-\frac{5}{2}}(\epsilon_{\text{dd}}) - J_{-\frac{5}{2}}(\epsilon_{\text{dd}})\right] + \cdots \tag{59}$$

so that $n_{\text{th}\perp}$ is enhanced for increasing ϵ_{dd} due to the fact that the function $Q_{-\frac{5}{2}}(\epsilon_{\text{dd}})$ shown in Fig. 1 increases faster than the function $J_{-\frac{5}{2}}(\epsilon_{\text{dd}})$ shown in Fig. 2. A comparison between the two different superfluid depletions shows that both $n_{\text{th}\perp}$ and $n_{\text{th}\parallel}$ coincide for vanishing dipolar interaction, and reproduce the isotropic contact interaction result [59] due to $Q_\alpha(0) = 1$ and $J_\alpha(0) = \frac{1}{3}$. The ratio $n_{\text{th}\parallel}/n_{\text{th}\perp}$ from (58) and (59) decreases for increasing values of the relative dipolar strength ϵ_{dd} as is shown in gray color in Fig. 10.

C. Superfluid ratios

Finally, after having discussed the different origins of the superfluid depletions for the dipolar Bose gas, we discuss now the ratios of the total superfluid depletions over the total condensate depletion Δn for different values of the respective parameters γ, ϵ_{dd}, and t. We consider both the clean case where the disorder strength $R_0 = 0$ and the dirty case with $R_0 = \frac{2\hbar^4\pi^{\frac{3}{2}}n^{\frac{1}{3}}}{5m^2}$. Figure 11 shows the clean case with the ratios $n_{\text{th}\parallel}/\Delta n$ (upper) and $n_{\text{th}\perp}/\Delta n$ (lower) versus relative dipolar strength ϵ_{dd}. For increasing ϵ_{dd} and t, both $n_{\text{th}\parallel}/\Delta n$ and $n_{\text{th}\perp}/\Delta n$ are enhanced, but increasing γ decreases both $n_{\text{th}\parallel}/\Delta n$ and $n_{\text{th}\perp}/\Delta n$.

Figure 12 shows in the dirty case the ratios $(n_{\text{th}\parallel} + n_{\text{R}\parallel})/\Delta n$ (upper) and $(n_{\text{th}\perp} + n_{\text{R}\perp})/\Delta n$ (lower) versus relative dipolar strength ϵ_{dd}. For increasing ϵ_{dd} we will get enhancement for $(n_{\text{th}\perp} + n_{\text{R}\perp})/\Delta n$, regardless of the temperature t, while for $(n_{\text{th}\parallel} + n_{\text{R}\parallel})/\Delta n$ we will get enhancement for small values of t and then upon increasing t, we observe a decrease up to $\epsilon_{\text{dd}} = 0.3$ and then an increase, which shows that the disorder changes the situation a lot compared to the clean case in Fig. 11. Increasing γ will cause a more complex interplay on both $(n_{\text{th}\parallel} + n_{\text{R}\parallel})/\Delta n$ and $(n_{\text{th}\perp} + n_{\text{R}\perp})/\Delta n$, as is seen in Fig. 12.

VI. CONCLUSION

In this paper, we have investigated the dipolar dirty boson problem, where both the short-range isotropic contact interaction as well as the anisotropic long-range dipole-dipole interaction are present within a Bogoliubov theory at low temperatures below the critical temperature. In the zero-temperature limit, we discussed analytically the nontrivial results for both the condensate and the superfluid depletions as well as the interplay between the long-range anisotropic dipolar interaction and the external random potential which yields an anisotropic superfluidity [37,39]. At finite temperature, we obtained, due to the long-range anisotropic dipole-dipole interaction, different analytical and numerical results for condensate and the superfluid depletions, where the latter depends sensitively on whether the flow direction is parallel or perpendicular to the dipolar polarization axis. It still remains to investigate how the Josephson relation between condensate and superfluid densities reads for a disordered dipolar superfluid [60].

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