

Interplay of Coulomb blockade and Aharonov-Bohm resonances in a Luttinger liquid

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We consider a ring of strongly interacting electrons connected to two external leads by tunnel junctions. By studying the positions of conductance resonances as a function of gate voltage and magnetic flux, the interaction parameter g can be determined experimentally. For a finite ring the minimum conductance is strongly influenced by device geometry and electron-electron interactions. In particular, if the tunnel junctions are close to one another, the interaction-related orthogonality catastrophe is suppressed and the valley current is unexpectedly large. [S0163-1829(98)06704-6]

When the size of an electronic system is reduced, a rich variety of new ‘‘mesoscopic’’ phenomena becomes experimentally observable. Some of the new phenomena are essentially classical, owing their existence to the granularity of the electric charge and the system size dependence of various energy scales. The most widely studied example of this type is Coulomb blockade. A different category of mesoscopic phenomena is entirely quantum mechanical, and is due to the fact that the phase-coherence length at low temperatures is comparable to the system size, giving rise to a number of interference effects. A particular example is the existence of a persistent current in the ground state of a mesoscopic ring. In this paper we study the interplay between two mesoscopic phenomena, Coulomb blockade, and Aharonov-Bohm interference, using an exactly solvable model.

We consider a system consisting of a small ring of interacting electrons connected to two noninteracting leads by tunnel junctions. The tunnel junctions are at positions x_L and x_R , respectively. The ring is capacitively coupled to an external gate electrode and may be pierced by a magnetic flux. We consider a small ac voltage applied to the right lead, and wish to evaluate the current at the left junction. A straightforward application of Kubo formula¹ yields the current

$$\langle I_L(t) \rangle = -i \frac{V(t)}{\hbar\Omega} \int_{-\infty}^t dt' e^{-i\Omega(t'-t)} \text{Tr} \{ \hat{\rho}_G [\hat{I}_R(t'), \hat{I}_L(t)] \},$$

where $V(t) = V_0 e^{-i\Omega t}$ is the applied voltage and $\hat{\rho}_G$ the equilibrium density matrix. The quantity on the right-hand side is recognized as the retarded current-current correlation function. It is most readily evaluated in imaginary time.

The Hamiltonians for the leads and the connection between the leads and the ring are given by

$$H_L = \sum_k \epsilon_{kL} c_L^\dagger(k) c_L(k), \tag{1}$$

$$H_{LT} = t_L c_L^\dagger(x_L) \psi(x_L) + t_L^* \psi^\dagger(x_L) c_L(x_L), \tag{2}$$

where c_L are operators on the left lead and ψ are operators on the ring. The Hamiltonians for the right lead are defined

analogously. Thus we take the leads to be noninteracting, and couple them to the ring with tunnel junctions at positions x_L and x_R . To calculate the current-current correlation function we define the generating functional $Z[J_L, J_R] = \text{Tr} \exp(-\beta H - (1/\hbar) \int_0^{\hbar\beta} d\tau [J_L(\tau) I_L(\tau) + J_R(\tau) I_R(\tau)])$, and integrate out the free fermions in the leads. That yields

$$\begin{aligned} Z[J_L, J_R] = & Z_L Z_R \text{Tr} \exp \left\{ -\beta H_{\text{ring}} - \frac{1}{\hbar^2} \int_0^{\hbar\beta} d\tau \right. \\ & \times \int_0^{\hbar\beta} d\tau' \left[|t_L|^2 \left(1 - \frac{ie}{\hbar} J_L(\tau) \right) \left(1 + \frac{ie}{\hbar} J_L(\tau') \right) \right. \\ & \times \psi^\dagger(\tau, x_L) G_L(\tau - \tau'; x_L, x_L) \psi(\tau', x_L) + |t_R|^2 \\ & \times \left(1 + \frac{ie}{\hbar} J_R(\tau) \right) \left(1 - \frac{ie}{\hbar} J_R(\tau') \right) \psi^\dagger(\tau, x_R) \\ & \left. \left. \times G_R(\tau - \tau'; x_R, x_R) \psi(\tau', x_R) \right] \right\}, \tag{3} \end{aligned}$$

where $Z_{L/R}$ and $G_{L/R}(\tau; x, x')$ are the partition functions and free fermion propagators in the leads, and H_{ring} is the Hamiltonian for an isolated ring. The imaginary time-ordered correlation function $\chi(\tau_1 - \tau_2) = -\langle T_\tau [I_L(\tau_1) I_R(\tau_2)] \rangle$ is obtained by differentiating $Z[J_L, J_R]$ with respect to J_L and J_R . To simplify the notation, we introduce the four-operator expectation value $A(\tau_1, \tau, \tau_2, \tau') = A(\tau_1 - \tau_2, \tau_1 - \tau, \tau_2 - \tau')$ $= \langle T_\tau [\psi^\dagger(\tau, x_L) \psi(\tau_1, x_L) \psi^\dagger(\tau_2, x_R) \psi(\tau', x_R)] \rangle$ and its Fourier transform $A(i\omega_n, i\omega'_n, i\omega''_n)$. To the lowest nonzero order in the tunneling matrix elements, χ is given by

$$\begin{aligned} \chi(i\Omega_n) = & \frac{e^2}{\hbar^4} |t_L t_R|^2 \frac{1}{(\hbar\beta)^2} \sum_{i\omega_n, i\omega'_n} \{ A(i\Omega_n, -i\omega_n, -i\omega'_n) \\ & \times [G_L(i\omega_n - i\Omega_n; x_L, x_L) - G_L(i\omega_n; x_L, x_L)] \\ & \times [G_R(i\omega'_n; x_R, x_R) - G_R(i\omega'_n + i\Omega_n; x_R, x_R)] \}. \tag{4} \end{aligned}$$

If we regard the leads as infinite (rather than semi-infinite), the propagators in the leads are easy to evaluate and yield $G_L(i\omega_n; x_L, x_L) = -i(\hbar/2)D_L(\epsilon_F)\text{sgn}(\omega_n)$ where $D_L(\epsilon)$ is the density of states in the left lead.

To evaluate the four-operator product $A(i\omega_n, i\omega'_n, i\omega''_n)$, we must specify the Hamiltonian for the ring. We choose to work with the simplest exactly solvable interacting model, the spinless Luttinger model. In the bosonized form the Hamiltonian reads²

$$H_{\text{ring}} = \frac{\pi\hbar}{2L} \left[\frac{v}{g} (\hat{N} - N_0)^2 + gv(\hat{J} - J_0)^2 \right] + \sum_{q \neq 0} \hbar v |q| b_q^\dagger b_q,$$

where \hat{N} and \hat{J} are zero modes associated with the total charge and total current. Since the numbers of clockwise and counterclockwise moving electrons on the ring must both be integers, the quantum numbers N and J must satisfy $(-1)^N = (-1)^J$. The gate voltage and magnetic flux determine the parameters $N_0 = CV_g/e$ and $J_0 = 2\Phi/\Phi_0$, which in turn determine the ground-state charge and current. The parameter g is a measure of the interaction strength, and equals 1 for noninteracting electrons.³ For future use we also define the shorthand notation $\gamma = \frac{1}{2}(g + g^{-1}) - 1$, which vanishes in the noninteracting limit.

Due to time ordering the exact expression for A is quite complicated, although in principle straightforward. Since we consider only the lowest order in the tunneling Hamiltonian, our analysis is valid only sufficiently far from the conductance resonances. Therefore, we can assume that the ground state is separated from the excited states by an energy gap $\delta\epsilon$ that is larger than $k_B T$. This approximation is basically similar to the one used by Ref. 4 for an interacting ring connected to superconducting leads. We also neglect events with all four imaginary times approximately equal, since their contribution is negligible. That allows us to evaluate A in an approximate fashion, and gives

$$\begin{aligned} A(i\Omega_n, -i\omega_n, -i\omega'_n) \\ \approx \hbar\beta [\delta_{\Omega_n, 0} G(i\omega_n, 0) G(i\omega'_n, 0) \\ - \delta_{\Omega_n, \omega_n - \omega'_n} G(i\omega_n, x_L - x_R) G(i\omega_n - i\Omega_n, x_R - x_L)] \end{aligned}$$

where $G(\tau, x) = -\langle T_\tau [\psi(\tau, x) \psi^\dagger(0, 0)] \rangle$ is the Green's function for interacting electrons on the ring and $G(i\omega_n, x)$ is its Fourier transform with respect to the imaginary time variable τ . For noninteracting electrons we can apply Wick's theorem, and find that this expression is exact. Substituting this into the expression (4) gives, after proper analytic continuations, the dc conductance

$$\begin{aligned} \sigma &\approx \frac{e^2}{h} \frac{|t_L t_R|^2}{\hbar^2} D_L(\epsilon_F) D_R(\epsilon_F) \int_{-\infty}^{\infty} d\omega \left(-\frac{\partial n_F(\omega)}{\partial \omega} \right) \\ &\times G^{\text{ret}}(\omega, x_L - x_R) G^{\text{adv}}(\omega, x_R - x_L) \\ &\approx \frac{e^2}{h} \frac{|t_L t_R|^2}{\hbar^2} D_L(\epsilon_F) D_R(\epsilon_F) |G^{\text{ret}}(\omega = 0, x_L - x_R)|^2, \quad (5) \end{aligned}$$

where the last expression is valid if the system is further than $k_B T$ from a resonance.

This expression can also be understood using a scattering matrix approach regarding the ring as a (complicated) scatterer for the free electrons in the leads as discussed in a specific case by Jagla and Balseiro.⁵ From that point of view our basic approximation is that one scattering event is completed before another one takes place—the approximation breaks down near resonance, when the dwell time for extra electrons in the ring is large.

Now we turn to evaluating the retarded Green's function for interacting electrons in a finite ring at a finite temperature. We use the low-energy expansion $\psi(x) = \psi_+(x) + \psi_-(x)$, where $\psi_\pm(x)$ are clockwise and counterclockwise moving fermions so that $G(\tau, x) = G_{++}(\tau, x) + G_{--}(\tau, x)$. Following Haldane,² the correlation functions can be evaluated exactly and we obtain ($p = \pm 1$) (Ref. 6)

$$\begin{aligned} &\langle \psi_p(x, \tau) \psi_p^\dagger(0) \rangle \\ &= \frac{i}{2L} e^{-ip\pi x/L} e^{(\pi/L)(N_0 g^{-1}v + pJ_0 g v)\tau} \\ &\times \frac{\vartheta_3(ig^{-1}\alpha N_0 - x_N, e^{-2g^{-1}\alpha}) \vartheta_3(ig\alpha p J_0 - x_J, e^{-2g\alpha}) + \vartheta_2(ig^{-1}\alpha N_0 - x_N, e^{-2g^{-1}\alpha}) \vartheta_2(ig\alpha p J_0 - x_J, e^{-2g\alpha})}{\vartheta_3(ig^{-1}\alpha N_0, e^{-2g^{-1}\alpha}) \vartheta_3(ig\alpha p J_0, e^{-2g\alpha}) + \vartheta_2(ig^{-1}\alpha N_0, e^{-2g^{-1}\alpha}) \vartheta_2(ig\alpha p J_0, e^{-2g\alpha})} \\ &\times \frac{1}{\vartheta_1\left(\frac{\pi(iv\tau - px)}{L}, e^{-\alpha}\right)} \left| \frac{(a/L)}{2\vartheta_1\left(\frac{\pi(iv\tau - px)}{L}, e^{-\alpha}\right)} \right|^\gamma |\vartheta_1'(0, e^{-\alpha})|^{\gamma+1}, \quad (6) \end{aligned}$$

where $\gamma = \frac{1}{2}(g + g^{-1}) - 1$ and we introduced the shorthand notation $\alpha = \pi v \hbar \beta / L$, $x_N = (\pi/L)(ig^{-1}v\tau - px)$, and $x_J = (\pi/L)(igv\tau - px)$. Here a is a short distance cutoff for the interaction, and is of the order of the lattice spacing. The Jacobi theta functions⁷ ϑ_3 and ϑ_2 arise from the $q=0$

modes, with N and J both even or odd, respectively, and ϑ_1 arises from the bosons with $q \neq 0$. The appearance of doubly periodic elliptic functions is natural, since the Green's function must be periodic in x and antiperiodic in τ . The Jacobi functions also appear in the partition function of an isolated

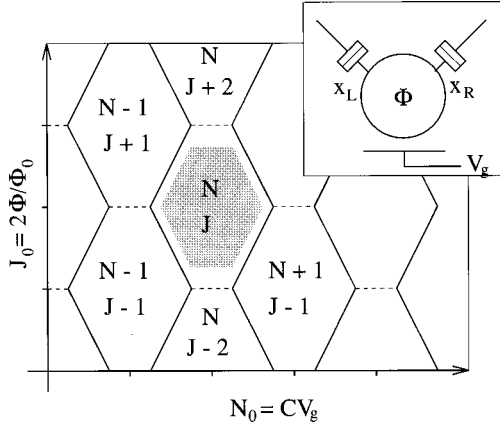


FIG. 1. Positions of conductance resonances in the (V_g, Φ) plane for interacting electrons (repulsive interactions, $g = 1/\sqrt{2}$). The labels N and J denote the ground-state charge and current as a function of the external parameters, and the shaded area indicates the domain of validity of our analysis. The line segments with different slopes correspond to fluctuations in the numbers of clockwise and counterclockwise moving electrons, respectively. Inset: device geometry.

ring (essentially the x - and τ -independent parts of the above expression) and therefore in most equilibrium properties of mesoscopic rings like persistent currents.⁸

The parameters that are most readily accessible in an experiment are the gate voltage and the magnetic flux. They enter only the $q=0$ part of the Hamiltonian which we can rewrite as $H_0 = \frac{1}{2}E_c(\hat{N} - N_0)^2 + (\pi\hbar v_F/2L)(\hat{J} - J_0)^2$, where $v_F = gv$ is the Fermi velocity of a noninteracting system with the same density, and $E_c = \pi\hbar v_F/g^2L$ is the charging energy. The conductance resonances correspond to values of the gate voltage and magnetic flux at which the ground-state quantum numbers N and J change (degenerate ground state). Hence the positions of conductance resonances can be determined from a simple charging energy model with a single-particle Aharonov-Bohm term—note, however, that E_c is not simply given by the geometric capacitance (it is nonzero even for noninteracting electrons). In the (V_g, Φ) plane the resonance positions form a network, the shape of which depends on the interaction parameter g . We suggest therefore that the interaction parameter can be experimentally measured by studying the trajectories of conductance maxima as a function of the gate voltage and magnetic flux. For noninteracting systems the resonance positions form a lattice of diamond-shaped parallelograms, whereas for repulsive interactions ($g < 1$) there are some values of the gate voltage such that a resonance condition is not met for any Φ , as indicated in Fig. 1. For attractive interactions, there are ranges of Φ such that the total current in the ring, J , is independent of V_g . In that case electrons can tunnel into and out of the ring only as pairs of clockwise and counterclockwise movers, which is reminiscent of Cooper pair tunneling through a superconducting grain.⁹ From now on we only consider repulsive interactions.

We use expressions (5) and (6) to analyze the dependence of the conductance on the external parameters. The dependences on V_g and Φ are qualitatively similar, and in Fig. 2

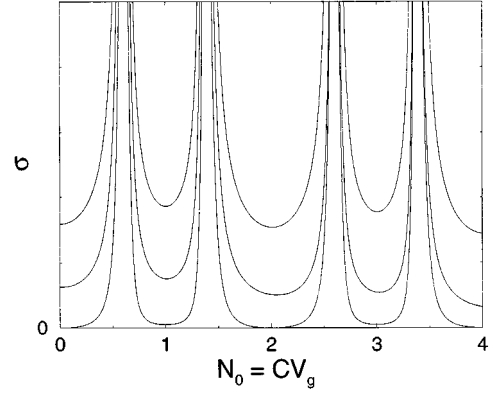


FIG. 2. Conductance vs V_g for $x_L - x_R = 0.02L$, $x_L - x_R = 0.05L$ and $x_L - x_R = 0.5L$ (from top to bottom). The temperature is $T = 0.1(\hbar v/L)$, and the interaction parameter is $g = 12$.

we show the conductance as a function of V_g . Near a resonance we obtain the limiting behavior $|G^{\text{ret}}(0, x)|^2 \sim C/(\delta\epsilon)^2$, where $\delta\epsilon$ is the energy cost of changing the number of electrons in the ring by one (we still assume $\delta\epsilon > k_B T$). In a generic case a resonance corresponds to a degeneracy for the addition or removal of either a clockwise or counterclockwise moving particle, and the prefactor C is independent of device geometry. However, since we have two control parameters V_g and Φ , we can use them to bring both clockwise and counterclockwise modes to resonance simultaneously. These special parameter values correspond to slope changes in the trajectories of conductance resonances in Fig. 1. Since at these double resonances the Green's function receives significant contributions from wave vectors k_{F+} and $-k_{F-}$, its absolute square has components with wave vector $k_{F+} - (-k_{F-}) = 2\pi N/L$. Thus we find that near a double resonance the amplitude of the conductance maximum varies periodically with the separation between the tunnel junctions as $(1 + \cos[(2\pi N/L)(x_L - x_R)])$. Away from a double resonance one channel dominates, and the amplitude of these oscillations is exponentially small. Since the wave vector of these oscillations is essentially $2k_F$, they can be observed only in low-density systems or using local probes like scanning tunneling microscopy. The interference effects are smeared out by temperature which leads to a different temperature dependence of the conductance for different device geometries near a double resonance: if $N|x_L - x_R| \approx nL$, where n is an integer, the conductance decreases with temperature due to reduced interference, whereas for $N|x_L - x_R| \approx (n + \frac{1}{2})L$ the conductance increases with T .

From expression (6) we see that the conductance is reduced due to interactions by a factor $(a/L)^{2\gamma}$ which can be attributed to an orthogonality catastrophe that was previously studied in the context of quantum dots in the fractional quantum Hall regime.¹⁰ The exponent governing the resonance line shape for small $\delta\epsilon$, $\sigma \sim 1/(\delta\epsilon)^2$, is independent of the interaction parameter g , which is a consequence of a finite minimum energy of the bosonic modes. The resonant contribution dominates for $\delta\epsilon \lesssim \delta\epsilon_c = (2\pi\hbar v/L)|\sin[\pi(x_L - x_R)/L]|^\gamma$ (up to logarithmic correc-

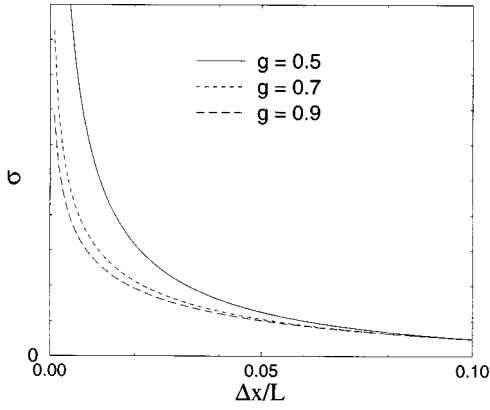


FIG. 3. Conductance vs $|x_L - x_R|$ near a conductance minimum for interaction parameters $g = 0.9, 0.7,$ and 0.5 . The temperature is $T = 0.1(\hbar v/L)$.

tions); for $\delta\epsilon \gg \delta\epsilon_c$, the valley conductance levels off to a constant value proportional to $(a/|x_L - x_R|)^{2\gamma}$, as seen in Fig. 3. For large separations $\Delta x = |x_L - x_R|$, the crossover point $\delta\epsilon_c$ exceeds half of the resonance spacing, and the crossover is not observed. The two limiting behaviors can be combined to give the approximate line shape

$$\sigma(\delta\epsilon) \sim \Gamma_L \Gamma_R \left[\frac{1}{(\delta\epsilon)^2} \left(\sqrt{\frac{a}{L}} \right)^{2\gamma} + \left(\frac{L}{2\pi\hbar v} \right)^2 \left| \frac{L}{\sqrt{2}a} \sin\left(\frac{\pi\Delta x}{L} \right) \right|^{-2\gamma} \times \left(\frac{1 - e^{\gamma \ln|\sin(\pi\Delta x/L)|}}{\gamma} \right)^2 \right], \quad (7)$$

where $\gamma = \frac{1}{2}(g + g^{-1}) - 1$ and $\Gamma_{L/R} = |t_{L/R}|^2 D_{L/R}(\epsilon_F)$ are the linewidths for a noninteracting system. The last factor gives rise to a logarithmic dependence on Δx in the noninteracting limit.

At $T=0$ the smallest $\delta\epsilon$ that we can consider is determined by when terms that are higher order in the tunneling Hamiltonian become significant, due to the orthogonality catastrophe that happens when the first term in Eq. (7) is of

order unity. Therefore, we expect that at $T=0$ the height of the resonance peak is independent of g , and the peak width is given by $\delta\epsilon$ such that $\Gamma_L \Gamma_R [\sqrt{2}(a/L)]^{2\gamma} (\delta\epsilon)^{-2} \approx 1$. A simple Breit-Wigner formula $\sigma \sim \Gamma^2 / (\delta\epsilon)^2 + \Gamma^2$ gives a peak-to-valley ratio $2(\Gamma/\Delta)^2$, where Δ is the separation between adjacent resonances and Γ is the resonance width. In the present case this simple connection does not hold: the line width at $T=0$ is reduced by a factor $(a/L)^\gamma$, whereas the valley current (for small Δx) is only suppressed by factor $(a/\Delta x)^{2\gamma}$. The former suppression factor can be identified as the lifetime of a charged excitation of the ring while the second one is the off-resonance probability of transmission through the ring. The valley current is therefore anomalously large for small Δx , since effectively the system size is replaced by Δx and the orthogonality catastrophe is less severe. Since at $T=0$ the peak widths are reduced by interactions, the resonance peak heights at a finite temperature are suppressed due to thermal broadening. Thus the main effect of interactions is to change the peak-to-valley ratio in a way that depends on device geometry and temperature.

The experimental possibilities for the study of nanostructures like the one we consider are developing rapidly. New techniques like conducting organic molecules and carbon compounds are emerging to complement the conventional semiconductor structures. In particular, it was recently demonstrated¹¹ that carbon nanotubes exhibit coherent electron transport, and can be used to fabricate nanoscale ring structures. We believe these devices can be used to study the system we have analyzed experimentally.

In conclusion, we have considered tunneling through a finite strongly interacting system within the framework of an exactly solvable model. We find that the positions of conductance resonances in the (V_g, Φ) plane can be used to determine the interaction parameter g . We conclude that at $T=0$ the heights of resonance peaks are unaffected by interactions, but, due to the narrowness of $T=0$ resonances, the peak conductance at a finite temperature is reduced by interactions. The valley current depends on both interactions and the device geometry. Near a double resonance we find that the heights of resonance peaks depend on device geometry due to interference between different current carrying processes.

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