Supplemental material for
Non-equilibrium Floquet steady states of time-periodic driven Luttinger liquids

S. Fazzini,[1] P. Chudzinski,[2] C. Dauer,[1] I. Schneider,[1] and S. Eggert[1]

[1] *Physics Department and Research Center OPTIMAS, University of Kaiserslautern, 67663 Kaiserslautern, Germany*

[2] *School of Mathematics and Physics, Queen's University Belfast, Belfast, UK*

Here we give details on the Floquet Bogoliubov transformation, its relation to Floquet theory, the explicit form of the transformed ground state state, density-density correlations, and the application of Floquet's theorem to Mathieu functions.

Relation of the time-dependent transformation to Floquet theory

The goal is to find all possible steady state solutions $|u_n(t)\rangle = |u_n(t+T)\rangle$ under time-periodic driving at each time t, which are defined by the Floquet eigenvalue equation

$$(H - i\partial_t)|u_n(t)\rangle = \epsilon_n|u_n(t)\rangle, \tag{1}$$

where ϵ_n are real quasi energies. It should be noted that it is not always possible to find steady state solutions, but if they exist they form a complete basis in the original Hilbert space. The underlying Floquet theory has been discussed in a number of review articles [1–4], where different approaches are presented: By Fourier transforming into frequency space, the eigenvalue problem becomes static in an extended Hilbert space. Different frequency components can be perturbatively decoupled using a Magnus expansion, which is helpful in defining a so-called Floquet Hamiltonian H_F. The Floquet Hamiltonian is useful since it determines the quasi-energies and the stroboscopic time evolution. The eigenstates of H_F are the steady states $|u_n(0)\rangle$ at one instant in time only, so for the full time evolution it is necessary to additionally know the micromotion operator $U(t) = \sum_n |u_n(t)\rangle\langle u_n(0)|$, which is in general more difficult.

Our novel approach is now to solve the Floquet eigenvalue problem in one single step by mapping it to a static problem in the original Hilbert space

$$\tilde{H}|n\rangle = (QHQ^\dagger - iQ\partial_t Q^\dagger)|n\rangle = \epsilon_n|n\rangle. \tag{2}$$

If solutions to the original problem in Eq. (1) exist the unitary transformation Q can formally always be written as

$$Q(t) = \sum_n |n\rangle\langle u_n(t)|, \tag{3}$$

which transforms the entire basis of steady state solutions at each time into a diagonal static basis. This new transformation Q therefore does three things at once: It maps the system to a static problem in the original Hilbert space, it diagonalizes the eigenvalue problem, and it provides the time-dependent steady states for all times. All this is done without using a Fourier transform into an extended Hilbert space. Needless to say, each of the above steps is normally highly non-trivial, so finding such a transformation Q into a diagonal rotating frame is very ambitious indeed. Note, that $Q(t) = Q(t+T)$ is time periodic, but we need not assume that $Q(t)$ becomes the identity at the initial time or any other time.

The operator Q must therefore not be confused with the time-evolution operator W

$$W(t) = \sum_n |u_n(t)\rangle\langle u_n(0)|e^{-i\epsilon_n t} = U(t)e^{-iH_F t}, \tag{4}$$

which can be used to study the time-dependence of a given initial state. In particular, knowing the time evolution cannot be used to construct Q, but the time evolution can always be expressed as

$$W(t) = Q^\dagger(t)e^{-i\tilde{H}t}Q(0). \tag{5}$$

Moreover, the Floquet Hamiltonian can be obtained by $H_F = Q^\dagger(0)\tilde{H}Q(0)$, but again just knowing H_F cannot be used to extract the steady states for all times unless Q is known. Finally, also the micromotion operator $U(t) = Q^\dagger(t)Q(0)$ and all steady states $|u_n(t)\rangle = Q^\dagger(t)|n\rangle$ can be obtained with Q, so such a transformation truely contains a complete solution of the many-body driven system.

Explicit form of the Floquet Bogoliubov transformation

The model of interest can conveniently be expressed in terms of SU(1,1) generators

$$H(t) = \lambda_1 2J_0 + \lambda_2(J_+ + J_-), \tag{6}$$

where

$$2J_0 = b_L^\dagger b_L + b_R b_R^\dagger, \quad J_+ = J_-^\dagger = b_L^\dagger b_R^\dagger, \tag{7}$$

and $\lambda_1 = v_F q(1+g_4)$ and $\lambda_2 = v_F q g_2$ are the time-periodic coupling parameters. For the static case it is known that the transformation $U_1 = e^{r(J_+ - J_-)}$ can be used for diagonalization, using the following relations for transformed operators $\tilde{\Lambda} = U_1 \Lambda U_1^\dagger$ [5–7]

$$\tilde{b}_R = b_R \cosh r - b_L^\dagger \sinh r \quad (8)$$
$$\tilde{b}_L = b_L \cosh r - b_R^\dagger \sinh r \quad (9)$$
$$\tilde{J}_0 = J_0 \cosh 2r - \frac{J_+ + J_-}{2} \sinh 2r \quad (10)$$
$$\tilde{J}_\pm = -J_0 \sinh 2r + \frac{J_+ + J_-}{2} \cosh 2r \pm \frac{J_+ - J_-}{2} \quad (11)$$
$$\tilde{J}_+ + \tilde{J}_- = -2J_0 \sinh 2r + (J_+ + J_-) \cosh 2r \quad (12)$$

For the time-dependent transformation, we need a more general ansatz parametrized in terms of three real time-periodic parameters θ, ϕ, r

$$Q(t) = e^{i\theta J_0} e^{r(J_+ - J_-)} e^{-i\phi J_0} \quad (13)$$
$$Q^\dagger = Q^\dagger = e^{i\phi J_0} e^{-r(J_+ - J_-)} e^{-i\theta J_0}. \quad (14)$$

Using relations Eqs. (8)-(12) together with gauge transformations, we find that the general time-dependent Bogoliubov transformation can be written as

$$\beta_\chi = Q^\dagger b_\chi Q = \gamma_1 b_\chi + \gamma_2 b_{\bar{\chi}}^\dagger \quad (15)$$
$$Q b_\chi Q^\dagger = \gamma_1^* b_\chi - \gamma_2 b_{\bar{\chi}}^\dagger \quad (16)$$

with $\chi = L, R$ and

$$\gamma_1 = e^{i(\theta - \phi)/2} \cosh r \quad (17)$$
$$\gamma_2 = e^{i(\theta + \phi)/2} \sinh r \quad (18)$$

With this parametrization the transformed operators $\tilde{\Lambda} = Q \Lambda Q^\dagger$ can again be straightforwardly derived from Eqs. (15)-(18)

$$\tilde{J}_0 = \cosh 2r J_0 - \tfrac{1}{2} \sinh 2r (e^{i\theta} J_+ + h.c.) \quad (19)$$
$$\tilde{J}_+ + \tilde{J}_- = -2 \cos\phi \sinh 2r \; J_0 + \quad (20)$$
$$\left[(\cos\phi \cosh 2r - i \sin\phi) e^{i\theta} J_+ + h.c.\right] \quad (21)$$
$$-iQ\partial_t Q^\dagger = (-\dot{\theta} + \dot{\phi} \cosh 2r) J_0 \quad (22)$$
$$+ \left[(i\dot{r} - \tfrac{\dot{\phi}}{2} \sinh 2r) e^{i\theta} J_+ + h.c.\right] \quad (23)$$

Note, that the three real parameters θ, ϕ, r give a general one-to-one parametrization of the complex functions γ_1 and γ_2 which obey $|\gamma_1|^2 - |\gamma_2|^2 = 1$. The functions γ_1 and γ_2 have been extensively discussed in the paper so the transformation Q is already explicitly known, but what is left to show in the following is that the Hamiltonian in Eq. (6) indeed becomes static and diagonal when using those functions.

The defining differential equation is given in Eq. (9) of the paper in terms of γ_1 and γ_2

$$i\dot{\gamma}_1 = (\Delta - \lambda_1)\gamma_1 + \lambda_2 \gamma_2 \quad (24)$$
$$i\dot{\gamma}_2 = (\Delta + \lambda_1)\gamma_2 - \lambda_2 \gamma_1 \quad (25)$$

where Δ is a real constant which is fixed by the constraint that both γ_1 and γ_2 are periodic as discussed in the paper. In terms of the parametrization θ, ϕ, r, the differential equations become after multiplying by $\exp(-i\frac{\theta \pm \phi}{2})$ respectively

$$i\dot{r} \sinh r - \frac{\dot{\theta} - \dot{\phi}}{2} \cosh r = (\Delta - \lambda_1) \cosh r + \lambda_2 e^{i\phi} \sinh r$$
$$i\dot{r} \cosh r - \frac{\dot{\theta} + \dot{\phi}}{2} \sinh r = (\Delta + \lambda_1) \sinh r - \lambda_2 e^{-i\phi} \cosh r$$

The imaginary parts of both equations give the same relation

$$\dot{r} = \lambda_2 \sin\phi \quad (26)$$

The real parts give

$$0 = (\Delta + \dot{\theta}/2 - \lambda_1 - \dot{\phi}/2) \cosh r + \lambda_2 \cos\phi \sinh r \quad (27)$$
$$0 = (\Delta + \dot{\theta}/2 + \lambda_1 + \dot{\phi}/2) \sinh r - \lambda_2 \cos\phi \cosh r \quad (28)$$

For later use we take $(27)\times \sinh r - (28)\times \cosh r$, which gives

$$0 = -(\lambda_1 + \dot{\phi}/2) \sinh 2r + \lambda_2 \cos\phi \cosh 2r \quad (29)$$

Likewise $(28)\times \sinh r - (27)\times \cosh r$ gives

$$\Delta = -\dot{\theta}/2 + (\lambda_1 + \dot{\phi}/2) \cosh 2r - \lambda_2 \cos\phi \sinh 2r \quad (30)$$

We now turn to identify the different parts in the transformed Hamiltonian

$$\tilde{H} = QHQ^\dagger - iQ\partial_t Q^\dagger \quad (31)$$

Collecting all the terms of $\tilde{H}$ from Eqs. (19)-(23) we find that the prefactor of the diagonal part $2J_0$ reads

$$(\lambda_1 + \tfrac{\dot{\phi}}{2}) \cosh 2r - \lambda_2 \cos\phi \sinh 2r - \tfrac{\dot{\theta}}{2} \quad (32)$$

which is exactly Δ according to Eq. (30) and therefore time-independent. The prefactor of the off-diagonal part $e^{i\theta} J_+$ is given by

$$-\lambda_1 \sinh 2r + \lambda_2 (\cos\phi \cosh 2r - i \sin\phi) + i\dot{r} - \tfrac{\dot{\phi}}{2} \sinh 2r. \quad (33)$$

Using Eq. (26) for the imaginary part and Eq. (29) for the real part, we see that this expression is indeed zero, so that we have shown that the model in Eq. (6) transforms to

$$\tilde{H} = QHQ^\dagger - iQ\partial_t Q^\dagger = 2\Delta J_0 = \Delta(b_L^\dagger b_L + b_R b_R^\dagger) \quad (34)$$

where the constant Δ is determined by the constraint of periodicity and Floquet's theorem as described in the text.

The transformed ground state

We give an explicit expression of the transformed ground state $|u_0(t)\rangle = Q^\dagger |0\rangle$ and show that it indeed satisfies the condition

$$\beta_{L,R}(t) |u_0(t)\rangle = 0 \;\; \forall t. \quad (35)$$

With Eq. (14) the calculation of $Q^\dagger|0\rangle$ is split into three steps, one for each operator exponential. As $|0\rangle$ is an eigenstate of J_0, the first step yields $e^{-i\theta J_0}|0\rangle = e^{-i\theta/2}|0\rangle$. Using the relation [6]

$$e^{-r(J_+ - J_-)} = e^{-\tanh(r)J_+}e^{-2\ln(\cosh(r))J_0}e^{\tanh(r)J_-} \quad (36)$$

and $J_-|0\rangle = 0$, we find as an intermediate result

$$Q^\dagger|0\rangle = e^{-i\theta/2}e^{i\phi J_0}e^{-\tanh(r)J_+}e^{-\ln(\cosh(r))}|0\rangle, \quad (37)$$

With the definition of γ_1 and γ_2 in Eqs. (17) and (18) we further simplify $e^{-\ln(\cosh(r))} = 1/|\gamma_1|$ and $\tanh(r) = |\gamma_2|/|\gamma_1|$. The action of the last part of the transformation is found to be

$$e^{i\phi J_0}e^{-|\gamma_2|/|\gamma_1|\ J_+}|0\rangle = e^{i\phi/2}\sum_{n=0}^{\infty}(-|\gamma_2|/|\gamma_1|e^{i\phi})^n|n\rangle_{\mathrm{L}}|n\rangle_{\mathrm{R}}. \quad (38)$$

With $e^{i\phi}|\gamma_2|/|\gamma_1| = \gamma_2/\gamma_1$ we finally find an explicit expression for the transformed ground state

$$|u_0(t)\rangle = \frac{1}{\gamma_1}e^{-\frac{\gamma_2}{\gamma_1}b_L^\dagger b_R^\dagger}|0\rangle. \quad (39)$$

It is important to note that while the form of state (39) is similar to the results of a static Bogoliubov transformation [7] here all parameters are time-dependent. Using the transformation Q the state (39) solves the Floquet Eq. (5) in the main article with $\epsilon_0 = \Delta$. Moreover, we can show explicitly that the transformed ground state $|u_0(t)\rangle$ obeys condition Eq. (35) by applying $\beta_L(t) = \gamma_1(t)b_L + \gamma_2(t)b_R^\dagger$ to Eq. (39), which reads

$$\beta_L(t)|u_0(t)\rangle = \frac{1}{\gamma_1}\sum_{n=0}^{\infty}\left(-\frac{\gamma_2}{\gamma_1}\gamma_1 + \gamma_2\right)\left(-\frac{\gamma_2}{\gamma_1}\right)^n\sqrt{n+1}|n\rangle_L|n+1\rangle_R. \quad (40)$$

As the first bracket in (40) vanishes trivially, the state (39) is indeed the ground state of the $\beta_L(t)$ operator obeying Eq. (35) and analogously also for $\beta_R(t)$. This is an important result, as $|u_0(t)\rangle$ serves as base case for generating the entire set of steady states $|u_n(t)\rangle$ by application of $(\beta_L^\dagger(t))^{n_L}(\beta_R^\dagger(t))^{n_R}$ using Eq. (7) in the main article.

Correlation functions

It is well known how to calculate correlation functions of physical operators in terms of the diagonal boson model $\tilde{H}$ [8–10]. Of particular interest for ultra-cold gases is the density-density correlation, which we will consider here to exemplify the calculation. The fluctuating density is given in terms of the bosonic field $n(x) = \partial_x\phi(x)/\pi$, which has the mode expansion [8–10]

$$\partial_x\phi = \sum_{q>0}\left[\sqrt{\frac{\pi q}{2L}}\,e^{iqx}\left(b_{L,q}^\dagger + b_{R,q}\right) + h.c.\right] \quad (41)$$

For the density-density correlation function we find in the transformed ground state $|u_0(t)\rangle = Q^\dagger(t)|0\rangle$

$$\langle u_0|n(x)n(y)|u_0\rangle = \sum_{q>0}\frac{q}{L\pi}|\gamma_1 + \gamma_2^*|^2\cos q(x-y) \quad (42)$$

where we have used Eq. (15). If the parameters $\gamma_{1,2}$ are constant we recover the known asymptotic powerlaw behavior $\frac{1}{2\pi^2}|\gamma_1 + \gamma_2^*|^2/|x-y|^2$ [8–10]. However, if a resonance $q_\ell = \ell\omega/2\bar{v}$ is part of the linear TLL regime, the parameters $\gamma_{1,2}$ will become very large as discussed in the main article. Therefore, the sum in Eq. (42) will be dominated by the corresponding instability region, leading to a long-range density order of the form

$$\langle u_0|n(x)n(y)|u_0\rangle \propto \cos q_\ell(x-y). \quad (43)$$

Floquet solution in terms of Mathieu functions

The solution of the Mathieu equation

$$\ddot{y}(t) + (a - 2p\cos\omega t)y(t) = 0 \quad (44)$$

is usually discussed in terms of even and odd solutions, known respectively as Mathieu cosine $\mathcal{C}$ and Mathieu sine $\mathcal{S}$ functions. A general solution can be therefore written as

$$y(t) = c_1\mathcal{C}(a,p,\tau) + c_2\mathcal{S}(a,p,\tau)\,, \quad (45)$$

with $\tau = \omega t/2$. Floquet's theorem states that the solutions of a time-periodic differential equation can always be written in the form

$$y(t) = e^{\imath\nu\tau}P_\nu(\tau) \quad (46)$$

with $P_\nu(\tau) = P_\nu(\tau \pm \pi)$. We want to use the quantum number ν, which is commonly referred to as Mathieu characteristic exponent. Therefore, in this section we clarify the relation between the latter and the Mathieu functions. Comparing Eqs. (45) and (46) and employing the periodicity of $P_\nu(\tau)$, we get the following relation

$$c_1\mathcal{C}(a,p,\tau) + c_2\mathcal{S}(a,p,\tau) = e^{\mp\imath\nu\pi}\left(c_1\mathcal{C}(a,p,\tau\pm\pi) + c_2\mathcal{S}(a,p,\tau\pm\pi)\right)\,. \quad (47)$$

Evaluating this expression in $\tau = 0$ and normalizing the Mathieu functions such that $\mathcal{C}(a,p,0) = \mathcal{S}(a,p,\pi) = 1$, we obtain

$$c_1(e^{\pm\imath\pi\nu} - \mathcal{C}(a,p,\pi)) = \pm c_2\mathcal{S}(a,p,\pi) = \pm c_2\,, \quad (48)$$

from which we finally get

$$\cos\pi\nu = \mathcal{C}(a,p,\pi)\,, \qquad \text{and} \\ c_2 = \imath c_1 \sin\pi\nu\,. \tag{49}$$

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