## Lectures:

Monday, 13.5.: Electrons in a periodic potential: field operators, Bloch's Theorem
Thursday, 23.5.: Nearly free electron approximation; Wannier functions.

## Exercises:

All solutions must be handed in by **Tue. 21.5.** noon in box on 5th floor of Building 46 or electronically to laschwar@rptu.de

6) We have defined fermionic creation operators in „k-space“ $\hat{c}^\dagger(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int d^3\vec{r}\; e^{i\vec{k}\cdot\vec{r}}\,\hat{\Psi}^\dagger(\vec{r})$ and the corresponding inverse for the field operators $\hat{\Psi}^\dagger(\vec{r})$.

a) Express the Fourier transformation of the local density operator $\int d^3\vec{r}\; e^{i\vec{k}\cdot\vec{r}}\,\hat{n}(\vec{r})$ using k-space operators, where $\hat{n}(\vec{r}) = \hat{\Psi}^\dagger(\vec{r})\hat{\Psi}(\vec{r})$. What is the "action" in k-space?

b) Express the momentum operator $\vec{P} = -i\hbar\int d^3\vec{r}\,\hat{\Psi}^\dagger(\vec{r})\vec{\nabla}\hat{\Psi}(\vec{r})$ using k-space operators.

c) At $T=0$ a spherical symmetric Fermi sea is given by

$$\langle \hat{n}(k)\rangle = \left\langle \hat{c}^\dagger(\vec{k})\hat{c}(\vec{k})\right\rangle = \begin{cases} 1 & |\vec{k}| < k_F \\ 0 & |\vec{k}| > k_F \end{cases}$$

Determine the average density $n = \langle \hat{n}(\vec{r})\rangle = \left\langle \hat{\Psi}^\dagger(\vec{r})\hat{\Psi}(\vec{r})\right\rangle$ in this case. Note, that the volume per Fermion $V_s = 1/n$ corresponds to a sphere of radius of $r_s \approx 2.418/k_F$, which is a rough estimate for the average distance in a spinless 3D Fermi gas.

7) Consider the three components of the vector operator defined by the Pauli matrices $\vec{S} = \frac{\hbar}{2}\sum_{\alpha,\beta=\uparrow,\downarrow} b_\alpha^\dagger \vec{\sigma}_{\alpha\beta} b_\beta$ where $b_\alpha^\dagger$ are fermionic or bosonic creation operators for the quantum numbers $\alpha = \uparrow,\downarrow$.

a) Express $S^+ = S^x + iS^y$, $S^- = S^x - iS^y$ and $S^z$ in terms of $b_\uparrow^\dagger$ and $b_\downarrow^\dagger$. Evaluate the commutators $\left[S^z, S^\pm\right]$ and $\left[S^+, S^-\right]$. Do they agree with the usual $\mathrm{SU}(2)$ commutation relations for angular momentum operators?

b) Evaluate the product of two single particle operators to create the many-body operator $\vec{S}\cdot\vec{S} = \frac{1}{2}\left(S^+S^- + S^-S^+\right) + S^zS^z$ (i.e. keep all products of 2 creation and 2 annihilation operators in the expression). Using (anti-) commutation rules, separate the expression into "normal ordered form" plus a single particle operator $S^2$ (normal ordered form means that all the annihilation operators are on the right and the creation operators on left).