## Lectures:

Monday, 15.7.: High temperature expansion: Curie Weiss Susceptibility
Thursday, 18.7.: Mean field theory

## Exercises:

All solutions must be handed in by **Tue. 23.7.** noon in box on 5th floor of Building 46 or electronically to laschwar@rptu.de

22.) The Heisenberg model $H = \sum_{\langle i,j \rangle} J_{ij}\ \boldsymbol{S}_i \cdot \boldsymbol{S}_j$ is rotationally invariant which means that $H$ commutes with the three components of the total spin operator $\boldsymbol{S}_{\text{tot}} = \sum_j \boldsymbol{S}_j$. Therefore $(\boldsymbol{S}_{\text{tot}})^2$ and $S^z_{tot}$ are conserved with quantum numbers $s(s+1)$ and $s_z$, respectively. For a dimer of two coupled spin, the total spin can take on values $s=0$ (singlet) or $s=1$ (triplet), i.e. two coupled spin-1/2 give ½ ⊗ ½ = 0 ⊕ 1. Generally, two coupled spins of size $s_1$ and $s_2$ can be represented by the total quantum numbers $|s_1 - s_2| \leq s \leq |s_1 + s_2|$, i.e. $s_1 \otimes s_2 = |s_1 - s_2| \oplus |s_1 - s_2 + 1| \oplus \ldots \oplus |s_1 + s_2|$ (see also the discussion of the addition of angular momentum in a quantum mechanics text book). Since $-s \leq s_z \leq s$ this can be used to calculate the susceptibility $\chi = \frac{g^2 \mu_B^2}{k_B T} \left\langle (S^z_{tot})^2 \right\rangle$ as discussed in the lecture.

a) Argue that with four coupled spin-1/2 we get ½ ⊗ ½ ⊗ ½ ⊗ ½ = 0 ⊕ 0 ⊕ 1 ⊕ 1 ⊕ 1 ⊕ 2, i.e. two singlet states, three triplet states and one quintuplet state. Verify that this is consistent with the total dimension 16 of the Hilbert space.

b) Consider the Heisenberg Model for four coupled spin-1/2 operators (see drawing below) $H = J\,(\boldsymbol{S}_1 \cdot \boldsymbol{S}_2 + \boldsymbol{S}_2 \cdot \boldsymbol{S}_3 + \boldsymbol{S}_3 \cdot \boldsymbol{S}_4 + \boldsymbol{S}_4 \cdot \boldsymbol{S}_1) + J'\,(\boldsymbol{S}_1 \cdot \boldsymbol{S}_3 + \boldsymbol{S}_2 \cdot \boldsymbol{S}_4)$
Find constants $a,b,c$ so that $H$ can be written as a linear combination of total spin operators $H = a(\boldsymbol{S}_1+\boldsymbol{S}_2+\boldsymbol{S}_3+\boldsymbol{S}_4)^2 + b(\boldsymbol{S}_1 + \boldsymbol{S}_3)^2 + b(\boldsymbol{S}_2 + \boldsymbol{S}_4)^2 + c$
Therefore, the three total spin quantum numbers $s=0,1,2,$ $s_{13}=0,1$ and $s_{24}=0,1$ are conserved in this case.

c) List the sixteen eigenstates in terms of possible quantum numbers $s,$ $s_{13}$, $s_{24}$ and $s_z$ and provide their eigenenergies.

d) Calculate $\chi = \frac{g^2 \mu_B^2}{k_B T} \left\langle (S^z_{tot})^2 \right\rangle$. Show that for $J'=0$ the expression simplifies to $\chi = \frac{g^2 \mu_B^2}{k_B T} \frac{2}{1 + e^{J/k_B T}}$.