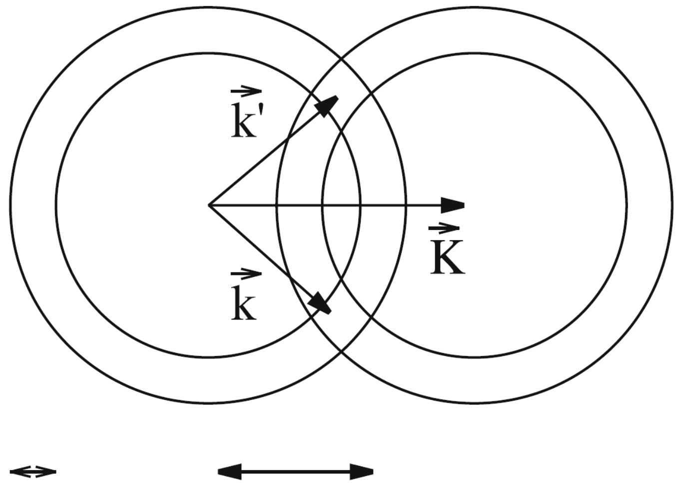
Attractive coupling electron-electron coupling

$$H_{\text{eff}} = H^0 + \frac{\lambda}{2}[H^1, S] = H^0 + \frac{1}{2}\sum_{\vec{q},\vec{k},\vec{k}_1}\sum_{\sigma,\sigma_1} V\hat{\Psi}^\dagger_{\vec{k},\sigma}\hat{\Psi}_{\vec{k}+\vec{q},\sigma}\hat{\Psi}^\dagger_{\vec{k}_1,\sigma_1}\hat{\Psi}_{\vec{k}_1-\vec{q},\sigma_1}$$

$$V = |T(\vec{q})|^2 \frac{2\hbar\omega_{\vec{q}}}{\Delta\varepsilon^2 - (\hbar\omega_{\vec{q}})^2}$$



Cooper pairs

$$H_{\text{int}} = \frac{1}{2}\sum_{\vec{q},\vec{k}_1,\vec{k}_2}\sum_{\sigma_1,\sigma_2} V\hat{\Psi}^\dagger_{\vec{k}_1,\sigma_1}\hat{\Psi}^\dagger_{\vec{k}_2,\sigma_1}\hat{\Psi}_{\vec{k}_2-\vec{q},\sigma_1}\hat{\Psi}_{\vec{k}_1+\vec{q},\sigma_1}$$

$$\left|\text{Cooper}\right\rangle = \sum_{k_F<k<k_F+\Lambda} u(\vec{k})\hat{\Psi}^\dagger_{\vec{k}\uparrow}\hat{\Psi}^\dagger_{-\vec{k}\downarrow}\left|FS\right\rangle$$

$$H\left|\text{Cooper}\right\rangle = E\left|\text{Cooper}\right\rangle$$

Self-consistency equation

$$\left(E - 2\varepsilon(\vec{k})\right)u(\vec{k}) = \sum_{k_F < q < k_F + \Lambda} V u(\vec{q})$$

Gap equation

$$1 = \sum_{k_F < k < k_F + \Lambda} \frac{|V|}{2\varepsilon(\vec{k}) - E}$$

$$\Delta_{\text{Cooper}} = 2\varepsilon_F - E \approx 2\varepsilon_0 \exp\left(-2 / Vg(\varepsilon_F)\right)$$

BCS Hamiltonian

$$H = \sum_{\mathbf{k},\sigma} \varepsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - V \sum_{\mathbf{k},\mathbf{k}'} c^{\dagger}_{\mathbf{k}'\uparrow} c^{\dagger}_{-\mathbf{k}'\downarrow} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}$$

MFT Theory

$$H_{\text{eff}} = \sum_{\mathbf{k},\sigma} \varepsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - V \sum_{\mathbf{k},\mathbf{k}'} \left\langle c^{\dagger}_{\mathbf{k}'\uparrow} c^{\dagger}_{-\mathbf{k}'\downarrow} \right\rangle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} - V \sum_{\mathbf{k},\mathbf{k}'} \left\langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \right\rangle c^{\dagger}_{\mathbf{k}'\uparrow} c^{\dagger}_{-\mathbf{k}'\downarrow} + V \sum_{\mathbf{k},\mathbf{k}'} \left\langle c^{\dagger}_{\mathbf{k}'\uparrow} c^{\dagger}_{-\mathbf{k}'\downarrow} \right\rangle \left\langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \right\rangle$$

Mean field: Order parameter

$$\Delta = V \sum_{\mathbf{k}'} \left\langle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \right\rangle , \quad \Delta^{*} = V \sum_{\mathbf{k}'} \left\langle c^{\dagger}_{\mathbf{k}'\uparrow} c^{\dagger}_{-\mathbf{k}'\downarrow} \right\rangle$$

$$H_{\text{eff}} = \sum_{\mathbf{k},\sigma} \varepsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \Delta^{*} \sum_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} - \Delta \sum_{\mathbf{k}} c^{\dagger}_{\mathbf{k}'\uparrow} c^{\dagger}_{-\mathbf{k}'\downarrow} + \frac{|\Delta|^{2}}{V}$$

Bogoliubov Ansatz

$$\alpha_{\mathbf{k}} = u_{\mathbf{k}} c_{\mathbf{k}\uparrow} - v_{\mathbf{k}} c^{\dagger}_{-\mathbf{k}\downarrow} \quad \alpha^{\dagger}_{\mathbf{k}} = u^{*}_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\uparrow} - v^{*}_{\mathbf{k}} c_{-\mathbf{k}\downarrow}$$

$$\beta_{\mathbf{k}} = u_{\mathbf{k}} c_{-\mathbf{k}\downarrow} + v_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\uparrow} \quad \beta^{\dagger}_{\mathbf{k}} = u^{*}_{\mathbf{k}} c^{\dagger}_{-\mathbf{k}\downarrow} + v^{*}_{\mathbf{k}} c_{\mathbf{k}\uparrow}$$

$$[\alpha_{\mathbf{k}}, \beta_{\mathbf{k}'}]_{+} = \left[\alpha^{\dagger}_{\mathbf{k}}, \beta^{\dagger}_{\mathbf{k}'}\right]_{+} = 0$$

$$\left[\alpha_{\mathbf{k}}, \alpha^{\dagger}_{\mathbf{k}'}\right]_{+} = \left[\beta_{\mathbf{k}}, \beta^{\dagger}_{\mathbf{k}'}\right]_{+} = \left(|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2\right) \delta_{\mathbf{k}\mathbf{k}'}$$

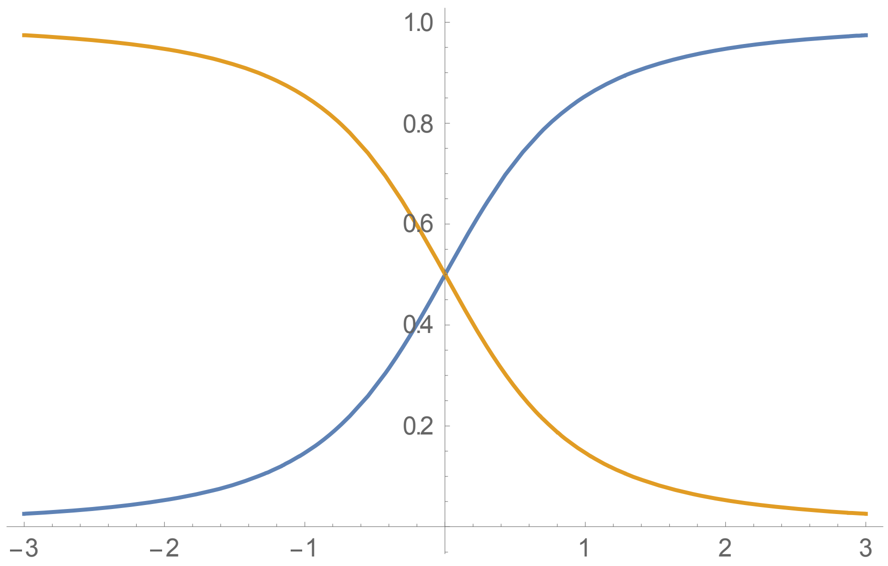
$$\left[\alpha_{\mathbf{k}}, \beta^{\dagger}_{\mathbf{k}'}\right]_{+} = 0$$

$$|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1$$

Bogoliubov Solution

$$u_{\mathbf{k}}^2 = \frac{1}{2}\left(1 + \frac{\varepsilon_{\mathbf{k}}}{\sqrt{\varepsilon_{\mathbf{k}}^2 + |\Delta|^2}}\right), \quad v_{\mathbf{k}}^2 = \frac{1}{2}\left(1 - \frac{\varepsilon_{\mathbf{k}}}{\sqrt{\varepsilon_{\mathbf{k}}^2 + |\Delta|^2}}\right)$$

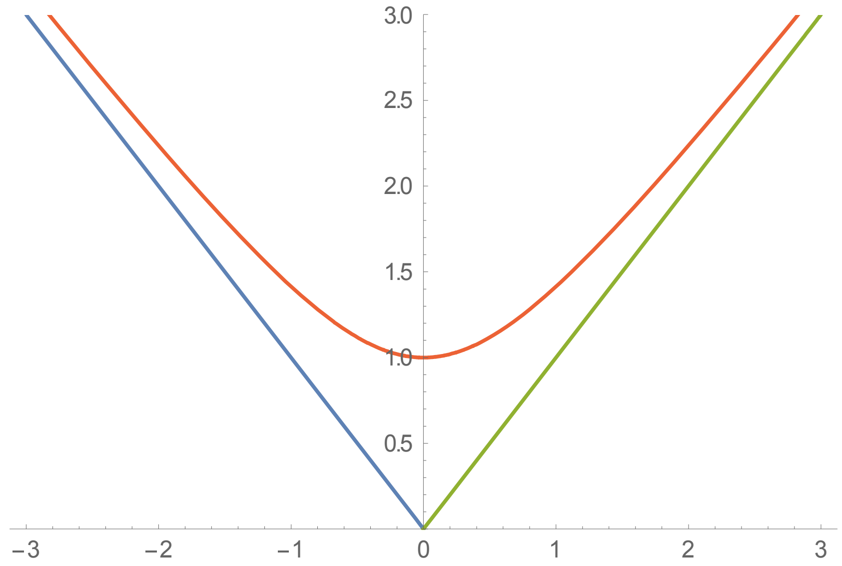
$$u_{\mathbf{k}} v_{\mathbf{k}} = \frac{1}{2}\sqrt{1 - \frac{\varepsilon_{\mathbf{k}}^2}{\varepsilon_{\mathbf{k}}^2 + |\Delta|^2}} = \frac{\Delta}{2\sqrt{\varepsilon_{\mathbf{k}}^2 + |\Delta|^2}}$$



Bogoliubov Dispersion

$$H_{\mathrm{eff}} = \sum_{\mathbf{k}} E_{\mathbf{k}} \left(\alpha^{\dagger}_{\mathbf{k}} \alpha_{\mathbf{k}} + \beta^{\dagger}_{\mathbf{k}} \beta_{\mathbf{k}} \right) + \sum_{\mathbf{k}} (\varepsilon_{\mathbf{k}} - E_{\mathbf{k}}) + \frac{|\Delta|^2}{V}$$

$$E_{\mathbf{k}} = \sqrt{\varepsilon_{\mathbf{k}}^2 + |\Delta|^2}$$



Self-consistency equation

$$\Delta = V \sum_{\mathbf{k}'} \left\langle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \right\rangle_{H_{\text{eff}}}$$

$$\begin{aligned}
\Delta &= V \sum_{\mathbf{k}} \left\langle \left(-v_{\mathbf{k}} \alpha_{\mathbf{k}}^{\dagger} + u_{\mathbf{k}} \beta_{\mathbf{k}} \right) \left(u_{\mathbf{k}} \alpha_{\mathbf{k}} + v_{\mathbf{k}} \beta_{\mathbf{k}}^{\dagger} \right) \right\rangle_{H_{\text{eff}}} \\
&= V \sum_{\mathbf{k}} \left(-v_{\mathbf{k}} u_{\mathbf{k}} \left\langle \alpha_{\mathbf{k}}^{\dagger} \alpha_{\mathbf{k}} \right\rangle_{H_{\text{eff}}} - u_{\mathbf{k}} v_{\mathbf{k}} \left\langle \beta_{\mathbf{k}}^{\dagger} \beta_{\mathbf{k}} \right\rangle_{H_{\text{eff}}} \right. \\
&\qquad \left. + u_{\mathbf{k}} v_{\mathbf{k}} + u_{\mathbf{k}}^{2} \left\langle \beta_{\mathbf{k}} \alpha_{\mathbf{k}} \right\rangle_{H_{\text{eff}}} - v_{\mathbf{k}}^{2} \left\langle \alpha_{\mathbf{k}}^{\dagger} \beta_{\mathbf{k}}^{\dagger} \right\rangle_{H_{\text{eff}}} \right)
\end{aligned}$$

$$\Delta = V \sum_{\mathbf{k}} \frac{\Delta}{\sqrt{\varepsilon_{\mathbf{k}}^{2} + |\Delta|^{2}}} \left(\frac{1}{2} - f(E_{\mathbf{k}}) \right) = \frac{V\Delta}{2} \sum_{\mathbf{k}} \frac{1}{E_{\mathbf{k}}} \tanh \frac{\beta E_{\mathbf{k}}}{2}$$

Determining the gap

$$1 = \frac{V}{2} \sum_{\mathbf{k}} \frac{1}{\sqrt{\varepsilon_{\mathbf{k}}^2 + |\Delta|^2}} \tanh \frac{\beta \sqrt{\varepsilon_{\mathbf{k}}^2 + |\Delta|^2}}{2}$$

$$1 = V\rho_0 \int_0^{\hbar\omega_D} \frac{d\varepsilon}{\sqrt{\varepsilon^2 + \Delta_0^2}} = V\rho_0 \operatorname{arsinh} \frac{\varepsilon}{\Delta_0}\Big|_0^{\hbar\omega_D} = V\rho_0 \operatorname{arsinh} \frac{\hbar\omega_D}{\Delta_0}$$

$$\Delta(T=0) = \Delta_0 = \hbar\omega_D \frac{1}{\sinh \frac{1}{V\rho_0}} \approx 2\hbar\omega_D e^{-\frac{1}{V\rho_0}}$$

Connection of gap, coherence length and superfluid density

$$\lambda_L^2 = \frac{mc^2}{4\pi n_s e^2} \qquad\qquad \xi^2 = \frac{2\pi^2\hbar^2 n_s}{mB_c^2}$$