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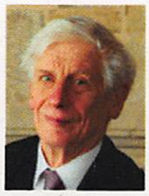
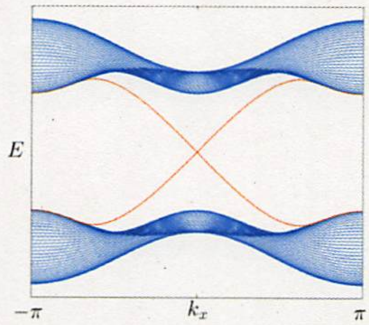
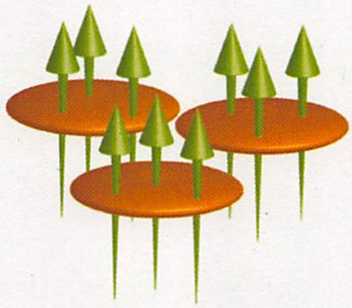


Topological quantum systems
WS 2018/2019 4SWS





Prof. Dr. Michael Fleischhauer

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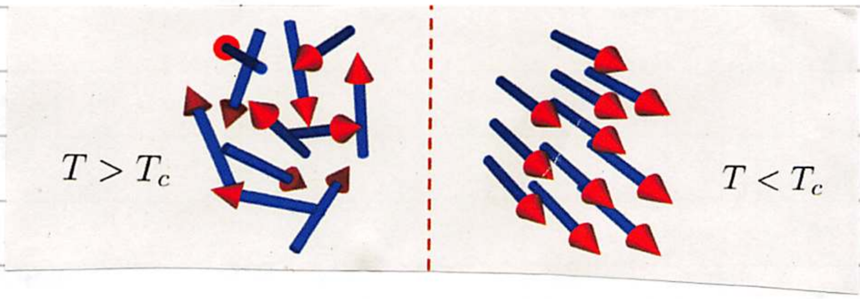
0. Introduction

0.1. Ginzburg-Landau paradigma of phases of matter

Heisenberg model $\quad H = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$

H has $O(3)$ symmetry $\quad D^{-1} H D = H \quad D = \exp(i\theta \vec{n} \cdot \vec{S}_{tot})$

paramagnet $\qquad$ ferromagnet



in ferromagnetic phase ground state breaks spontaneously $O(3)$ symmetry

$\longrightarrow$ spontaneous symmetry breaking

$[\vec{S}_{tot}, H] = 0 \quad$ but $\quad \vec{S}_{tot} |\Psi_0\rangle \neq 0 \quad$ for $T < T_c$

Ginzburg Landau paradigma: different phases of matter (consisting of the same constituents) can be classified by spontaneously broken symmetries

here free energy $\quad F(\vec{M}) = (\partial_t \vec{M})^2 + \alpha_1(T) \vec{M}^2 + \alpha_2(T) \vec{M}^4$

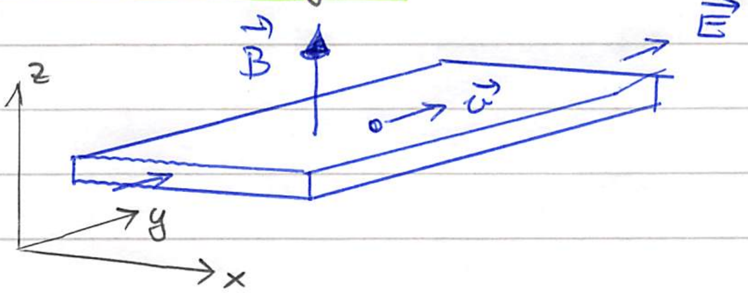
$\alpha_1 > 0 \quad \alpha_2 = \alpha(T - T_c) \quad \alpha > 0$

$F \overset{!}{=} \min \qquad T > T_c \quad \vec{M} = 0 \qquad T < T_c \quad |\vec{M}| = \sqrt{\frac{-\alpha_1}{\alpha_2}} \neq 0$

0.2 Beyond Ginzburg Landau: Quantum Hall effect

(A) classical Hall effect of 2D electron gas

Lorentz force + friction



(1) $\vec{F} = -e\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$

$= m\dot{\vec{v}}$

stationary conditions $\dot{\vec{v}} = 0$

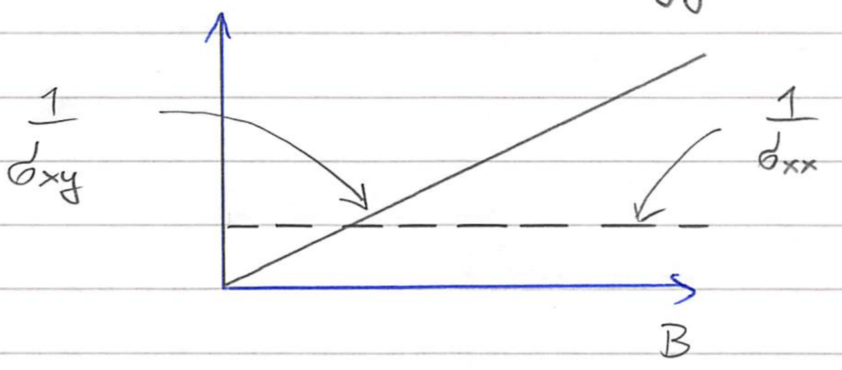
(2) $\vec{E} = -\frac{\vec{v}}{c} \times \vec{B}$ $\qquad \vec{B} = (0,0,B)$ $\qquad \vec{E} = (E_x, E_y, 0)$

current $\qquad \vec{j} = -e\varrho\vec{v}$

(3) $j_x = -\frac{e\varrho c}{B} E_y \qquad j_y = \frac{e\varrho c}{B} E_x$

Hall resistivity

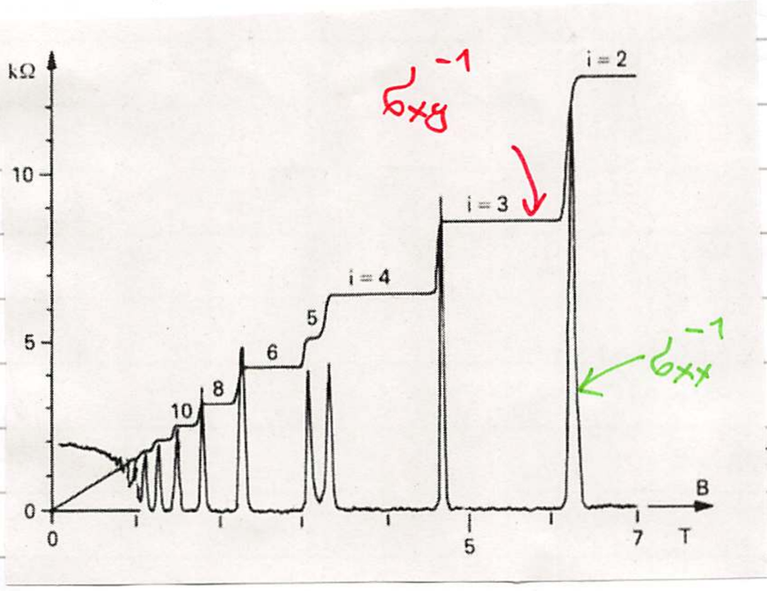
(4) $\vec{j} = \underline{\underline{\sigma}}\, \vec{E} \qquad \sigma_{xy}^{-1} = \frac{E_x}{j_y} = \frac{B}{e\varrho c}$



after adding a damping term $-m\frac{\vec{v}}{\tau}$ to eq (1)

(B) the discoveries of von Klitzing (1980) and Tsui, Störmer and Gossard (1982)

- Klaus von Klitzing: resistivity in high magnetic field at low T



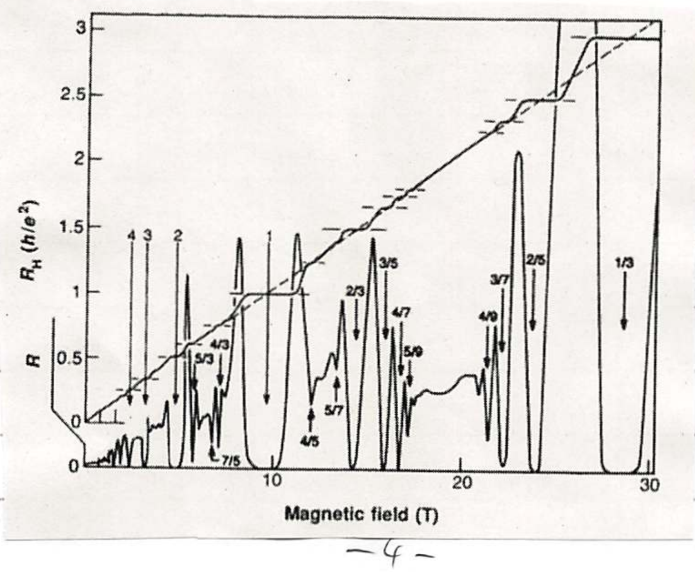
Nobel price 1985

plateaus at

(5) $$\sigma_{xy}^{-1} = \frac{1}{n}\frac{h}{e^2} \qquad \frac{h}{e^2} = 25813{,}807\,\Omega$$

(6) and $$\sigma_{xx}^{-1} \to 0 \qquad \left(\sigma_{xx}^{-1} \sim \exp\left(-\frac{\Delta}{2k_BT}\right) \underset{T \ll \Delta}{\to} 0 \right)$$

- H. L. Störmer, D. C. Tsui, A. G. Gossard experiment; R. Laughlin theory



Nobel prize Störmer, Tsui, Laughlin; 1998 plateaus at fractional numbers

(7) $$\sigma_{xy}^{-1} = \frac{1}{f}\frac{h}{e^2} \qquad f = \frac{p}{q} \text{ rational fractions}$$

$\Rightarrow$ integer (IQHE) and fractional (FQHE) quantum Hall effect

• phenomena robust against disorder

problem: phase transitions but Hamiltonian does not have symmetries that could be spontaneously broken

$$\Rightarrow \boxed{\text{beyond Landau Ginzburg}}$$

phases can be characterized by discrete quantum numbers
we will see that these are

$$\boxed{\text{topological quantum numbers}}$$

How does topology come into the game?
Quantum states have a phase.

0.3. Berry phase

Consider a Hamiltonian that depends smoothly on a parameter $\vec{\lambda}$. $H = H(\vec{\lambda})$. Then lets define "instantaneous" eigenstates which we choose to be changing smoothly (at least localy)

(8) $$H(\lambda)\,|n(\lambda)\rangle = E_n^{(\lambda)}|n(\lambda)\rangle \qquad \lambda \,\hat{=}\, \vec{\lambda}$$

The ground state $|\psi_0(\lambda)\rangle$ is only defined up to a phase $e^{i\phi}$, which is arbitrary.

adiabatic theorem:

If $\lambda = \lambda(t)$ changes sufficiently slowly in time a system prepared at time $t=0$ in the instantaneous ground state $\|\psi(t=0)\rangle = \|\psi_0(\lambda(0))\rangle$ it remains in the ground state at later times t

proof:

(9) $$|\psi(t)\rangle = \sum_n c_n(t) e^{i\alpha_n(t)} |n(\lambda(t))\rangle$$

where

(10) $$\alpha_n(t) = -\frac{1}{\hbar}\int_0^t d\tau\, E_n(\tau)$$

from $$i\hbar \frac{d}{dt}|\psi(t)\rangle = H(\lambda(t))\,|\psi(t)\rangle$$ follows

$$\sum_n \left(\dot{c}_n(t) e^{i\alpha_n(t)} |n(t)\rangle + c_n e^{i\alpha_n(t)} |\dot{n}(t)\rangle\right) = 0$$

(11) $$\dot{c}_m(t) = -\sum_n c_n(t)\, e^{i(\alpha_n - \alpha_m)} \langle m|\dot{n}\rangle$$

$\langle m|\dot{n}\rangle = ?$ $\quad (\langle m|n\rangle = \delta_{mn})$

$$\frac{d}{dt}\Big|\quad H(\lambda)|n(\lambda)\rangle = E_n |n(\lambda)\rangle$$

$$\langle m|\Big|\quad \dot{H}|n\rangle + H|\dot{n}\rangle = \dot{E}_n|n\rangle + E_n|\dot{n}\rangle \qquad |m\rangle \neq |n\rangle$$

$$\langle m|\dot{H}|n\rangle + E_m\langle m|\dot{n}\rangle = E_n\langle m|\dot{n}\rangle$$

(12) $$\langle m|\dot{n}\rangle = \frac{\langle m|\dot{H}|n\rangle}{E_n - E_m}$$

$\Rightarrow$

$$\dot{C}_m(t) = -\sum_{n\neq m} C_n(t)\, e^{i(\alpha_n(t)-\alpha_m(t))} \frac{\langle m|\dot{H}|n\rangle}{E_n - E_m}$$

let system be in eigenstate $|n(t=0)\rangle$ for $t=0$

$$C_n(0)=1 \qquad C_{m\neq n}(0)=0$$

then

$$C_m(\delta t) = -i\hbar \frac{\langle m|\dot{H}|n\rangle}{(E_m - E_n)^2}\left(e^{i(E_m-E_n)\frac{\delta t}{\hbar}} - 1\right)$$

so $|C_m(\delta t)| \ll 1$ if

(13) $$\boxed{\langle m|\dot{H}|n\rangle \ll \frac{|E_m - E_n|^2}{\hbar}} \qquad m\neq n$$

<u>adiabatic condition</u>

characteristic rate of change of hamiltonian must be small compared to energy separation from other states

$\Rightarrow$ systems with energy gap!

We can use the above calculation to determine time evolution of state in adiabatic limit which turns out to be non-trivial. We know from conservation of probability in <u>adiabatic limit</u>

(14) $$C_n(0)=1 \Rightarrow |C_n(t)|=1$$

ansatz $\qquad C_n(t) = e^{i\gamma_n(t)}$

(15) $\qquad \dot{C}_n = i\dot{\gamma}_n C_n(t) \underset{\substack{\uparrow \\ \text{eq.}(11)}}{=} - C_n(t)\langle n|\dot{n}\rangle$

(16) $\qquad \boxed{\gamma_n(t) = i\int_0^t d\tau\, \langle n(\tau)|\frac{d}{d\tau}|n(\tau)\rangle}$ $\qquad$ Berry phase

so the total dynamics reads if $|\psi(t=0)\rangle = |n(t=0)\rangle$

(17) $\qquad |\psi(t)\rangle = e^{i(\alpha_n(t)+\gamma_n(t))}|n(t=0)\rangle$

dynamical phase $\qquad$ Berry phase

Lets reformulate (16) using $\lambda(t) \to \vec{\lambda}(t)$

$$\dot{\gamma}_n = i\langle n(\vec{\lambda})|\vec{\nabla}_{\vec{\lambda}}\, n(\vec{\lambda})\rangle\, \dot{\vec{\lambda}}(t) \qquad \vec{\nabla}_\lambda = \frac{\partial}{\partial\vec{\lambda}}$$

(18) $\qquad \gamma_n(t) = i\int_{\vec{\lambda}_i}^{\vec{\lambda}_f} d\vec{\lambda}\, \langle n(\vec{\lambda})|\frac{\partial}{\partial\vec{\lambda}}|n(\vec{\lambda})\rangle$

The Berry phase depends on gauge transformations,

(19) $\qquad |n(\vec{\lambda})\rangle \longrightarrow e^{i\phi(\vec{\lambda})}|n(\vec{\lambda})\rangle \equiv |\tilde{n}(\vec{\lambda})\rangle$

(20) $\qquad \tilde{\gamma}_n(t) = \gamma_n(t) - \int d\vec{\lambda}\cdot\vec{\nabla}\phi(\lambda)$

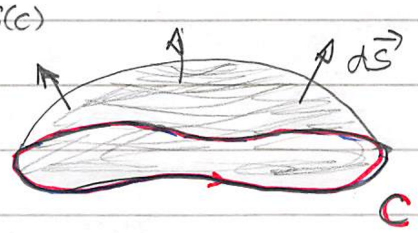
If we choose a closed path, however, i.e. $\vec{R}(0) = \vec{R}(T)$

(21) $$\boxed{\gamma_n = i \oint_C d\vec{R} \; \langle n(\vec{R}) | \vec{\nabla}_R n(\vec{R}) \rangle}$$

is invariant under any smooth gauge trafo

(21) looks like a contour integral over a gradient field.
Assume (for the sake of discussion) $\vec{R} \in \mathbb{R}^3$
Stokes theorem implies

(22) $$\gamma_n = i \iint_{S(C)} d\vec{S} \left(\vec{\nabla}_R \times \langle n | \vec{\nabla}_R n \rangle \right)$$



$\gamma_n \neq 0$ if there is a flux through S

Berry connection (drop index "n" from now on)

(23) $$\boxed{A_j \equiv i \langle n | \partial_j n \rangle = A_j^*}$$ like gauge potential

note $\partial_j \langle n | n \rangle = 0 \Rightarrow \langle \partial_j n | n \rangle = - \langle n | \partial_j n \rangle$

Berry curvature (like magnetic field)

(24) $$\boxed{F_{jk} \equiv i \left(\partial_j A_k - \partial_k A_j \right) = i \left(\partial_j \langle n | \partial_k n \rangle - \partial_k \langle n | \partial_j n \rangle \right)}$$

(25) $$\boxed{B_i = \varepsilon_{ijk} F_{jk} = i\,\varepsilon_{ijk}\left(\partial_j \langle n|\partial_k n\rangle - \partial_k \langle n|\partial_j n\rangle\right)}$$

one easily finds in $\mathbb{R}^3$ $\quad (A_j = A_j^*)$

$$\vec{B} = \mathrm{Im}\left(\vec{\nabla} \times \langle n|\vec{\nabla} n\rangle\right) = \dots$$

$$= \mathrm{Im} \sum_{\substack{m \\ (m \neq n)}} \langle \vec{\nabla} n|m\rangle \times \langle m|\vec{\nabla} n\rangle \qquad \text{with eq (12)}$$

(26) $$\vec{B} = \mathrm{Im} \sum_{m \neq n} \frac{\langle n|\vec{\nabla} H|m\rangle \times \langle m|\vec{\nabla} H|n\rangle}{(E_m - E_n)^2}$$

From this we find the following important result after reintroducing the index n for the Berry phase of state $|n\rangle$. Note that the terms in (26) are antisymmetric in m and n

(27) $$\boxed{\sum_n \gamma_n = 0}$$ <u>Sum of all Berry phases vanishes</u>

So far we have assumed that the state $|n(\lambda)\rangle$ nondegenerate for all values of λ

▶ d-fold degeneracy (also if only at some values λ)

(non-Abelian) <u>Berry connection</u>

d-dimensional subspace: $\{|n_1\rangle, \dots, |n_d\rangle\}$

due to (partial) degeneracy transitions between eigenstates in sub-space possible; for full degeneracy:

$$\begin{bmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ c_d(t) \end{bmatrix} = \underline{\underline{U}}(t) \begin{bmatrix} c_1(0) \\ c_2(0) \\ \vdots \\ c_d(0) \end{bmatrix} = T e^{i\int_0^t d\tau \underline{\underline{A}}(\tau)} \begin{bmatrix} c_1(0) \\ c_2(0) \\ \vdots \\ c_d(0) \end{bmatrix}$$

$A_j \rightarrow \underline{\underline{A}}_j$ j index in space of $\vec{r}$ e.g. $\mathbb{R}^3$

(25) $$\boxed{\underline{\underline{A}}_j \Big|_{ke} = i \langle n_k | \partial_j n_e \rangle}$$ $k, e \in (0, 1, \dots, d)$ $d \times d$ matrix

(26) note $$\underline{\underline{A}}_j = \underline{\underline{A}}_j^+$$

matrix-valued Berry connection

may be non-Abelian if

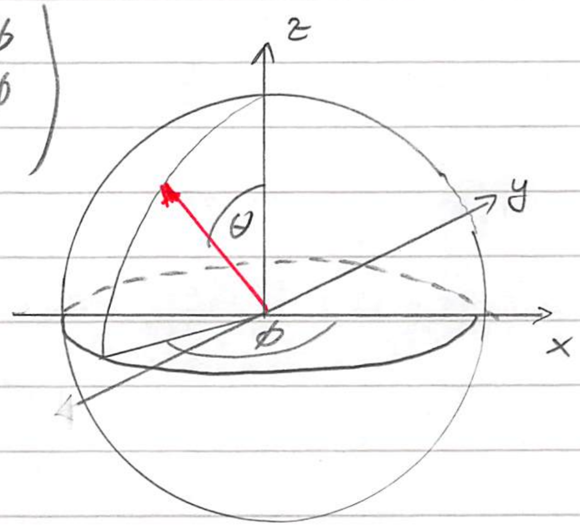
(27) $$\left[\underline{\underline{A}}(\vec{r}_1), \underline{\underline{A}}(\vec{r}_2) \right] \neq 0$$

0.4. Degeneracy points in parameter space Chern number

instructive example $\mathcal{H} = \mathbb{C}^2$

(28) $$H = \vec{h}(\vec{R}) \cdot \vec{\sigma} \qquad \vec{R} \in \mathbb{R}^3$$

(29) $\vec{n}(\vec{R}) = \vec{e}_{\vec{R}} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$



eigenstates

$E_- = -1 \quad |E_-\rangle = \begin{pmatrix} \sin\frac{\theta}{2} e^{-i\phi} \\ -\cos\frac{\theta}{2} \end{pmatrix}$

$E_+ = +1 \quad |E_+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\phi} \\ \sin\frac{\theta}{2} \end{pmatrix}$

- Berry connection for $|E_-\rangle$

$$A_\theta = i\langle E_-|\partial_\theta|E_-\rangle = 0 \qquad A_\phi = i\langle E_-|\partial_\phi|E_-\rangle = \sin^2\frac{\theta}{2}$$

$\vec{A}$ defined everywhere <u>except at south pole</u> since for $\theta = \pi$

$$|E_-\rangle = \begin{pmatrix} e^{-i\phi} \\ 0 \end{pmatrix} \qquad \phi \text{ not defined}$$

- we could choose different gauge, e.g.

$$|E_\pm\rangle' = e^{i\phi}|E_\pm\rangle$$

$$|E_-'\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} e^{-i\phi} \end{pmatrix} \qquad |E_+'\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{-i\phi} \end{pmatrix}$$

$$A_\theta' = 0 \qquad A_\phi' = -\cos^2\frac{\theta}{2}$$

$\vec{A}'$ is undefined at north pole since for $\theta = 0$

$$|E_-'(\theta=0)\rangle = \begin{pmatrix} 0 \\ -e^{i\phi} \end{pmatrix} \qquad \phi \text{ not defined}$$

$\Rightarrow$ for the example (29) we cannot find a gauge where the Berry connection is well defined in the whole parameter space

$\Rightarrow$ (i) $\nexists$ globally smooth gauge!

- Let's now extend the parameter space to the full unit sphere i.e. including the interior $|\vec{R}| \leq 1$

(30) $$\vec{h}(\vec{R}) = \vec{R} \qquad |\vec{R}| \leq 1$$

then

$$\vec{B}_\pm = \mp \frac{1}{2} \frac{\vec{R}}{R^3}$$

$\hat{=}$ monopoles at origin $\vec{R} = 0$

(31) $$\oiint_{O(K)} d\vec{S} \cdot \vec{B}_\pm = \iiint_K dV \, \vec{\nabla} \cdot \vec{B}_\pm = \pm 1$$

(ii) Surface integral of Berry curvature over closed surface with contains origin in its interior is non vanishing

(iii) the energies are degenerate at the origin

$$E_+(\vec{R}=0) = E_-(\vec{R}=0)$$

The three points (i), (ii), (iii) reflect the same physics

general statement

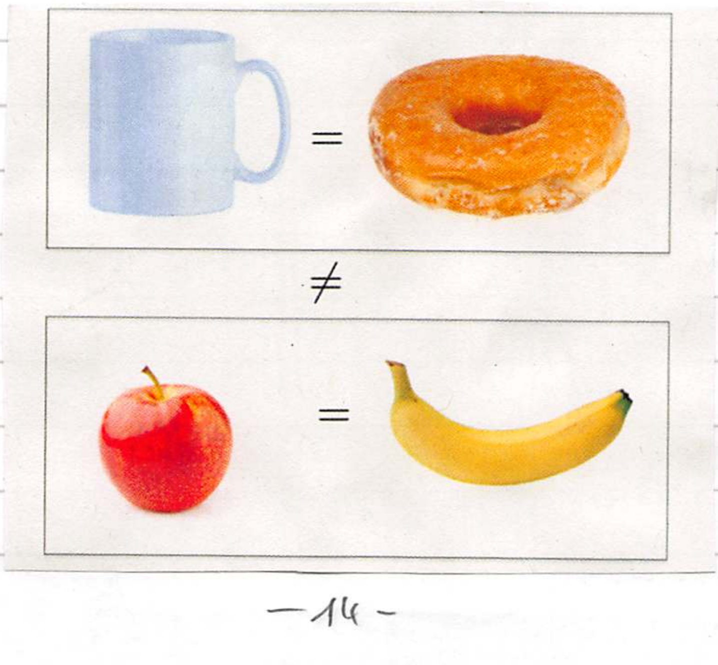
If the spectrum of $H(\vec{R})$ is gapped in a simply connected part of parameter space except for a number of isolated degeneracy points

$$\boxed{\oiint d\vec{R} \cdot \vec{B} = C \qquad C \in \mathbb{Z},} \tag{32}$$

C is called (first) Chern number

0.5 Some elementary notions from topology

"A topologist is a mathematician which cannot distinguish a coffee cup from a donut but from a banana or an apple"



"topology is the mathematical study of the properties that are preserved through (smooth) deformations, twistings and stretching of objects. Tearing, however, is not allowed."

Eric. W. Weisstein "topology"

objects that can be smoothly deformed into each other are called homotop

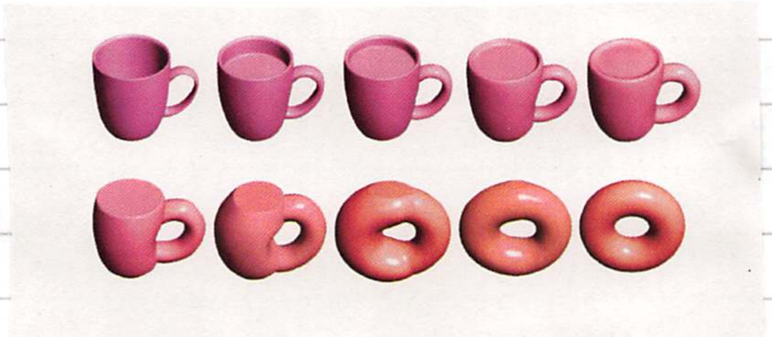
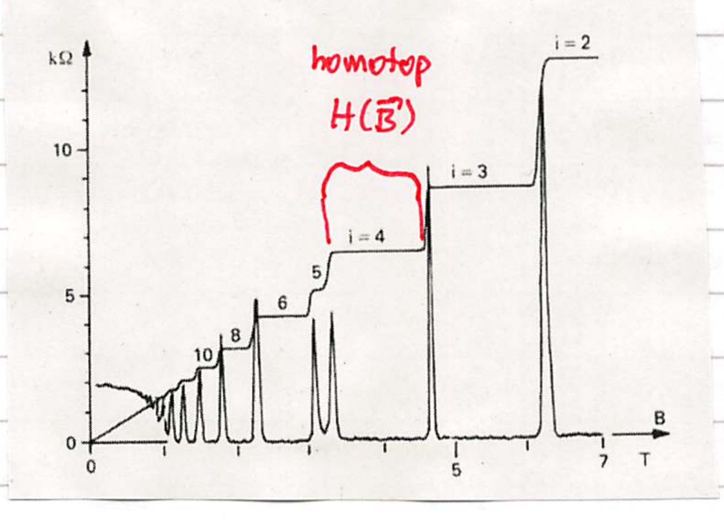
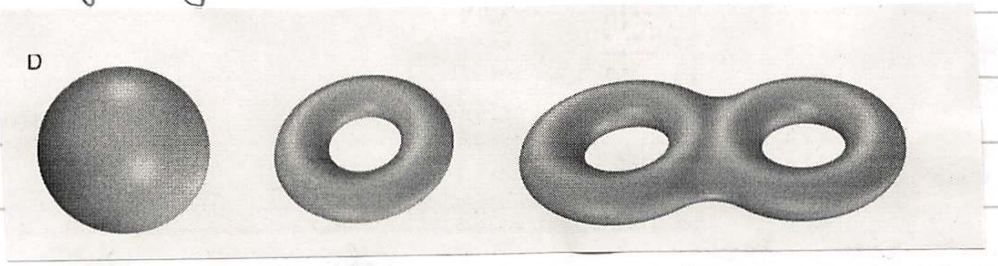


Figure: Homotopy between mug and torus.

we will see that all Hamiltonians of a quantum Hall system for magnetic fields corresponding to one Hall plateau are homotop, i.e. belong to the same topological phase



topologically distinct surfaces in $\mathbb{R}^3$



(a) (b) (c)

surface of a unit sphere

$$(\varphi, \vartheta) \xrightarrow{f} \vec{R} = \begin{pmatrix} \sin\vartheta \cos\varphi \\ \sin\vartheta \sin\varphi \\ \cos\vartheta \end{pmatrix}$$

a **homotopy**
between two functions f and g between topological spaces X and Y is a continuous function $h(\lambda)$ with $\lambda \in [0,1]$ such that

(33) $$h(\lambda=0) = f \qquad h(\lambda=1) = g$$

obviously no such function exists between the surface of the sphere (a) and the torus (b)

example: $f, g : \mathbb{R} \to \mathbb{R}^2$

$$f(x) = (x, x^3) \qquad g(x) = (x, e^x)$$

$h(x, \lambda) = (x, (1-t)\,x^3 + t e^x)$ is a homotopy between $f(x)$ and $g(x)$

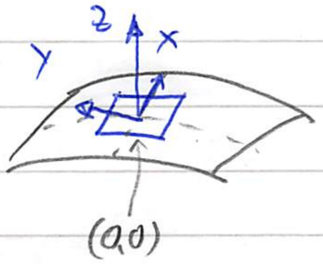
We will see that we can associate **integer numbers** to topological spaces. Since deformations are continuous these numbers should be preserved under deformations

$\Rightarrow$ **topological invariants**

We will see that the phase transition in the QHE is associated to a change of a topological invariant.

<u>example of topological invariant: Gauss-Bonnet theorem</u>

2d compact manifold smoothly embedded in 3d



$$\left(\frac{\partial z}{\partial x}\right)\Big|_{0,0} = \left(\frac{\partial z}{\partial y}\right)\Big|_{0,0} = 0$$

surface in neighborhood of (0,0)

(34) $$z(x,y) \simeq \frac{1}{2}\,(x,y)\underbrace{\begin{pmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial x \partial y} \\ \frac{\partial^2 z}{\partial y \partial x} & \frac{\partial^2 z}{\partial y^2} \end{pmatrix}}_{\text{Hesse matrix } H}\begin{pmatrix} x \\ y \end{pmatrix}$$

eigenvalues of H = inverse radii of curvature

(34) $$\lambda_1, \lambda_2 = r_1^{-1}, r_2^{-1}$$

$$\det(H) = \lambda_1 \lambda_2 \qquad \text{Gaussian curvature}$$

- example: surface of sphere

$$z = r\cos\vartheta \quad x = r\sin\vartheta\cos\varphi \quad y = r\sin\vartheta\sin\varphi$$

$$\Rightarrow \cdots \Rightarrow \qquad \lambda_1 = \lambda_2 = r^{-1}$$

lets average the Gaussian curvature over the whole manifold

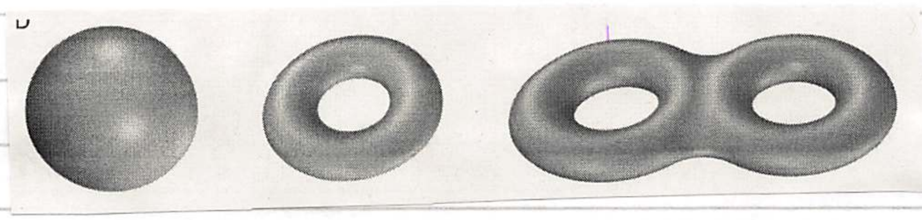
$$\int dA\, \lambda_1\lambda_2 = 4\pi r^2 \frac{1}{r^2} = 4\pi$$

if we would do the same for the torus we would find zero!

Gauß-Bonnet

(35) $$\boxed{\int dA\, \lambda_1\lambda_2 = 2\pi\chi = 2\pi(2-2g)}$$

$$\left.\begin{array}{ll} \chi & \text{Euler characteristics} \\ g & \text{genus} \end{array}\right\} \begin{array}{l} \text{topological} \\ \text{invariants} \end{array} \in \mathbb{Z},$$



$g=0$ $\qquad$ $g=1$ $\qquad$ $g=2$

Chern number as topological invariant

lets come back to our example in section 0.4.

(36) $$H = \vec{h}(\vec{R})\cdot\vec{\sigma} \quad \text{with} \quad \vec{h}(\vec{R}) = \vec{R}$$

$$E_\pm = \pm R \qquad |E_-\rangle = \begin{pmatrix} \sin\frac{\theta}{2} e^{-i\phi} \\ -\cos\theta/2 \end{pmatrix} \quad |E_+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\phi} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

all Hamiltonians are topologically equivalent for arbitrary values of $R > 0$. The topological invariants associated to the space of all $\{|E_+\rangle\}$ or $\{|E_-\rangle\}$ is always $C_\pm = \pm 1$

Now lets change H by adding a shift and $R = 1$

(37) $$\vec{h} \to \vec{h}'(x) = \vec{x} + \vec{e}_R$$

then one finds

(38) $$C_\pm(|\vec{x}| < 1) = \pm 1 \qquad C_\pm(|\vec{x}| > 1) = 0$$

so something happens when $|\vec{x}| = 1$. Here a degeneracy occurs

(39) $$E_+(|\vec{x}| = 1) = E_-(|\vec{x}| = 1)$$

and the Chern number changes. This is what we will call a <u>topological phase transition.</u>