9. Classification of noninteracting topological systems: the tenfold way

See eg. S. Ryu, A. P. Schnyder, A. Furusaki, A. W. Ludwig New J. Phys. 12, 065010 (2010)

generic hamiltonian of non-interacting fermions

- not superconducting (isolators)

(313) $$H = \sum_{A,B} \psi_A^+ h_{AB} \psi_B \qquad \{\psi_A, \psi_B^+\} = \delta_{AB}$$

- superconductor

(314) $$H = \sum_{A,B} (\psi_A^+, \psi_A) \tilde{h}_{AB} \begin{pmatrix} \psi_B^+ \\ \psi_B \end{pmatrix}$$

We have seen (in case of (313)) that these types of Hamiltonians can have topological properties described by $\mathbb{Z}$ or $\mathbb{Z}_2$ quantum numbers.

> Is the existence of these quantum numbers related to some general (symmetry) properties of H?

We know already that local perturbations such as disorder (breaking of trivial translational invariance) or breaking of some trivial symmetries such as inversion do in general not destroy topological properties. Topology is a property of translational non-invariant hamiltonians.

$\Rightarrow$ need classification scheme beyond usual symmetries described by unitary operators U_S with

(315) $$[H, U_S] = 0 \quad \Leftrightarrow \quad U_S H U_S^{-1} = H \qquad (U_S^{-1} = U_S^{+})$$

We already know one generalized symmetry, namely time-reversal $T = UK$, which is anti-unitary

(316) $$THT^{-1} = H \qquad \text{but } T^{-1} \neq T^{+} \text{ in general}$$

9.1. Time reversal for fermions - a short repetition

chapter 4

(317) $$T = e^{-i\pi S_y} K$$

(318) $$T^2 = e^{-2i\pi S_y} = \begin{cases} -1 & \text{odd number} \\ +1 & \text{even number of fermions} \end{cases}$$

transformation of elementary operators in 2nd quantization

(319) $$T c_{j\uparrow} T^{-1} = c_{j\downarrow} \qquad T c_{j\downarrow} T^{-1} = -c_{j\uparrow}$$

(320) $$T \tilde{c}_{k\uparrow} T^{-1} = \tilde{c}_{-k\downarrow} \qquad T \tilde{c}_{k\downarrow} T^{-1} = -\tilde{c}_{-k\downarrow}$$

than $$T H T^{-1} \stackrel{!}{=} H$$

implies

(321) $$\underline{\underline{h}}(\bar{k}) = i\sigma_y^T \, \underline{\underline{h}}^*(-\bar{k}) \, i\underline{\underline{\sigma}}_y$$

9.2. Particle-hole transformation

Now we will get to know another generalized symmetry transformation, the particle-hole transformation.
To motivate the transformation lets consider the BCS hamiltonian of superconductivity. In a nutshell in BCS theory the interaction term is replaced by a mean-field term

(322) $$c^+ c\, c^+ c \longrightarrow \langle c^+ c^+ \rangle c c = \Delta^* c c$$

The resulting mean-field Hamiltonian, the so-called Bogoliubov-de Gennes (BdG) Hamiltonian reads

(323) $$H = \frac{1}{2} \sum_{\bar{k}} \left(\tilde{c}^+(\bar{k}), \tilde{c}(-\bar{k}) \right) \underline{\underline{h}}_{BdG}(\bar{k}) \begin{pmatrix} \tilde{c}(\bar{k}) \\ \tilde{c}^+(-\bar{k}) \end{pmatrix}$$

where

(324) $$\underline{\underline{h}}_{BdG}(\bar{k}) = \begin{pmatrix} h_0 & \Delta \\ -\Delta^* & -h_0 \end{pmatrix}$$

One often calls

(325) $$\Psi(k) = \begin{pmatrix} \tilde{c}(k) \\ \tilde{c}^+(-k) \end{pmatrix}$$ Nambu spinor

One recognizes that the following holds

$$\tilde{c}(\vec{k}) \to \tilde{c}^+(-\vec{k}) \iff H \to -H$$

real-space particle hole transformation

(326) $$\boxed{C\, c_{\mu j}\, C^{-1} \equiv \sum_\nu (U_c^*)_{\mu\nu}\, c^+_{\nu j}}$$

$U_c^+ = U_c^{-1}$ unitary matrix

momentum space

$$C\, \tilde{c}_\mu(k)\, C^{-1} = \frac{1}{\sqrt{N}} \sum_j e^{-\frac{2\pi i}{N} jk}\, C\, c_{\mu j}\, C^{-1}$$

$$= \frac{1}{\sqrt{N}} \sum_{j\nu} e^{-\frac{2\pi i}{N} jk} (U_c^*)_{\mu\nu}\, c^+_{\nu j}$$

$$= \sum_\nu (U_c^*)_{\mu\nu}\, \tilde{c}^+_\nu(-k)$$

(327) $$\boxed{C\, \tilde{c}_\mu(k)\, C^{-1} = \sum_\nu (U_c^*)_{\mu\nu}\, \tilde{c}^+_\nu(-k)}$$

What can we say about U_c?
Two-fold application of C

$$C^2 c_A^+ C^{-2} = \sum_B (U_c)_{AB}\, C\, c_B\, C^{-1} = \sum_B (U_c U_c^*)_{AB}\, c_B^+$$

consider single-particle state $|\psi\rangle = c_A^+ |0\rangle$

$$C^2 |\psi\rangle \stackrel{!}{=} e^{i\varphi} |\psi\rangle = \underbrace{U_c U_c^*}_{=A} |\psi\rangle$$

$$A = U_c U_c^* = U_c (U_c^T)^{-1} \quad \text{is diagonal, i.e.} \quad A \overset{(*)}{=} A^T$$

$$\Rightarrow \quad U_c = A U_c^T \qquad U_c^T = U_c A^T \overset{(*)}{=} U_c A$$

$$U_c = A U_c A \qquad \Rightarrow \quad A = \pm 1$$

so

(328) $$\boxed{C^2 = \pm 1}$$

particle-hole symmetric hamiltonian (2nd quantized)

(329) $$\boxed{CHC^{-1} = H}$$

(330) $$H = \sum_{AB} \psi_A^+ h_{AB} \psi_B$$

- where for non-superconducting systems (creation, annihilation operators of fermions)

$$\psi^+, \psi \;\hat{=}\; \tilde{c}^+(k), \tilde{c}(k)$$

- PHS $\quad CHC^{-1} = H$

$$C \sum_{k,AB} \tilde{c}_A^+(k) h_{AB}(k) \tilde{c}_B(k) C^{-1} = \sum_{AB,k} C\tilde{c}_A^+(k) C^{-1} h_{AB}(k) C \tilde{c}_B(k) C^{-1}$$

$$= \sum_{\substack{kAB\\CD}} U_{AC} \tilde{c}_C(-k) h_{AB}(k) U_{BD}^* \tilde{c}_D^+(-k)$$

$$= -\sum_{kABCD} \tilde{c}_D^+(-k) U_{DB}^+ h_{BA}^T(k) U_{AC} \tilde{c}_C(-k)$$

$$= -\sum_{kABCD} \tilde{c}_D^+(k) \left(U_{DB}^+ h_{BA}^T(-k) U_{AC} \right) \tilde{c}_C(k) \overset{!}{=} \sum_{kCD} \tilde{c}_D^+(k) h_{DC} \psi_c$$

$$(331)\quad \boxed{U_C^\dagger \underline{\underline{h}}^T(k)\, U_C = U_C^\dagger \underline{\underline{h}}^*(k)\, U_C = -\underline{\underline{h}}(-k)}$$

- Superconducting system $\quad \psi(k) = \left(C(k), C^\dagger(-k)\right)^T$

$$H = \sum_k \psi^\dagger(k)\, h_{BdG}(k)\, \psi(k)$$

$$(332)\quad U_C^\dagger\, \underline{\underline{h}}^T_{BdG}(k)\, U_C = -\underline{\underline{h}}_{BdG}(-k)$$

"T" with respect to particle-hole only

9.3 combined particle-hole and time reversal

The combination of PH and TR transformation is called chiral transformation (or sublattice transformation)

$$(333)\quad S = TC$$

For a (second quantized) many-particle Hamiltonian to be chiral symmetric

$$(334)\quad SHS^{-1} = H$$

the single particle Hamiltonian must then obey

$$(335)\quad \boxed{U_T^* U_C^*\, \underline{\underline{h}}\, U_T U_C^* = -\underline{\underline{h}}}$$

$$(336)\quad \boxed{\{\underline{\underline{h}}, \Gamma\} = 0} \qquad \Gamma = U_T U_C^*$$

If either TRS or PHS is given than Chiral Symmetry follows automaticaly. Only if neither TRS nor PHS is given the presence or absence of Chiral Symmetry is relevant (Chiral = Sublattice "SLS")

TRS	PHS	SLS
± 1	$\emptyset$	$\emptyset$
$\emptyset$	± 1	$\emptyset$
± 1	± 1	1
$\emptyset$	$\emptyset$	$0,1$

$\Rightarrow$ 10 symmetry classes

				δ = spatial dimension							
Class	T	C	S	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	+	+	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	+	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	−	+	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	−	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	−	−	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	−	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	+	−	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

example 1 Qi-Wu-Zhang model

(337) $\underline{\underline{h}}(\bar{k}) = \sin k_x \sigma_x + \sin k_y \sigma_y + [u + \cos k_x + \cos k_y]\sigma_z$

- if $\sigma_x, \sigma_y, \sigma_z$ isospin TRS $\Leftrightarrow$ $\underline{\underline{h}}^*(\bar{k}) = \underline{\underline{h}}(-\bar{k})$
  broken by σ_x term

- if $\sigma_x, \sigma_y, \sigma_z$ real spin TRS $\Leftrightarrow$ $\underline{\underline{h}}(k) = \underline{\underline{\sigma}}^y \underline{\underline{h}}(-k)^* \underline{\underline{\sigma}}^y$
  broken by σ_z term

no TRS

- PH-symmetry

(338) $\underline{\underline{h}}^T(\bar{k}) = \sin k_x \sigma_x - \sin k_y \sigma_y + [u + \cos k_x + \cos k_y]\sigma_z$

(339) $-\underline{\underline{h}}(-\bar{k}) = \sin k_x \sigma_x - \sin k_y \sigma_y - [u + \cos k_x + \cos k_y]\sigma_z$

$U_c = \sigma_x$ does not work since $\sigma_x \sigma_y \sigma_x = -\sigma_y$
$U_c = \sigma_y$ does not work since $\sigma_y \sigma_x \sigma_y = -\sigma_x$
$U_c = \sigma_z$ — " — since $\sigma_z \sigma_x \sigma_z = -\sigma_x$ no PHS

- Chiral symmetry

$\underline{\underline{\sigma}}_x \underline{\underline{h}}(\bar{k}) \underline{\underline{\sigma}}_x = \sin k_x \sigma_x - \sin k_y \sigma_y - [u + \cos k_x + \cos k_y]\sigma_z$

$\underline{\underline{\sigma}}_y \underline{\underline{h}}(\bar{k}) \underline{\underline{\sigma}}_y = -\sin k_x \sigma_x + \sin k_y \sigma_y - [u + \cos k_x + \cos k_y]\sigma_z$

$\underline{\underline{\sigma}}_z \underline{\underline{h}}(k) \underline{\underline{\sigma}}_z = -\sin k_x \sigma_x - \sin k_y \sigma_y + [u + \cos k_x + \cos k_y]\sigma_z$

no SLS

class A