3. Chern insolators (CI's)

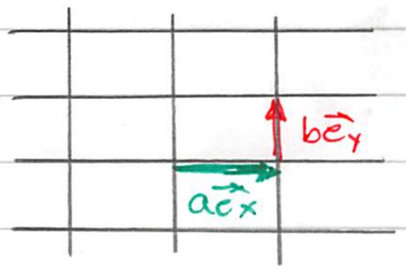
3.1 The Harper Hofstadter model

The Harper Hofstadter model describes electrons on a rectangular lattice in a constant transverse magnetic field.

(A) Bloch states in a magnetic field

- without magnetic field

lattice constants a ($\vec{e}_x$) and b ($\vec{e}_y$)



(135) $$V(\vec{r}) = V(\vec{r} + \vec{R})$$

(136) $$\vec{R} = n a \vec{e}_x + m b \vec{e}_y \qquad n, m \in \mathbb{Z}$$

(137a) lattice translation $$\hat{T}_{\vec{R}} = e^{\frac{i}{\hbar} \vec{R} \hat{\vec{p}}} \qquad [T_{\vec{R}}, H] = 0$$

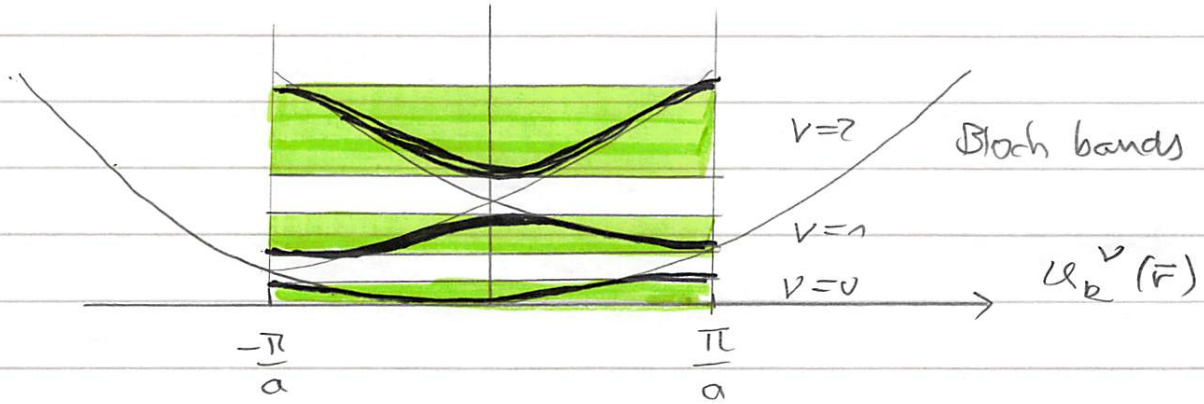
(137b) $$\hat{T}_{\vec{R}_1} \hat{T}_{\vec{R}_2} = \hat{T}_{\vec{R}_2} \hat{T}_{\vec{R}_1} = \hat{T}_{\vec{R}_1 + \vec{R}_2}$$

eigenstates of H are Bloch functions

(138) $$\psi_{\vec{k}}(\vec{r}) = e^{i \vec{k} \cdot \vec{r}} u_{\vec{k}}(\vec{r}) \qquad u_{\vec{k}}(\vec{r} + \vec{R}) = u_{\vec{k}}(\vec{r})$$

$\vec{k}$ is lattice (quasi) momentum

(139) $$-\frac{\pi}{a} \le k_x \le \frac{\pi}{a} \qquad -\frac{\pi}{b} \le k_y \le \frac{\pi}{b}$$



- with magnetic field : translational invariance (137) is no longer given!
  $\vec{A}(\vec{r})$ is not translational invariant.
  e.g. $\vec{A} = -By\vec{e}_x$

but

(140) $$T_{\vec{R}}\vec{A}(\vec{r}) = \vec{A}(\vec{r}+\vec{R}) = \vec{A}(\vec{r}) + \underset{\uparrow}{\delta\vec{A}(\vec{r})} = \vec{A}(\vec{r}) + \vec{\nabla}\underset{\uparrow}{\xi_{\vec{R}}(\vec{r})}$$

consider as gauge trafo

$\Rightarrow$ Hamiltonian is invariant under combined translation and gauge transformation

(141) $$\overline{T}_{\vec{R}} \equiv e^{i\frac{e}{\hbar c}\xi_{\vec{R}}(\vec{r})}\, T_{\vec{R}}$$ magnetic translation

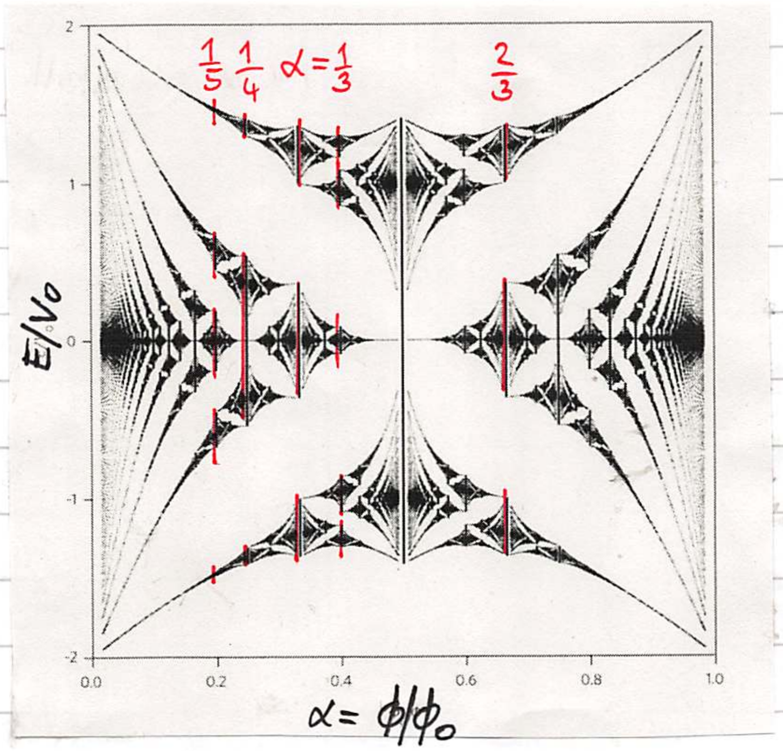
but magnetic translations are in general no Abelian group

(142) in general $$\left[\overline{T}_{\vec{R}_1}, \overline{T}_{\vec{R}_2}\right] \neq 0$$ ☹

(B) Hofstadter Butterfly, magnetic unit cell

Due to (142) there is in general no Bloch theorem and the spectrum of the Harper Hofstadter model has a

"Hofstadter butterfly"



fractal spectrum

ϕ : flux per plaquette

(140) $\alpha \equiv \dfrac{\Phi}{\Phi_0}$

D. R. Hofstadter
Phys. Rev. B 14, 2239
(1976)

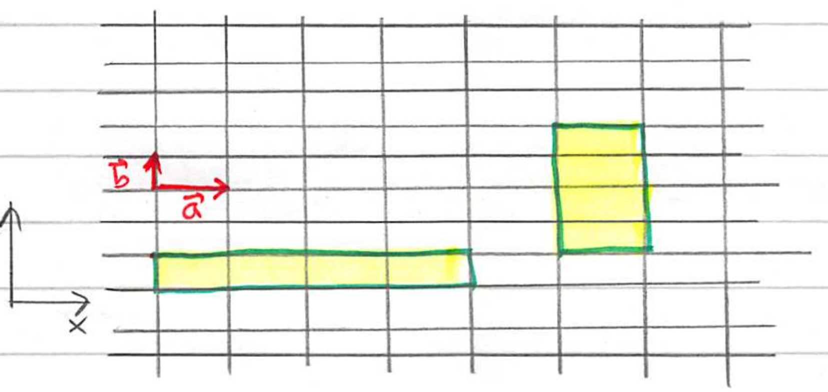
However at certain rational values of α the spectrum seems to have bands!

so consider

$$\boxed{\alpha = \frac{\Phi}{\Phi_0} = \frac{p}{q}} \qquad (141)$$

p, q relative prime integers.

Here we can introduce a magnetic unit cell consisting of q normal unit cells



$q = 4$

magnetic unit cells with flux

$$\boxed{\Phi = p\,\Phi_0}$$

integer multiple of Φ_0

Let's define <u>magnetic lattice vectors</u>

$$(1.42) \quad \boxed{\vec{R}_M \equiv nq\vec{a} + m\vec{b} = nqa\vec{e}_x + mb\vec{e}_y} \qquad \alpha = \frac{p}{q}$$

The corresponding magnetic translation operators now form an Abelian group. Let's choose the Landau gauge $\vec{A} = -By\vec{e}_x$

$$\overline{T}_{\vec{R}} = e^{-\frac{i}{\ell_0^2} Y \cdot x} \, T_{\vec{R}} = e^{-\frac{i}{\ell_0^2} Y \cdot x} \, e^{\frac{i}{\hbar} \vec{R} \cdot \vec{p}}$$

$$\overline{T}_{\vec{R}_1} \overline{T}_{\vec{R}_2} = e^{-\frac{i}{\ell_0^2} Y_1 x} \, e^{\frac{i}{\hbar} \vec{R}_1 \vec{p}} \, e^{-\frac{i}{\ell_0^2} Y_2 x} \, e^{\frac{i}{\hbar} \vec{R}_2 \vec{p}}$$

where $\vec{R}_1 = (X_1, Y_1) \qquad \vec{R}_2 = (X_2, Y_2)$

Baker-Hausdorff

$$e^{-A} B e^{A} = B + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} [A, B]_m$$

$$[A, B]_m = \underbrace{[A, [A, [A \cdots [A, B]] \cdots]}_{m \text{ times}}$$

for $[A, [A, B]] = [B, [B, A]] = 0$ this gives

$$e^{A} e^{B} = e^{A+B} \, e^{\frac{1}{2}[A+B]}$$

With this we find

$$e^{\frac{i}{\hbar} \vec{R}_1 \vec{p}} \, e^{-\frac{i}{\ell_0^2} Y_2 x} \, e^{\frac{i}{\hbar} \vec{R}_2 \vec{p}} =$$

$$= e^{-\frac{i}{\ell_0^2} Y_1 x} \, e^{-\frac{i}{\ell_0^2} X_1 Y_2} \, e^{\frac{i}{\hbar}(\vec{R}_1+\vec{R}_2)\vec{p}}$$

$$\Rightarrow \quad \bar{T}_{R_1} \bar{T}_{R_2} = e^{-\frac{i}{\ell_0^2} X_1 Y_2} \, \bar{T}_{R_1+R_2}$$

If $\vec{R}_1$ and $\vec{R}_2$ are magnetic lattice vectors
$X_1 = n_1 q a$ and $Y_2 = m_2 b$

$$\frac{X_1 Y_2}{\ell_0^2} = n_1 m_2 \frac{qab}{\ell_0^2} = n_1 m_2 2\pi p$$

i.e

$$e^{-i\frac{X_1 Y_2}{\ell_0^2}} = 1$$

(143) $$\boxed{\bar{T}_{R_1} \bar{T}_{R_2} = \bar{T}_{R_1+R_2} = \bar{T}_{R_2} \bar{T}_{R_1}}$$

if $\vec{R}_1, \vec{R}_2$ are magnetic lattice vectors

Since the unit cell is now q times enlarged (in x-direction) the Brillouin zone (in k_x direction) is now reduced.

$\Rightarrow$ <u>q magnetic sub-bands</u>

(144) $$\boxed{-\frac{1}{q}\frac{\pi}{a} \leq k_x \leq \frac{1}{q}\frac{\pi}{a}} \qquad \boxed{-\frac{\pi}{b} \leq k_y \leq \frac{\pi}{b}}$$

• Remark: equivalently one could have choosen the magnetic unit cell in y-direction

(c) Harper equations and spectrum of Harper-Hofstadter model

We want to diagonalize the HHM-Hamiltonian

(145) $$H = -t \sum_{mn} \left(C^{+}_{m+1n} C_{mn} + C^{+}_{mn+1} C_{mn} e^{i2\pi \frac{p}{q} m} + h.a. \right)$$

(Set $a = b = 1$)

F-Trafo in the full Brillouin zone, i.e. without reduction to magnetic Brillouin zone would give

(146) $$H = -t \int_{-\pi}^{\pi} \frac{dk_x}{2\pi} \int_{-\pi}^{\pi} \frac{dk_y}{2\pi} \Big[\cos(k_x) \, \tilde{C}^{+}_{k_x k_y} \tilde{C}_{k_x k_y} + e^{-ik_y} \tilde{C}^{+}_{k_x + \frac{2\pi p}{q}, k_y} \tilde{C}_{k_x k_y} + h.a. \Big]$$

which is obviously not diagonal in k_x, k_y. However using the magnetic unit cell

(147) $$C_{m,n} \equiv \frac{1}{(2\pi)^2} \int_{-\pi/q}^{\pi/q} dk_x \int_{-\pi}^{\pi} dk_y \; e^{ik_x m} e^{ik_y n} \; \tilde{C}_{k_x k_y}$$

yields a decomposition of H into q sectors

(148) $$H = \frac{1}{(2\pi)^2} \int_{-\pi/q}^{\pi/q} dk_x \int_{-\pi}^{\pi} dk_y \sum_{\nu=0}^{q-1} H^{\nu}_{k_x k_y}$$

where

$$H^{\nu}_{k_x k_y} = -t\Big[2\cos\left(k_x + 2\pi\tfrac{p}{q}\nu\right)\tilde{c}^{+}_{k_x+2\pi\frac{p}{q}\nu,\,k_y}\,\tilde{c}_{k_x+2\pi\frac{p}{q}\nu,\,k_y}$$

(149) $$+\, e^{-ik_y}\,\tilde{c}^{+}_{k_x+2\pi\frac{p}{q}(\nu+1),\,k_y}\,\tilde{c}_{k_x+2\pi\frac{p}{q}\nu,\,k_y}$$

$$+\, e^{ik_y}\,\tilde{c}^{+}_{k_x+2\pi\frac{p}{q}(\nu-1),\,k_y}\,\tilde{c}_{k_x+2\pi\frac{p}{q}\nu,\,k_y}\Big]$$

One recognizes that $\sum_{\nu=0}^{q-1} H^{\nu}_{k_x k_y}$ is still not diagonal but it corresponds to a finite 1-dimensional tight-binding Hamiltonian with q "lattice sites" and hopping

$$k_x + 2\pi\tfrac{p}{q}\nu \;\underset{t e^{ik_y}}{\overset{t e^{-ik_y}}{\rightleftarrows}}\; k_x + 2\pi\tfrac{p}{q}(\nu+1)$$

and periodic boundary conditions. This can be solved like any 1D tight binding Hamiltonian

(150) $$H_{k_x k_y}\,|\psi\rangle = \sum_{\nu=0}^{q-1} H^{\nu}_{k_x k_y}\,|\psi\rangle = E_{k_x k_y}\,|\psi\rangle$$

Ansatz

$$|\psi\rangle = \sum_{\nu=0}^{q-1} \alpha_\nu\,\tilde{c}^{+}_{k_x+2\pi\frac{p}{q}\nu,\,k_y}\,|0\rangle$$

yields the so-called Harper equations

(151) $$\boxed{-2\cos\left(k_x + 2\pi\tfrac{p}{q}\nu\right)\alpha_\nu - e^{-ik_y}\alpha_{\nu-1} - e^{-ik_y}\alpha_{\nu+1} = \frac{E_{k_x k_y}}{t}\,\alpha_\nu}$$

with boundary conditions

$$\alpha_{j+q} = \alpha_j \tag{152}$$

The terms e^{-ik_y} can be removed by the transformation $\alpha_j = \beta_j e^{-ik_y j}$

$$\boxed{-\beta_{\nu-1} - \beta_{\nu+1} - 2\cos\left(k_x + 2\pi \frac{p}{q}\nu\right)\beta_\nu = \frac{E_{k_x k_y}}{t}\beta_\nu} \tag{153}$$

and

$$\beta_{j+q} = e^{ik_y q}\beta_j$$

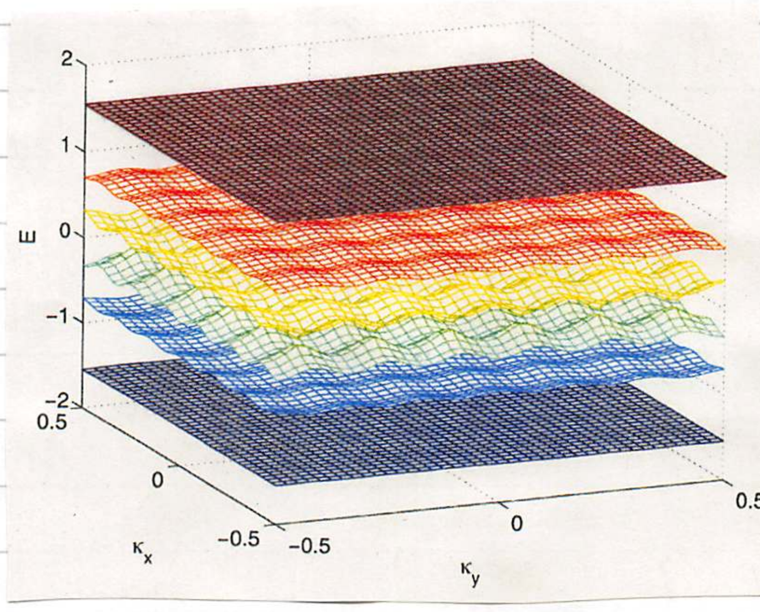
Since k_y disappeared and is only hidden in the boundary condition there holds

$$E(k_x, k_y) = E\left(k_x, k_y + \frac{2\pi}{q}n\right) \qquad n = 1, .., q-1 \tag{154}$$

$\Rightarrow$ <u>q-fold periodicity in the spectrum</u> with respect to k_y as expected by introduction of magnetic cells

magnetic subbands $q=6$

note periodicity (6-fold) in k_x and k_y



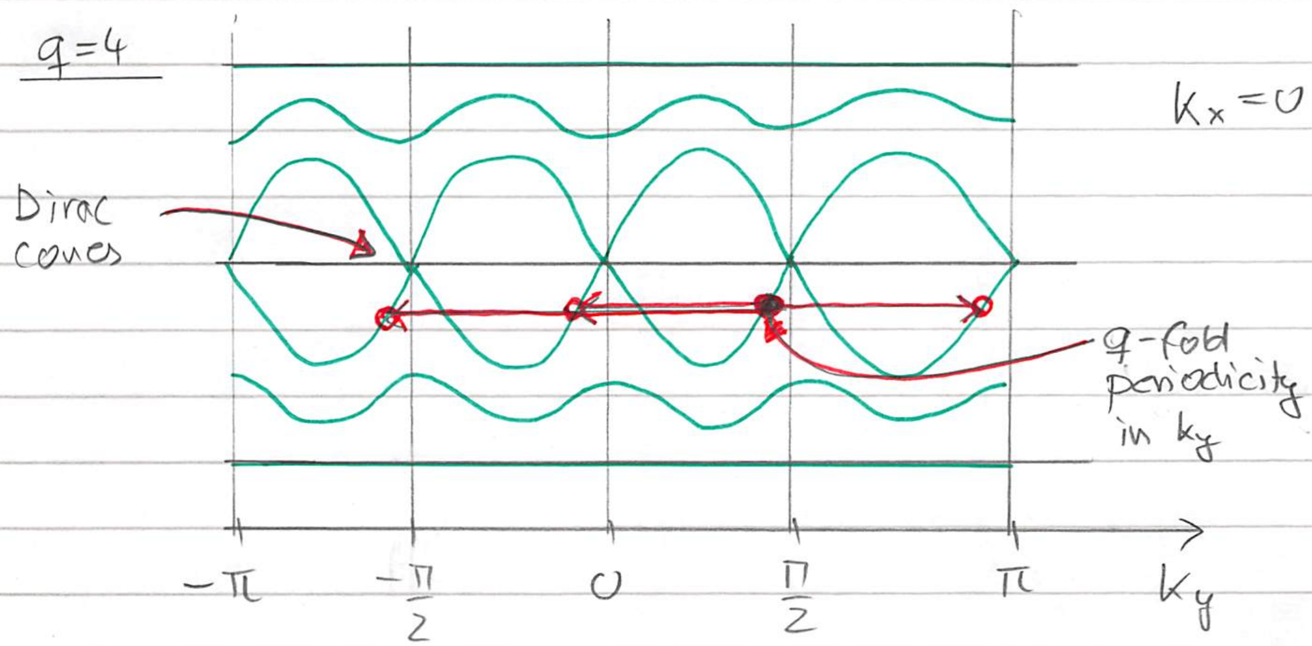
One can show that the characteristic polynomial following from the q algebraic equations (153) which determine the eigenenergies leads to $E=0$ solutions for even q

$$\text{for } q = 4n \qquad k_y = \frac{2m\pi}{q} \qquad k_x = 0 \tag{155}$$
$$\text{for } q = 4n+2 \qquad k_y = \frac{(2m+1)\pi}{q} \qquad k_x = \frac{\pi}{q} = -\frac{\pi}{q}$$

Close to these points the dispersion reads

$$\boxed{E = \pm \gamma q t^{q/2} \sqrt{k_x^2 + k_y^2}} \tag{156}$$

which is a Dirac spectrum

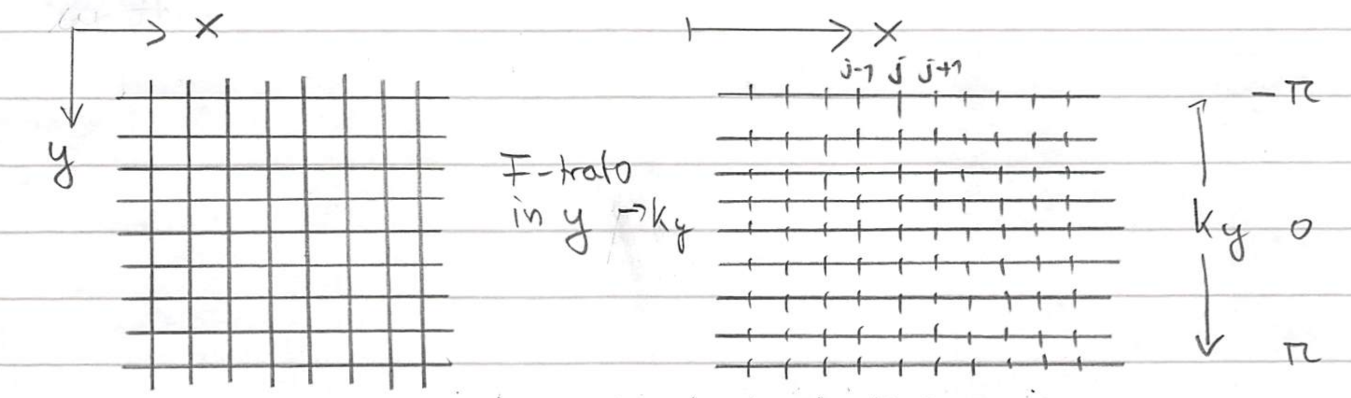


in chosen gauge ($\hat{=}$ choice of magnetic unit cell) i.e. also periodic in k_x!

$$-\frac{\pi}{q} \le k_x \le \frac{\pi}{q} \qquad -\pi \le k_y \le \pi$$

(D) Hall conductivity and Chern number

We consider the Harper Hofstadter model with rational $\alpha = p/q$, i.e. commensurate magnetic and bare lattice



magnetic unit cells

periodic (in k_y) set of 1D chains

$\Rightarrow \quad h_{HHM}(k_y)$

Center of mass of Wannier center $W_{nj}(k_y)$ for every value of k_y

(157)
$$\boxed{\begin{aligned} P_n(k_y) &\equiv \langle W_{nj}(k_y) \,|\, \hat{x} \,|\, W_{nj}(k_y) \rangle \\ &= \frac{1}{2\pi}\, \phi_n^{ZAK}(k_y) \end{aligned}}$$

n denotes magnetic subband

We call this quantity polarization of quasi momentum k_y

Consider a filled magnetic subband (insulator) and apply a small electric field in the y-direction

If the field E_y is weak the Wannier states become time dependent

$$|W_{nj}(k_y)\rangle \rightarrow |W_{nj}(k_y(t))\rangle$$

with

(158) $$\frac{dk_y}{dt} = E_y$$

Hall conductivity $\quad \sigma_{xy} = j_x / E_y \quad$ (charge $e=1$)

(159) $$j_x^{(n)} = \int_{BZ} dk_y \, \frac{dP_n(k_y)}{dt}$$

all states in BZ are occupied by one electron

(160)
$$\boxed{\begin{aligned} \sigma_{xy} &= \int_{BZ} dk_y \, \frac{dP_n(k_y)}{dt} \, \frac{dt}{dk_y} = \int_{BZ} dk_y \, \frac{\partial P_n(k_y)}{\partial k_y} \\ &= \frac{1}{2\pi} \int_{BZ} dk_y \, \frac{\partial \varphi_n^{ZAK}(k_y)}{\partial k_y} \\ &= \frac{1}{2\pi} \int_{BZ} dk_y \int_{BZ} dk_x \; \underbrace{i \left[\langle \partial_{k_y} u_n | \partial_{k_x} u_n \rangle - \langle \partial_{k_x} u_n | \partial_{k_y} u_n \rangle \right]}_{\text{Berry curvature}} \\ &= C \qquad \text{Chern number} \end{aligned}}$$

<u>quantization of Hall conductivity</u> in insulating bulk

(E) Streda formula and Hall conductivity in HHM

current density in 2D system ($\varepsilon_{k\ell}$ 2 dim. Levi Civita)

$$j_k = -\sigma_H \varepsilon_{k\ell} E_\ell \qquad k,\ell \in \{x,y\}$$

continuity equation

$$\frac{\partial}{\partial t}\rho = -\vec{\nabla}\cdot\vec{j} = -(\partial_x j_x + \partial_y j_y) = \sigma_H(\partial_x E_y - \partial_y E_x)$$

Maxwell equations

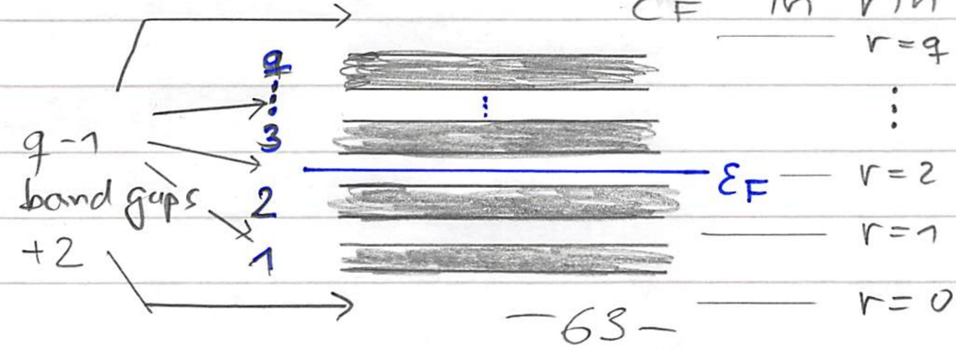
$$\partial_x E_y - \partial_y E_x = \frac{1}{c}\frac{\partial B_z}{\partial t} \Rightarrow \frac{\partial \rho}{\partial t} = \frac{\sigma_H}{c}\frac{\partial B}{\partial t}$$

(161) $$\boxed{\sigma_H = c\frac{\partial \rho}{\partial B}}$$ Streda formula

Hall conductivity is change of 2D charge density with magnetic field (while keeping voltage or Fermi energy constant). Changing B the number of available states below the Fermi energy changes

(162)
Now assume

- $\Phi = \frac{p}{q}\Phi_0$
- ε_F in rth sub-band gap



$r, p, q \in \mathbb{N}$
$0 \le r \le q$

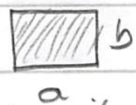
$\exists$ integer numbers s_r and t_r such that

(163) $$r = q s_r + p t_r \qquad |t_r| \leq \frac{q}{2}$$

consider finite lattice with M lattice sites. Then the number of occupied lattice sites is $N = r/q \; M$

$\Rightarrow$

(164) $$\rho = e\frac{N}{A} = e\frac{r}{q}\frac{M}{A} = \frac{er}{q\,ab}$$

ab 

area of unit cell

with Streda formula

$$\sigma_H = c\frac{\partial \rho}{\partial B} = ec\frac{1}{ab}\frac{\partial}{\partial B}\left(s_r + \frac{p}{q}t_r\right)$$

with $\frac{p}{q} = \frac{\Phi}{\Phi_0} = \frac{Bab}{\Phi_0} \Rightarrow$

(165) $$\boxed{\sigma_H = \frac{ec}{\Phi_0}t_r = \frac{e^2}{h}t_r}$$

Dauboux Theorem

t_r integer $|t_r| \leq \frac{q}{2}$

(i) $\boxed{q \text{ even}}$ set $\boxed{p=1}$ for simplicity

i.e. $0 \leq r = q s_r + t_r \leq q$

- $r < \frac{q}{2}$ $\qquad s_r = 0 \quad t_r = r$

contribution of band to Hall conductivity (in units of $\frac{e^2}{h}$)

(166) $$\boxed{J_r = t_{r+1} - t_r = 1}$$

increase by 1

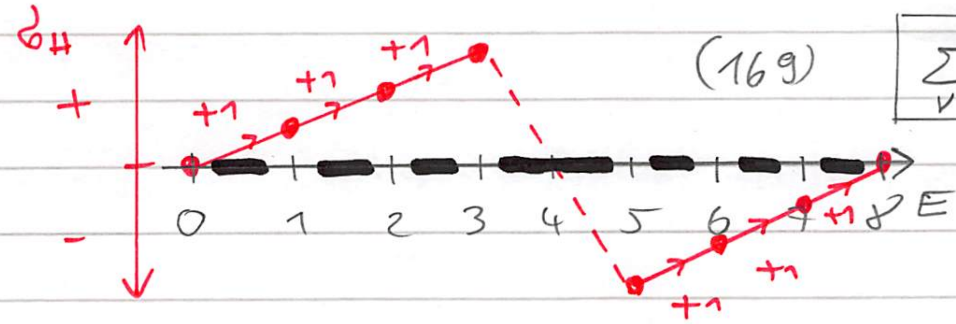
- $r > \frac{q}{2}$ $\quad S_r = 1 \quad t_r = r - q$

(167) $\boxed{J_r = t_{r+1} - t_r = 1}$ increase by 1

- $r = \frac{q}{2}$ not a true gap (Dirac points)

$$J_{\frac{q}{2}} = t_{\frac{q}{2}+1} - t_{\frac{q}{2}-1} = -\frac{q}{2} + 1 - \frac{q}{2} + 1$$

(168) $\boxed{J_{\frac{q}{2}} = 2 - q}$ jump!



(169) $\boxed{\sum_\nu \sigma_H^\nu = 0}$

$q = 8$

(ii) $\boxed{q \text{ odd}}$

- $r \leq \frac{q-1}{2}$ $\quad S_r = 0 \quad t_r = r$

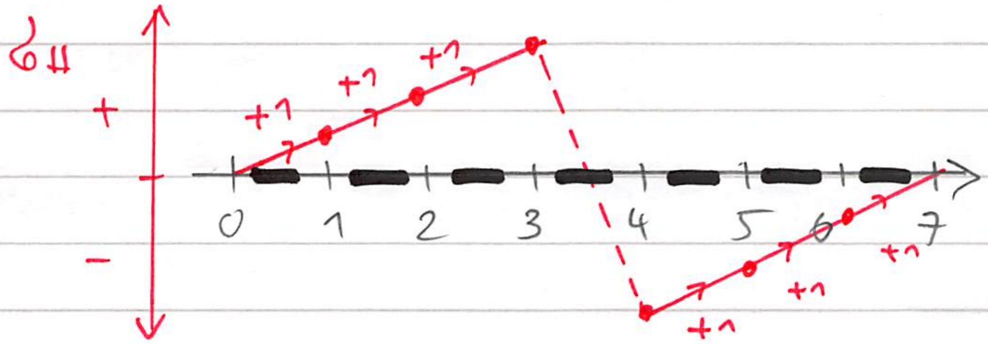
(170) $\boxed{J_r = t_{r+1} - t_r = 1}$ increase by 1

- $r \geq \frac{q+1}{2}$ $\quad S_r = 1 \quad t_r = r - q$

(171) $\boxed{J_r = t_{r+1} - t_r = 1}$ increase by 1

- central energy band

(172) $$\boxed{J_{\frac{q}{2}} = t_{\frac{q+1}{2}} - t_{\frac{q-1}{2}} = 1 - q}$$ jump



$q = 7$

(3F) Continuum limit: Landau levels

The Harper-Hofstadter model smoothly connects to the continuum model of a 2D electron gas in a magnetic field by considering the limit

(173) $$\alpha = \frac{p}{q} \longrightarrow 0 \qquad \text{i.e. } q \to \infty$$

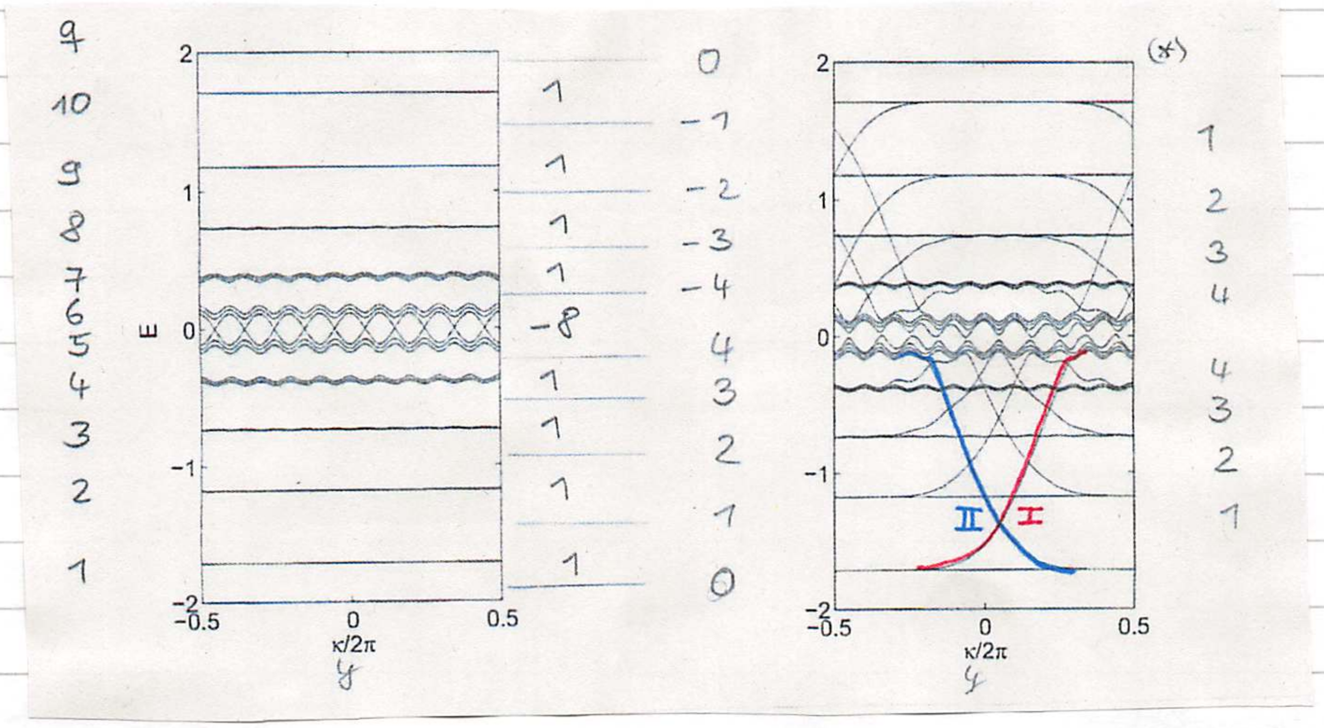
and having only a finite number of magnetic sub-bands occupied. In this limit all bands become flat (Landau levels)

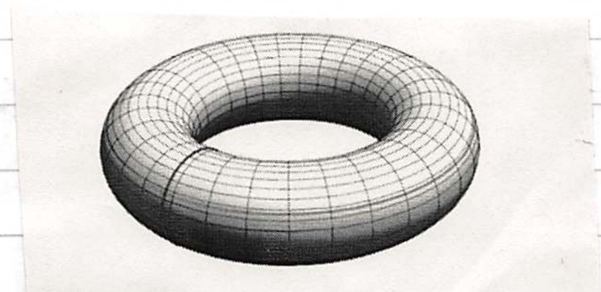
3.3 edge modes and bulk-boundary correspondence

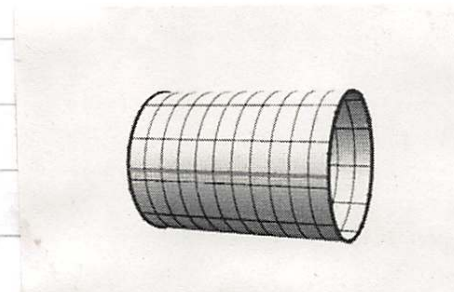
A question that is still open is what is the origin of the (almost) vanishing longitudinal resistance in a QH system or Chern insulator? At integer fillings these systems are <u>insulators</u> after all!

(A) some numerical findings

<u>periodic boundary conditions</u> <u>open boundary conditions</u>







One recognises isolated modes in the case of open boundary conditions. We will see that these are edge modes similar to what we have seen in the Qi-Wu-Zhang model. There are two types

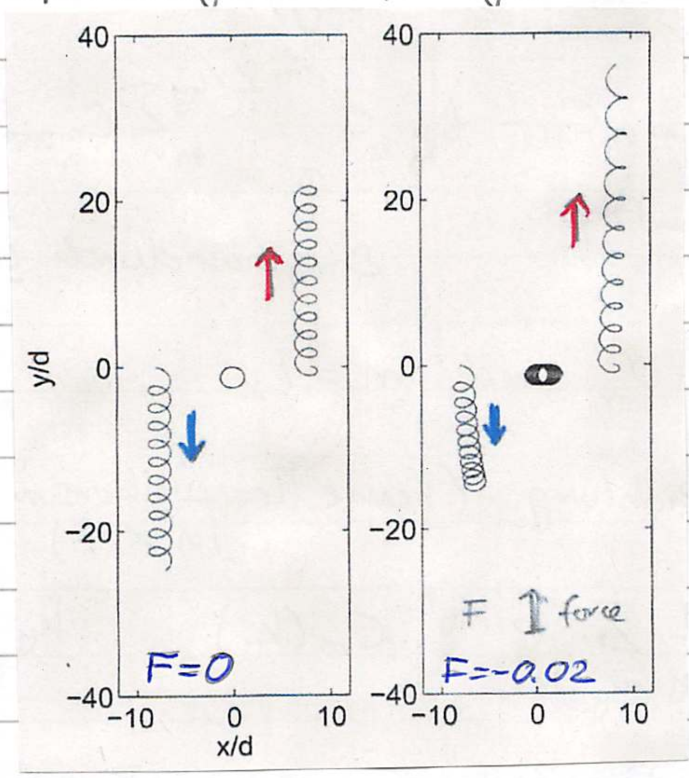
type I ● $d\omega/dk > 0$ transport in "+" direction

type II ● $d\omega/dk < 0$ transport in "−"

chiral edge modes

One also recognizes (as in the QWZ model) that the level crossings of edge modes are true level crossings. (in finite size systems with exponentially small gap, i.e. avoided crossing with exponentially small opening)

single-particle trajectories

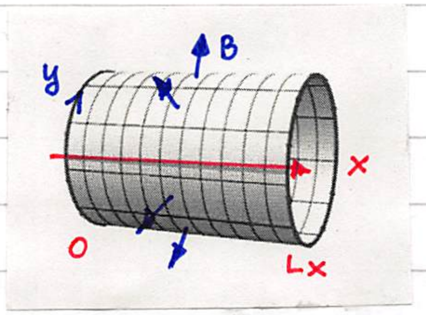


The total number of edge modes (per direction) is equal to the absolute value of the sum of the Hall conductivities (in units of e^2/h) of all bands below = sum of Chern numbers of bands below (all occupied bands in the CI)

bulk-boundary correspondence

(B) Calculating the spectrum with open boundaries: transfer matrix

consider cylinder geometry, periodic in y but with open boundaries in x direction in a magnetic field with flux per plaquette α



(174) $$H = -t_x \sum_{m,n} c^\dagger_{m+1\,n}\, c_{mn} - t_y \sum_{m,n} c^\dagger_{m\,n+1}\, c_{mn}\, e^{2\pi i \alpha m}$$

edges

$$m = 0 \qquad\qquad m = L_x$$

d.-F-trafo with respect to y

(175) $$c_{mn} = \frac{1}{L_y} \sum_{k_y} e^{i k_y n}\, \tilde{c}_m(k_y) \qquad\qquad k_y = \frac{2\pi}{L_y} n_y$$

this yields

(176) $$H = -t_x \sum_{m,k_y} \tilde{c}^\dagger_{m+1}(k_y)\, \tilde{c}_m(k_y) - t_y \sum_{m,k_y} \tilde{c}^\dagger_{m}(k_y)\, \tilde{c}_m(k_y)\, e^{-i k_y + 2\pi i \alpha m}$$

Now we want to solve the stationary Schrödinger equation

$$H|\psi\rangle = E|\psi\rangle$$

for a single-particle state

(177) $$|\psi\rangle = \sum_{m,k_y} \psi_m(k_y)\, \tilde{c}^\dagger_m(k_y)\, |0\rangle$$

this yields

$$-t_x\left(\psi_{m+1}(k_y)+\psi_{m-1}(k_y)\right)$$

$$-2t_y\cos(k_y-2\pi\alpha m)\,\psi_m(k_y) = E\,\psi_m(k_y)$$

This can be rewritten in the form of a recursive matrix equation

$$(178)\qquad \begin{pmatrix}\psi_{m+1}(k_y;\varepsilon)\\ \psi_m(k_y;\varepsilon)\end{pmatrix} = \underline{\underline{M}}_m(k_y;\varepsilon)\begin{pmatrix}\psi_m(k_y;\varepsilon)\\ \psi_{m-1}(k_y;\varepsilon)\end{pmatrix}$$

where $\varepsilon \equiv E/t_x$ and

$$(179)\qquad \underline{\underline{M}}_m = \begin{bmatrix} -\varepsilon-2\frac{t_y}{t_x}\cos(k_y-2\pi\alpha m) & -1\\ 1 & 0\end{bmatrix}$$

with transfer matrix

Now assume a rational $\alpha = P/q$ and a length L_x commensurate with a magnetic unit cell

$$L_x = q\,l \qquad l\in\mathbb{N}$$

$\Rightarrow$ transfer matrices are periodic in m with period q!

q-site transfer matrix

(180) $$\underline{\underline{M}}(k_y;\varepsilon) = \underline{\underline{M}}(\varepsilon) = \underline{\underline{M}}_q(\varepsilon)\underline{\underline{M}}_{q-1}(\varepsilon)\cdots\underline{\underline{M}}_1(\varepsilon)$$

here we drop the variable k_y for simplicity

$$\underline{\underline{M}}(\varepsilon) = \begin{pmatrix} M_{11}(\varepsilon) & M_{12}(\varepsilon) \\ M_{21}(\varepsilon) & M_{22}(\varepsilon) \end{pmatrix}$$

from (179) we see that

- M_{11} polynom in ε of degree q
- M_{12}, M_{21} —"— $q-1$
- M_{22} —"— $q-2$

Since $L_x = q\ell \Rightarrow$

(181) $$\begin{pmatrix} \psi_{L_x+1} \\ \psi_{L_x} \end{pmatrix} = \left(\underline{\underline{M}}(\varepsilon)\right)^{\ell} \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix}$$

Due to the open-boundary conditions

(182) $$\psi_{L_x} = \psi_0 \overset{!}{=} 0$$

and thus

$$\begin{pmatrix} \psi_{L_x+1} \\ 0 \end{pmatrix} \overset{!}{=} \underline{\underline{M}}^{\ell} \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix}$$

$\Rightarrow$

(183) $$\boxed{\left[\underline{\underline{M}}^{\ell}(\varepsilon)\right]_{21} \overset{!}{=} 0}$$ for every k_y

$(\underline{\underline{M}}^{\ell})_{21}$ is a polynom of degree $q\ell - 1 = L_x - 1$

$\Rightarrow$ $L_x - 1$ real roots $\mathcal{E}_n = \mathcal{E}_n(k_y)$ which are the eigenvalues of the problem with open BC

(c) edge modes

Let us now consider specificially the edge modes.
Eq. (183) can be fulfilled if we choose

(184) $$M_{21}(\mathcal{E}) = 0$$

since a product of triangular matrices is again a triangular matrix, $M_{21}(\mathcal{E})$ is a polynomial in $\mathcal{E}$ of degree $q-1$, i.e. (184) has $q-1$ solutions

$$\mathcal{E}_j \quad j = 1, 2, \ldots, q-1 \qquad \text{set} \quad \mathcal{E}_i < \mathcal{E}_j \quad \text{if } i < j$$

One can show that (184) does not have degenerate solutions i.e. all $\mathcal{E}_j$ are different from each other (for fixed value of k_y).

How to classify the $q-1$ solutions? One has from (181)

$$\begin{pmatrix} \psi_{q\ell+1}(\mathcal{E}_j) \\ 0 \end{pmatrix} = \begin{pmatrix} M_{11}(\mathcal{E}_j)^{\ell} & B \\ 0 & C \end{pmatrix} \begin{pmatrix} \psi_1(\mathcal{E}_j) \\ 0 \end{pmatrix}$$

$\Rightarrow$

(185) $$\frac{\psi_{q\ell+1}}{\psi_1} = M_{11}(\mathcal{E}_j)^{\ell}$$

(i) if $|M_{11}(\varepsilon_j)| < 1$ $\quad \frac{\psi_{ql+1}}{\psi_1} \xrightarrow[l \to \infty]{} 0$

$\Rightarrow$ wavefunction for $\varepsilon = \varepsilon_j$ is localized at (magnetic) unit cell $m = 1$

edge-mode on left edge

(ii) if $|M_{11}(\varepsilon_j)| > 1$ $\quad \frac{\psi_{ql+1}}{\psi_1} \xrightarrow[l \to \infty]{} \infty$

$\Rightarrow$ wave function for $\varepsilon = \varepsilon_j$ is localized at unit cell $m = ql+1 = L_x + 1$

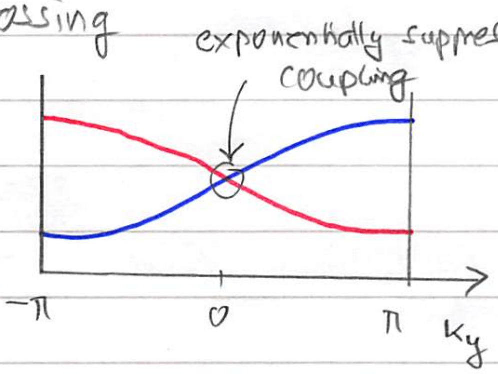
edge-mode on right edge

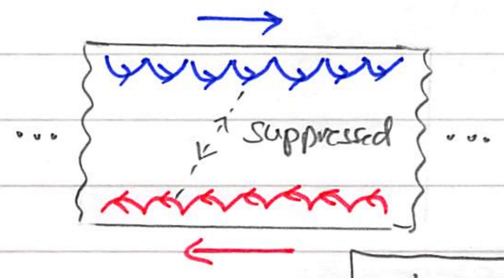
(iii) if $|M_{11}(\varepsilon_j)| = 1$ $\quad \left|\frac{\psi_{ql+1}}{\psi_1}\right| = 1$

$\Rightarrow$ wave function is delocalized in x-direction

bulk states

For reasons we still need to discuss, edge modes in Chern insolators are chiral and there is an (almost) perfect crossing





$\Rightarrow \boxed{\sigma_{xx} = 0}$ (184)