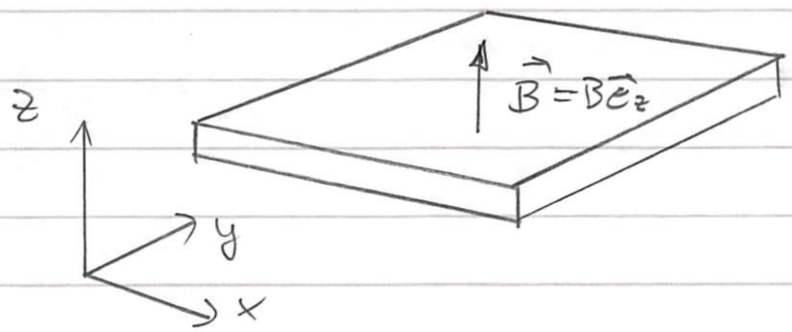
2. Integer quantum Hall system

2.1. Landau levels



free electrons in magnetic field (+2D confinement)

(101) $$H = \frac{1}{2m}\left(\vec{p} + \frac{e}{c}\vec{A}\right)^2 \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

(A) Landau gauge

(102) $$\vec{A} = B(-y, 0, 0)$$ (gauge breaks rotational symmetry)

(103) $$H = \frac{1}{2m}\left[\left(p_x - \frac{eB}{c}y\right)^2 + p_y^2\right]$$

$[H, p_x] = 0 \quad \rightsquigarrow \quad p_x = \hbar k_x$ conserved

characteristic length scale

(104) $$\boxed{l_0 = \sqrt{\frac{\hbar c}{eB}}}$$ magnetic length

introduce dimensionless variables

$$y \to y' = \frac{y}{l_0} - l_0 k_x \qquad p_y \to p_y' = \frac{l_0}{\hbar} p_y$$

fix k_x

(105) $$H[k_x] \to H'[k_x] = \hbar\omega_c \left[\frac{1}{2} y'^2 + \frac{1}{2} P_y'^2\right]$$

$\Rightarrow$ harmonic oscillator

(106) $$\boxed{\omega_c = \frac{eB}{mc}}$$ cyclotron frequency

(107) $$\boxed{E_n = (n+\tfrac{1}{2})\hbar\omega_c}$$ Landau levels

- degeneracy with respect to k_y
- y' coordinate of oscillator depends on k_x

(B) symmetric gauge

(108) $$\vec{A} = \frac{\vec{B}\times\vec{r}}{2} = \frac{B}{2}(-y, x, 0)$$ (gauge respects rotational symmetry)

in dimensionless units (l_m and $\hbar\omega_c$)

(109) $$H = \frac{1}{2}\left[\left(-i\frac{\partial}{\partial x} - \frac{y}{2}\right)^2 + \left(-i\frac{\partial}{\partial y} + \frac{x}{2}\right)^2\right]$$

introduce complex coordinates in (x,y) plane

(110) $$\boxed{z = x - iy \qquad z^* = x + iy = re^{i\theta}}$$

(111) $$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial z^*} \qquad \frac{\partial}{\partial y} = -i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*}\right)$$

where z and z^* are treated as independent variables define two types of ladder operators

(112) $$\boxed{\begin{aligned}\hat{a} &= \frac{1}{\sqrt{2}}\left(\frac{z}{2} + 2\frac{\partial}{\partial z^*}\right)\\ \hat{a}^+ &= \frac{1}{\sqrt{2}}\left(\frac{z^*}{2} - 2\frac{\partial}{\partial z}\right)\end{aligned}}$$

$$\frac{\partial}{\partial z^*} = \frac{1}{2}(\partial_x - i\partial_y)$$
$$\frac{\partial}{\partial z} = \frac{1}{2}(\partial_x + i\partial_y)$$

(113) $$\boxed{\begin{aligned}\hat{b} &= \frac{1}{\sqrt{2}}\left(\frac{z^*}{2} + 2\frac{\partial}{\partial z}\right)\\ \hat{b}^+ &= \frac{1}{\sqrt{2}}\left(\frac{z}{2} - 2\frac{\partial}{\partial z}\right)\end{aligned}}$$

(114) $[a, a^+] = [b, b^+] = 1 \qquad [a, b^{(+)}] = 0$

one finds

(115) $$\boxed{H = \hbar\omega_c\left(a^+a + \frac{1}{2}\right)}$$ again Landau levels

depends only on a^+a $\Rightarrow$ degeneracy with respect to b^+b

angular momentum in z-direction

$[H, \hat{L}_z] = 0$ by symmetry

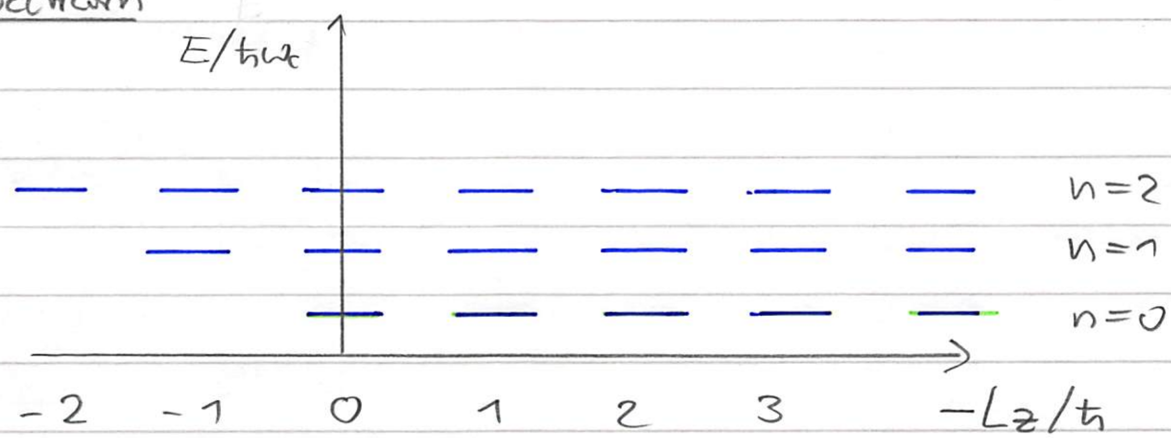
$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\theta} = -\hbar\left(z\frac{\partial}{\partial z} - z^*\frac{\partial}{\partial z^*}\right)$$

(116) $$\boxed{\hat{L}_z = \hbar(a^+a - b^+b)}$$

eigenvalues of $\hat{L}_z$: $\hbar m$

$$m = n, n-1, n-2, \ldots$$

<u>Spectrum</u>



<u>wavefunctions</u>

(117) $$|n,m\rangle = \frac{b^{+\,n+m}\, a^{+\,n}\, |0\rangle}{\sqrt{(n+m)!\, n!}} \qquad \begin{aligned} -L_z &= \hbar m \\ E &= \hbar\omega_c n \end{aligned}$$

(118) $$\psi_{nm}(z) = \mathcal{N}\left(\frac{z^*}{2} - 2\frac{\partial}{\partial z}\right)^n \left(\frac{z}{2} - 2\frac{\partial}{\partial z^*}\right)^{n+m} e^{-|z|^2/4}$$

(in normalized units)

2.2 Lowest Landau level (LLL)

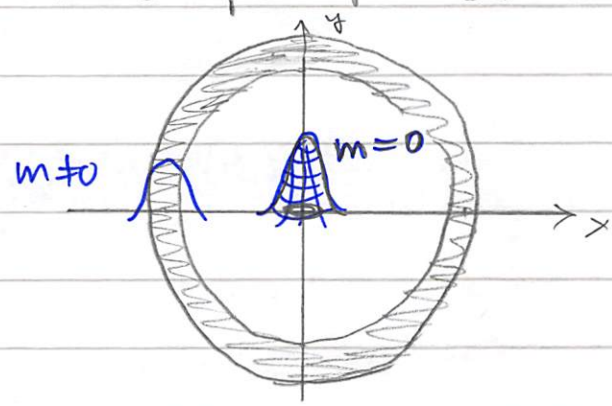
of particular interest for the physics at low temperature is the LLL $\boxed{n=0}$

(119) $$\psi_{0m}(z) = \mathcal{N}\left(\frac{z}{2} - 2\frac{\partial}{\partial z^*}\right)^m e^{-|z|^2/4}$$

so

(120) $$\psi_{0m}(z) = \mathcal{N}\, z^m e^{-|z|^2/4}$$

analytic in z apart from Gaussian



$$\Delta r_m = \frac{l_0}{\sqrt{2m}}$$

$$r_m = l_0\sqrt{2m}$$

$$A_m = 2\pi\, r_m\, \Delta r_m = 2\pi\, l_0^2$$

all eigenstates have same area

# of states in an area A ($\hat{=}$ degeneracy)

(121) $$\boxed{N = \frac{A}{2\pi l_0^2} = \frac{AB}{\Phi_0} = \frac{\Phi}{\Phi_0}}$$

since we have (spin-polarized) electrons every state can be occupied by one electron (Pauli principle)

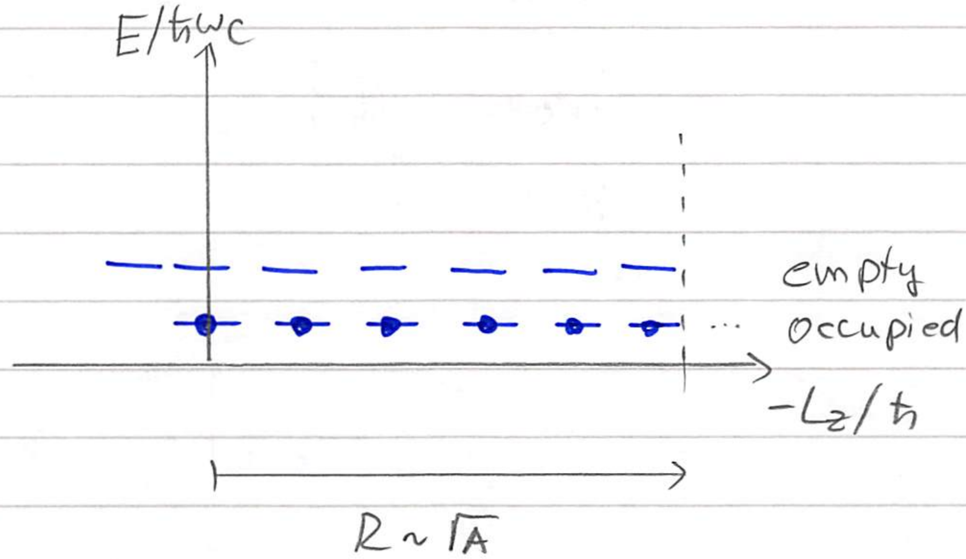
filling fraction

(122) $$\boxed{\nu = \frac{N_e}{N} = \frac{N_e 2\pi l_0^2}{A} = 2\pi \rho_e l_0^2 = \frac{\rho_e}{B/\Phi_0}}$$

where

$$\boxed{\Phi_0 = \frac{hc}{e}}$$ Dirac flux quantum

$\nu = 1$



many-body wavefunction of filled LLL $(\nu = 1)$
Slater determinant

$$(123) \quad \Psi_{\nu=1}(z_1, \dots, z_N) \sim \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & & z_N \\ z_1^2 & z_2^2 & \ddots & z_N^2 \\ \vdots & \vdots & & \vdots \\ z_1^N & z_2^N & & z_N^N \end{vmatrix} e^{-\frac{1}{4}\sum_{j=1}^{N} |z_j|^2}$$

every electron can have all angular momenta, i.e. all wavefunctions $z_i^m e^{-\frac{1}{4}|z_i|^2}$ with $m = 0, \dots, N$ occur

$$(124) \quad \boxed{\Psi_{\nu=1}(z_1, \dots, z_N) \sim \prod_{j<k} (z_j - z_k) \exp\left(-\frac{1}{4}\sum_j |z_j|^2\right)}$$

Pauli exclusion + anti-symmetry

normalization for $|z_j| \to \infty$

2.3 Discretization and lattice model

We will see now that the integer quantum Hall effect can be considered as limiting case of a Bloch hamiltonian with topological properties

discretization in real space (square lattice)

(125)
$$x \to x_j = j\,\Delta x$$
$$y \to y_\ell = \ell\,\Delta x$$

$$\psi(\vec{r}) = \psi(x,y) \to \psi_{j\ell}$$

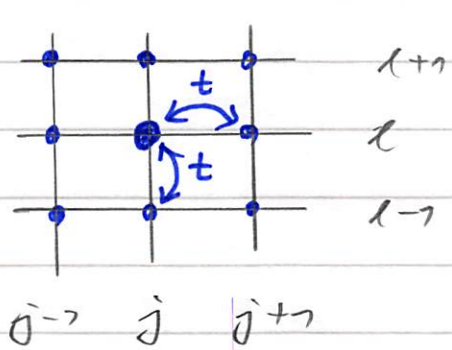
(126) $$\Delta\psi \to \frac{1}{\Delta x^2}\left(\psi_{j+1,\ell} + \psi_{j-1,\ell} + \psi_{j,\ell+1} + \psi_{j,\ell-1} - 4\psi_{j\ell}\right)$$

$$i\hbar\frac{\partial}{\partial t}\psi = H\psi = -\frac{\hbar^2}{2m}\Delta\psi$$

(127) $$H\psi_{\ell m} = -t\left(\psi_{j+1,\ell} + \psi_{j-1,\ell} + \psi_{j\ell+1} + \psi_{j\ell-1} - 4\psi_{j\ell}\right)$$

where

(128) $$t = \frac{\hbar^2}{2m\Delta x^2}$$



in second quantization

$$(129)\quad \boxed{\hat{H} = -t \sum_{j\ell} \left(\hat{c}^{+}_{j+1,\ell}\, \hat{c}_{j\ell} + \hat{c}^{+}_{j,\ell+1}\, \hat{c}_{j\ell} + h.a. \right)}$$

Lets add the interaction with the magnetic field in Landau gauge

$$\vec{A} = -By\,\vec{e}_x$$

$$(130)\quad H\psi = -\frac{\hbar^2}{2m}\left(\left(\frac{\partial}{\partial x} - \frac{ieBy}{\hbar c}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 \right)\psi$$

$$\left(\frac{\partial}{\partial x} - \frac{ieBy}{\hbar c}\right)\psi(x,y) = \frac{1}{\Delta x}\left(\psi_{j+1\ell} - \psi_{j\ell} - \frac{ieB\Delta x^2}{\hbar c}\ell\,\psi_{j\ell}\right)$$

$$(131)\quad \hat{=} \frac{1}{\Delta x}\left(\psi_{j+1\ell} - \psi_{j\ell} - \frac{ieB\Delta x^2}{\hbar c}\ell\,\psi_{j+1\ell}\right)$$

$$= \frac{1}{\Delta x}\left(\psi_{j+1\ell}\underbrace{\left(1 - \frac{ieB\Delta x^2}{\hbar c}\ell\right)}_{\cong\, \exp\left(-\frac{ieB\Delta x^2}{\hbar c}\ell\right)} - \psi_{j\ell}\right)$$

$\Rightarrow$ Peierls phase

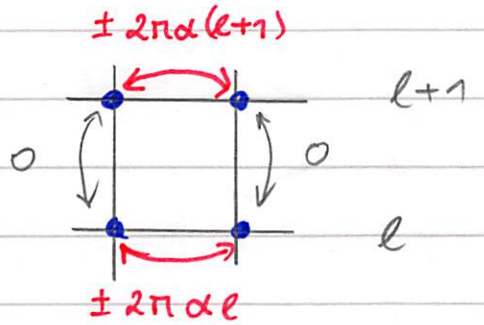
$\Rightarrow$

$$H\psi_{j\ell} = -t\left(e^{-2\pi i\alpha\ell}\psi_{j+1\ell} + e^{2\pi i\alpha\ell}\psi_{j-1,\ell} + \psi_{j\ell+1} + \psi_{j\ell-1} - 4\psi_{j\ell}\right)$$

$$(132)\quad \boxed{2\pi\alpha = \frac{eB}{\hbar c}\Delta x^2 = \left(\frac{\Delta x}{\ell_0}\right)^2}$$ Peierls phase

So in second quantization we obtain complex hopping amplitudes

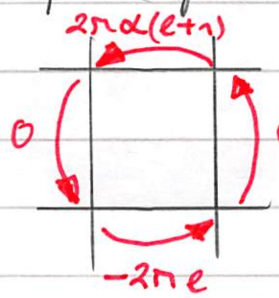
$$\boxed{\begin{aligned} H = &-t\sum_{jl}\left(e^{-2\pi i\alpha l}\,\hat{c}^{\dagger}_{j+1\,l}\,\hat{c}_{jl} + h.a.\right) \\ &-t\sum_{jl}\left(\hat{c}^{\dagger}_{j\,l+1}\,\hat{c}_{jl} + h.a.\right) + 4t\sum_{jl}\hat{c}^{\dagger}_{jl}\,\hat{c}_{jl} \end{aligned}} \tag{133}$$



in chosen gauge: phases when hopping in x-direction

the phase picked up upon a closed loop in any plaquette:

(134)



$$\boxed{\begin{aligned} \Delta\phi = 2\pi\alpha &= \left(\frac{\Delta x}{l_0}\right)^2 \\ &= \frac{B\Delta x^2}{\Phi_0} = \frac{\text{magn. flux per plaquette}}{\text{Dirac flux quantum}} \end{aligned}}$$

the continuum limit reproduces Landau-level physics

$$\Delta x \to 0 \qquad \alpha \to 0$$