12. Some comments on the general classification of topological systems

Topological quantum systems with interactions are far from being understood. In the following we want to scetch an idea by Xiao-Gang Wen et al PRB 82, 155138 (2010)

12.1. Symmetry breaking phase transitions and entanglement

1D Ising model in transverse magnetic field

(493) $$H = -B\sum_j \sigma_j^x - J\sum_j \sigma_{j+1}^z \sigma_j^z$$

has a Z_2-symmetry: $|\uparrow\rangle \leftrightarrow |\downarrow\rangle$ $\quad \sigma^z \to -\sigma^z$

$B \gg J$

(494) $$|\phi_{gs}^P\rangle = \prod_j \left(\frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}\right)_j = \prod_j |+\rangle_j$$

$J \gg B$

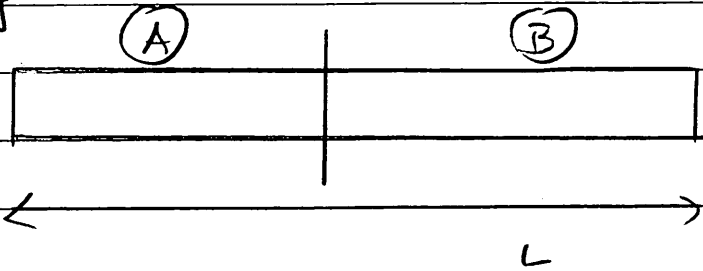
(495) $$|\phi_{gs}^F\rangle = \begin{cases} |\uparrow\rangle|\uparrow\rangle|\uparrow\rangle\cdots \\ \\ |\downarrow\rangle|\downarrow\rangle|\downarrow\rangle\cdots \end{cases}$$

phase transition between paramagnetic phase $|\phi_{gs}^P\rangle$ which is invariant under Z_2 trafo $\sigma^z \to -\sigma^z$

and $|\phi_{gs}^{F}\rangle$ which spontaneously breaks Z_2 symmetry

$\rightarrow$ symmetry breaking phase transition

state $|\phi_{gs}\rangle$ is product state in both limits $B \gg J$ and $J \gg B$. Close to the phase transition there will be some entanglement characterized by Schmidt spectrum or half system von-Neumann entropy



Schmidt theorem

$$(496) \qquad |\psi\rangle = \sum_{j=1}^{n} \sqrt{\lambda_j}\, |j\rangle_A |j\rangle_B$$

note: $|j\rangle_A$ and $|j\rangle_B$ can have completely different representations in local states

$$n \leq \min(\dim(\mathcal{H}_A), \dim(\mathcal{H}_B))$$

von-Neumann entropy of half system

$$\rho_A = \mathrm{Tr}_B\{|\psi\rangle\langle\psi|\} \qquad \rho_B = \mathrm{Tr}_A\{|\psi\rangle\langle\psi|\}$$

in Schmidt representation

$$(497) \qquad \rho_A = \sum_j \lambda_j\, |j\rangle_{A\,A}\langle j| \qquad \rho_B = \sum_j \lambda_j\, |j\rangle_{B\,B}\langle j|$$

$$S_A = -\mathrm{Tr}_A\{\varrho_A \ln \varrho_A\} = S_B = -\mathrm{Tr}_B\{\varrho_B \ln \varrho_B\}$$

(498)

$$= -\sum_{j=1}^{n} \lambda_j \ln \lambda_j$$

In general $n \leq p^{L/2}$ where p is local dimension in a lattice model and L is the total number of lattice sites. Thus in general

$$S_A = -\sum_{j=1}^{n} \lambda_j \ln \lambda_j \leq -\sum_{j=1}^{n} \frac{1}{n} \ln \frac{1}{n}$$

(499)

$$S_A \leq \ln n \leq \ln (p^{L/2}) = \frac{L}{2} \ln p$$

$\Rightarrow$ half system entropy can scale with the volume

(500) $$\boxed{S \leq L \ln \sqrt{p}} \quad \text{or} \quad S \lessapprox \eta V$$

which is clear because it is an <u>extensive</u> thdyn. quantity

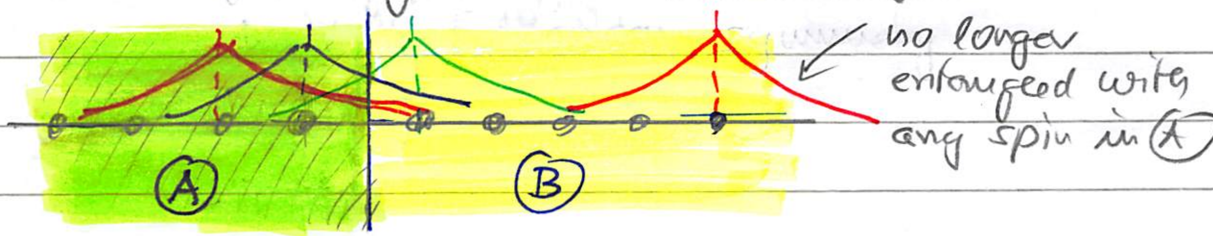
For <u>gapped states</u> one finds however in general an <u>area law</u>

(501) $$\boxed{S \leq \eta\, \mathcal{O}(L) \sim L^{d-1}} \begin{cases} \text{const} & 1D \\ \text{grows} & \geq 2D \end{cases}$$

for critical points one gets an extra factor $\ln V$

$\Rightarrow$ both phases of Ising model have much less entanglement as would be maximally allowed

This is because entanglement is short ranged



bi-partite entanglement between spins decays exponentially with distance

conjecture:	gapped ground state that is described by Landau spontaneous symmetry breaking is short-range entangled

$\Rightarrow$ ground states that cannot be described by symmetry breaking are not short-range entangled ?

12.2. Quantum Phases, quantum phase transitions, local unitary evolution (LU) and topological order

consider Hamiltonian depending on parameter λ

$$H(\lambda) \qquad 0 \leq \lambda \leq 1 \tag{502}$$

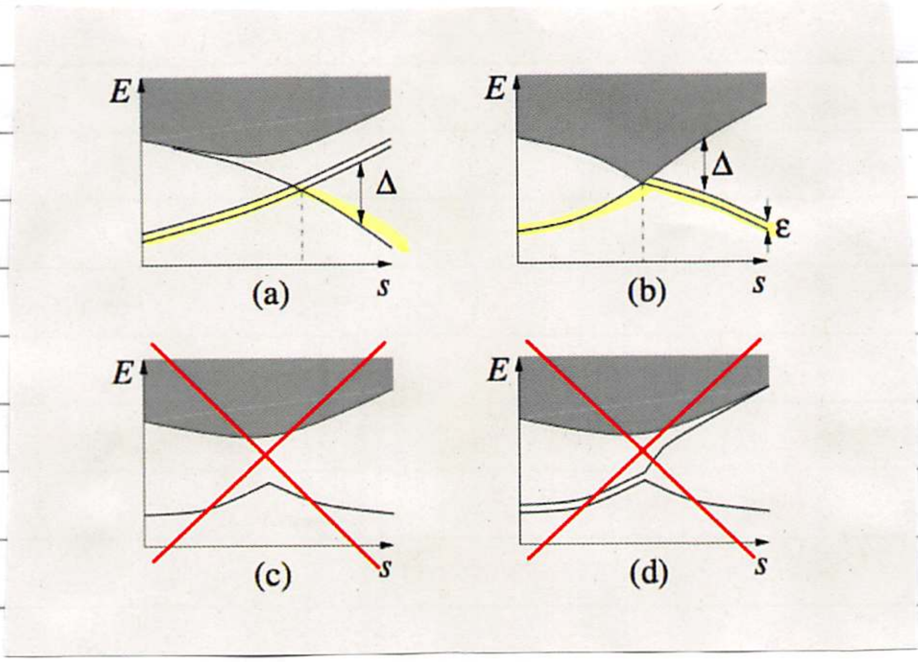
let $|\phi(\lambda)\rangle$ be the ground state of $H(\lambda)$. If a local observable exists which has a singularity or a non-analyticity at a value λ_c in the thermodynamic limit $L \to \infty$

$$\langle \hat{O} \rangle_\lambda = \langle \phi(\lambda) | \hat{O} | \phi(\lambda) \rangle \tag{503}$$

than the system is said to have a <u>quantum phase transition</u>,

If the energy gap of $H(\lambda)$ is finite for all $\lambda \in (0,1)$ than there is no phase transition

first order QPT　　　　　　continuous QPT



forbidden

> The states $|\phi(0)\rangle$ and $|\phi(1)\rangle$ are said to be in the same phase (equivalence class) if we can find a smooth deformation of the <u>local</u> Hamiltonian $H(\lambda)$ such that $|\phi(0)\rangle$ and $|\phi(1)\rangle$ are the ground states for $\lambda=0$ and $\lambda=1$ and there was no gap closing along the path.

(504) $\boxed{|\phi(1)\rangle \hat{=} |\phi(0)\rangle \quad \text{iff} \quad |\phi(1)\rangle = T e^{-i\int_0^1 d\lambda\, H(\lambda)} |\phi(0)\rangle}$

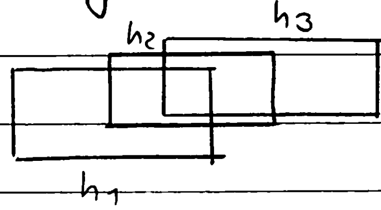
Here

(505) $$U_L = T e^{-i\int_0^1 d\tau\, H(\tau)}$$

is called a LU, a <u>local unitary evolution</u>.

Rem: A Hamiltonian is called <u>local</u> if it can be written as a sum of terms which act only on subspaces with finite size or with a coupling which decays exponentially

(506) $$H = \sum_j h_j$$



<u>Def.</u>: A state has <u>short-range entanglement</u> if and only if it can be transformed into a product state through a LU transformation (SRE)

~~all product states~~ $\qquad\qquad\qquad\qquad$ set of ~~short range entangled~~ states

$$|\phi\rangle = \prod_j |\phi_j\rangle \longleftarrow \boxed{LU} \longrightarrow |\tilde{\phi}\rangle$$

one can make a connection to <u>finite depth quantum circuits</u>

Let U_i^L be unitary operators that act on compact local set of sites; piecewise LU operator

(507) $$U_{pwL} = \prod_j U_j^L$$

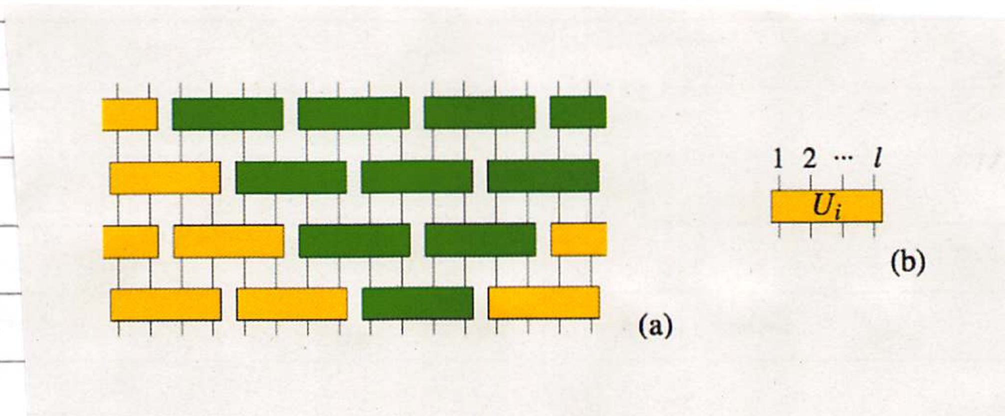
Quantum information:

time evolution with local hamiltonian over finite time can be simulated by a finite depth circuit with piecewise LU operators

(508) $$U^M_{circ} = U^{(1)}_{pwL} \cdots U^{(M)}_{pwL}$$

(509) $$|\phi(1)\rangle \hat{=} |\phi(0)\rangle \quad \text{iff} \quad |\phi(1)\rangle = U^M_{circ} |\phi(0)\rangle$$

M



finite depth circuit of LU's

Def: Quantum states that cannot be connected to product states by a finite depth quantum circuit of LU's are called long-range entangled (LRE)

Wen's classification

> Long-range entangled states have intrinsic topological order (LRE)

12.3 Symmetry protected topological order

One can show:

The ground states of local hamiltonians in 1D are all SRE

$\Rightarrow$ no intrinsic topological order in 1D ☹

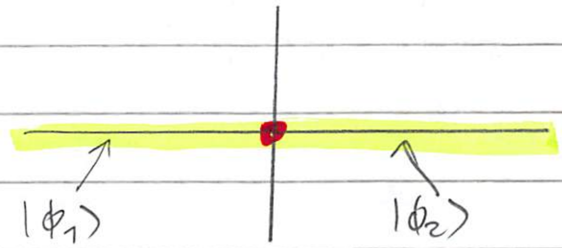
but there is a second class of states with topological order if we require certain symmetries

> The states $|\phi(0)\rangle$ and $|\phi(1)\rangle$ are said to be in the same symmetry-protected phase (equivalence class) if we can find a smooth deformation of the Hamiltonian $H(\lambda)$ such that for all λ; $[H(\lambda), S]=0$ where S is the symmetry operator(s) and there is no gap closing

$$|\phi(1)\rangle \underset{S}{\hat{=}} |\phi(0)\rangle \quad \text{iff} \quad |\phi(1)\rangle = Te^{-i\int_0^1 d\lambda H(\lambda)} |\phi(0)\rangle$$
$$\text{and} \quad [H(\lambda), S]=0$$

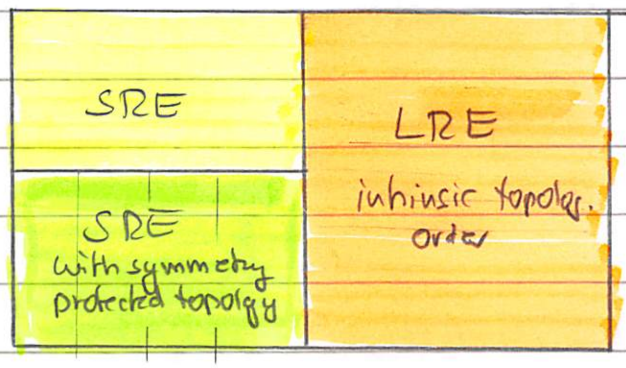
this restricts the allowed LU's to those that obey the symmetry: LU_S

example 1D Rice-Mele vs. SSH model



$|\phi_1\rangle$ and $|\phi_2\rangle$ can be connected by LU's

but not by LUs with chiral symmetry



Subjects not covered

- topological superconductors
  (Majorana edge states; topolog. quantum computing)
- topological excitations
  (skyrmions)
- spin liquids

... and much more