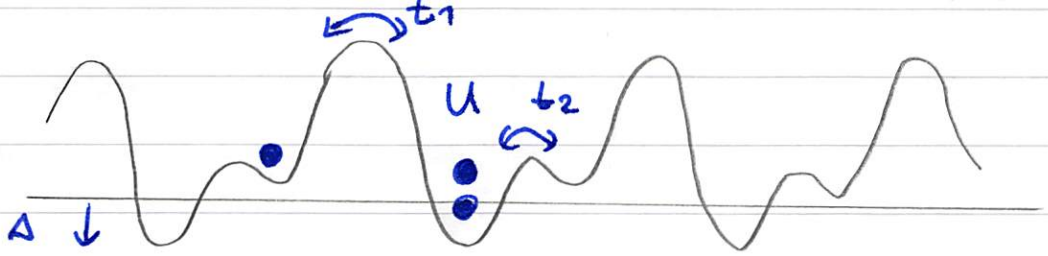
11. Topological lattice models with interactions and fractional Chern insulators

11.1. Rice-Mele model with interactions: The superlattice Bose-Hubbard model

We first consider the simplest topological model in 1D: the SSH and Rice Mele model with on-site interactions. On-site interactions are only possible for bosons which leads us to the superlattice Bose-Hubbard model (SLBHM)

$$H = \underbrace{-t_1 \sum_{j\,\text{even}} (a_{j+1}^\dagger a_j + \text{h.a.}) - t_2 \sum_{j\,\text{odd}} (a_{j+1}^\dagger a_j + \text{h.a.}) \pm \frac{\Delta}{2} \sum_j (-1)^j a_j^\dagger a_j}_{\text{single-particle RM}} + \frac{U}{2} \sum_j a_j^{\dagger 2} a_j^2 \tag{430}$$

repulsive interactions



Single particle Hamiltonian has chiral symmetry for $\Delta = 0$ and non-trivial winding number of Zak phase when encircling origin ($t_1 - t_2 = 0$, $\Delta = 0$) in a parameter path. The SSH model ($\Delta = 0$) has edge states. What survives with interactions?

consider model with chemical potential i.e. allow varying particle number

(431) $K = H - \mu N \qquad N = \sum_j a_j^+ a_j$

(A) Hardcore limit and relation to fermionic HTM

hardcore limit $U \to \infty$

$\Rightarrow$ only $n=0$ or $n=1$ particles per lattice site

(432) $\frac{U}{2} a^{+2} a^2 = \frac{U}{2} a^+ a (a^+ a - 1)$

hardcore bosons = spin 1/2 particle

(433) $\boxed{[a_i, a_j^+] = \delta_{ij} \qquad a_i^2 \overset{!}{=} 0}$

(434) $\boxed{a_i \equiv \sigma_i^- \qquad a_i^+ \equiv \sigma_i^+}$

Wigner-Jordan mapping to fermions

(435) $\boxed{\sigma_e^- = \exp\{-\pi i \sum_{j<e} c_j^+ c_j\} c_e}$ non-local transformation

(436) $\{c_e, c_{e'}^+\} = \delta_{ee'} \qquad \{c_e, c_{e'}\} = 0$

check (assume $e' < e$)

$$\sigma_e^- \sigma_{e'}^- = e^{-\pi i \sum_{j<e} c_j^+ c_j} c_e \, e^{-\pi i \sum_{n<e'} c_n^+ c_n} c_{e'}$$

$$= e^{-\pi i \sum_{j=e'}^{e-1} c_j^+ c_j} c_e c_{e'} = -e^{-\pi i \sum_{j=e'}^{e-1} c_j^+ c_j} c_{e'} c_e$$

$$\sigma_{e'}^- \sigma_e^- = e^{-\pi i \sum_{m<e'} c_m^\dagger c_m} c_{e'} e^{-\pi i \sum_{j<e} c_j^\dagger c_j} c_e$$

since $e > e'$!

$$= e^{-\pi i \sum_{j=e'+1}^{e} c_j^\dagger c_j} \underbrace{c_{e'} e^{-\pi i c_{e'}^\dagger c_{e'}} c_e}$$

$$= e^{-\pi i (c_{e'}^\dagger c_{e'} + 1)} c_{e'}$$

$$= e^{-\pi i} e^{-\pi i \sum_{j=e'}^{e} c_j^\dagger c_j} c_{e'} c_e = \sigma_e^- \sigma_{e'}^-$$

$\Rightarrow [\sigma_e^-, \sigma_{e'}^-] = 0$ □

Wigner-Jordan trafo of Hamiltonian in h.c.-limit

$$a_j^\dagger a_j = \sigma_j^\dagger \sigma_j = e^{+\pi i \sum_{e<j} c_e^\dagger c_e} c_j^\dagger \underbrace{e^{-\pi i \sum_{e<j} c_e^\dagger c_e} c_j}_{\text{commute}}$$

(437)

$$= c_j^\dagger c_j$$

$$a_{j+1}^\dagger a_j = e^{\pi i \sum_{e<j+1} c_e^\dagger c_e} c_{j+1}^\dagger \underbrace{e^{-\pi i \sum_{e<j} c_e^\dagger c_e} c_j}_{\text{commute}}$$

(438)

$$= e^{\pi i c_j^\dagger c_j} c_{j+1}^\dagger c_j = c_{j+1}^\dagger c_j \quad ☺$$

$\underbrace{\qquad\qquad}_{c_j^2 = 0}$

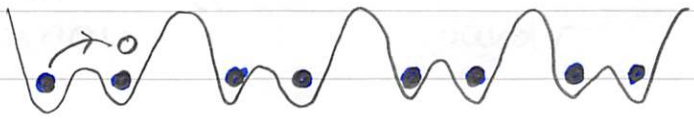
in the hard core limit the SLBHM is isomorph to the fermionic RMM ⇒ identical topological features

- for $\Delta = 0$ Chiral symmetry

(B) Going away from the hard core limit

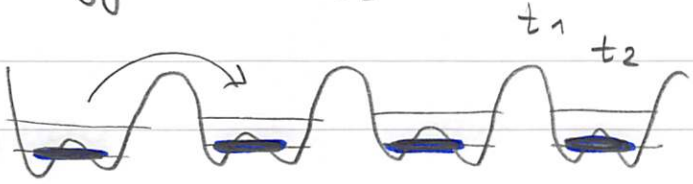
- small t/U : Mott-insulator (MI)
  hopping suppressed

$\nu = 1$



- cost energy U
- kinetic gain t_2

$\nu = \frac{1}{2}$

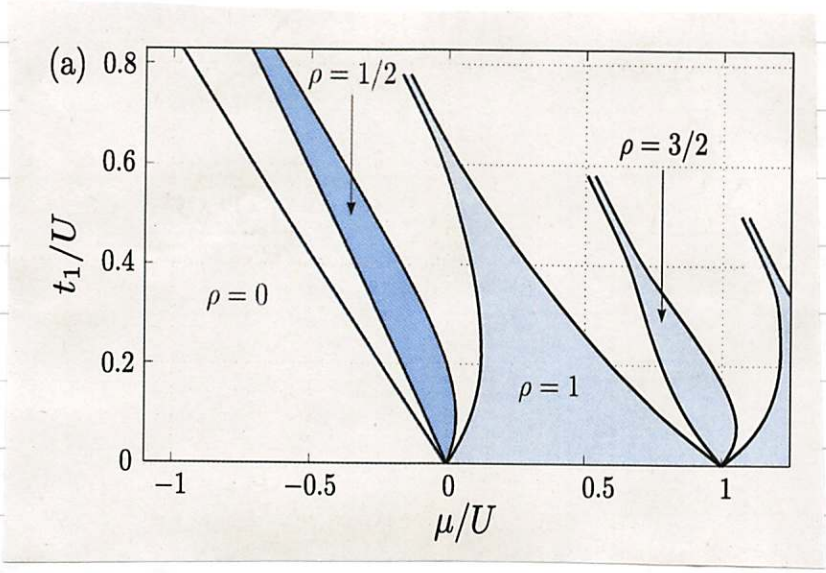


symmetric/antisymmetric

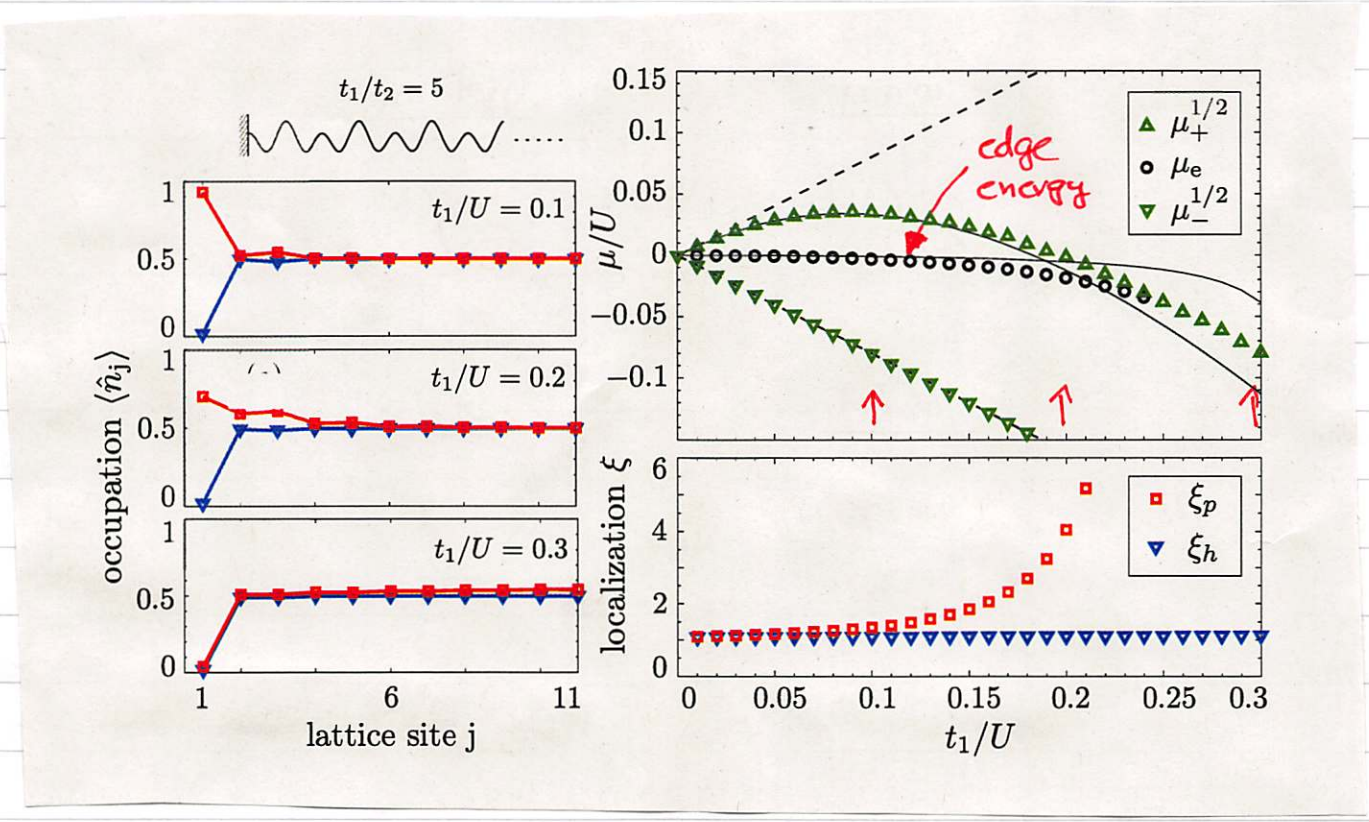
- cost energy t_2
- kinetic gain t_1

- large t/U : motion allowed
  Superfluid phase (SF)

phase diagram of SCBHM ; $\Delta = 0$



edge states in topologically non-trivial case



edge states no longer at zero energy ⇒ absence of PHS

(439)
$$\mu_e = -2t_2^2 \frac{U-2t_1}{(U+t_1)(U-3t_1)}$$

(C) Many-body topological invariant

⇒ Q. Niu & D.J. Thouless J. Phys. A 17 2453 (1984)

for finite interactions the single-particle lattice momentum is no longer a good quantum number ⇒ need generalized quantity

In chapter 1 we have seen that a nontrivial winding of the Zak phase was related to particle transport. Eq. (74) for non-interacting gapped fermion systems

$$\Delta n = \nu = \frac{1}{2\pi} \oint d\lambda \frac{\partial \phi_{Zak}}{\partial \lambda} \tag{440}$$

We have also seen that the Zak phase is directly related to the polarization (in appropriate units)

$$\Delta P = \frac{1}{2\pi} \oint d\lambda \frac{\partial \phi_{Zak}}{\partial \lambda} \tag{441}$$

with

$$P = \frac{1}{2\pi} \mathrm{Im} \ln \langle \exp\left(i \frac{2\pi}{L} \hat{x}\right) \rangle \tag{442}$$

Both transport and polarization are well defined also for interacting systems. Let's have follow the derivation by Niu and Thouless

Consider N-particle Hamiltonian (1st quantization) with pair interaction ($\hbar = m = 1$)

$$H(t) = \sum_{j=1}^{N} \left(-\frac{1}{2} \frac{\partial^2}{\partial x_j^2} + V(x_j, t) \right) + \sum_{i>j}^{N} U(x_i - x_j) \tag{443}$$

where $V(x,t)$ is a time-periodic potential for the particles. Assume twisted boundary conditions

$$\Psi(x_1, x_2, \ldots, x_j + L, \ldots x_N) = e^{i\alpha L} \Psi(x_1, x_2 \ldots, x_N) \quad \forall j \tag{444}$$

Let $|E_n(\alpha;t)\rangle$ be the instantaneous eigenstates of $H(t)$ with boundary condition (444) and assume that in the full period $0 \le t \le T$ there is a finite many-body gap for all $-\pi \le \alpha L \le \pi$. Then if the time change is sufficiently slow we can realize an adiabatic <u>many-body Thouless pump</u>. The density matrix evolves

(445) $$\rho(t) = \rho_{inst.}(t) + \delta\rho(t)$$

(446) $$\rho_{inst}(t) = |E_0(\alpha;t)\rangle\langle E_0(\alpha;t)|$$

when we started in the ground state $|E_0(\alpha, t=0)\rangle$

(447) $$i\dot{\rho} = i(\dot{\rho}_{inst} + \underbrace{\dot{\delta\rho}}_{\approx 0}) = [H(t), \rho(t)] = [H(t), \delta\rho(t)]$$

Now

$$\langle E_0|[H, \delta\rho]|E_n\rangle = (E_0 - E_n)\,\delta\rho_{on}$$

$\Rightarrow$

$$\delta\rho_{on} = \frac{\langle E_0|[H, \delta\rho]|E_n\rangle}{E_0 - E_n} = i\frac{\langle E_0|\dot{\rho}_{inst}|E_n\rangle}{E_0 - E_n}$$

$$= i\frac{\langle \partial_t E_0|E_n\rangle}{E_0 - E_n} \qquad n \neq 0$$

The charge transport in one period T can be obtained by integrating over the particle current

(448) $$\Delta N = \frac{1}{L}\int_0^T dt\,\langle P\rangle = \frac{1}{L}\int_0^T dt\,\mathrm{Tr}\{\rho(t)P\}$$

The trace is carried out over the instantaneous eigenstates $|E_n(\alpha,t)\rangle$; $P = \sum_{j=1}^{N} p_j$ total momentum (note $\hbar = m = 1$)

$$\Delta n = \frac{1}{L}\int_0^T dt\, (1+\delta g_{00}) \langle E_0(\alpha,t)|\, P\, |E_0(\alpha,t)\rangle$$

(449) $$+\frac{1}{2L}\int_0^T dt \sum_{n\neq 0} \left(\delta g_{0n} P_{n0} + P_{0n}\delta g_{n0}\right) + \text{terms } \delta g_{nm} \approx 0$$

In the instantaneous eigenstates the total momentum vanishes, so

(450) $$\Delta n(\alpha) = \frac{1}{2L}\int_0^T dt \sum_{n\neq 0} \left(i\frac{\langle \partial_t E_0 | E_n\rangle}{E_0 - E_n} P_{n0} + \text{c.c.}\right)_\alpha$$

The index "α" indicates the chosen boundary conditions. For the further discussion we perform a <u>canonical transformation</u>

(457)
$$H(t) \longrightarrow H(t,\alpha) = T(\alpha)\, H(t)\, T^{-1}(\alpha)$$
$$|E_n\rangle \longrightarrow |\tilde{E}_n(t,\alpha)\rangle = T(\alpha)\, |E_n(t,\alpha)\rangle$$

where in position representation $|\phi\rangle = T(\alpha)\,|\psi\rangle$

(452) $$\phi(x_1,\ldots x_N) = e^{-i\alpha(x_1+x_2+\cdots+x_N)}\, \psi(x_1,\ldots x_N)$$

Obviously ϕ now fulfills periodic boundary conditions.

With this

$$\tilde{P} = T(\alpha)\, P\, T^{-1}(\alpha) = T(\alpha) \sum_{j=1}^{N} \left(-i \frac{\partial}{\partial x_j}\right) T^{-1}(\alpha)$$

(453)

$$= \sum_{j=1}^{N} \left(-i \frac{\partial}{\partial x_j} + \alpha\right)$$

Thus the twisted boundary conditions are canonically equivalent to adding a gauge field α. With this

(454) $$\langle \tilde{E}_n | \tilde{P} | \tilde{E}_0 \rangle = \langle \tilde{E}_n | \frac{\partial H(t,\alpha)}{\partial \alpha} | \tilde{E}_0 \rangle$$

With

$$\frac{\partial}{\partial \alpha} | \qquad H(t,\alpha) | \hat{E}_0 \rangle = E_0(t) | \hat{E}_0 \rangle$$

$$\frac{\partial H}{\partial \alpha} | \hat{E}_0 \rangle + H | \partial_\alpha \hat{E}_0 \rangle = E_0(t) | \partial_\alpha \hat{E}_0 \rangle$$

$$\langle \tilde{E}_n | \frac{\partial H}{\partial \alpha} | \tilde{E}_0 \rangle = - \langle \tilde{E}_n | H | \partial_\alpha \tilde{E}_0 \rangle + E_0 \langle \tilde{E}_n | \partial_\alpha \tilde{E}_0 \rangle$$

(455)

$$= (E_0 - E_n) \langle \tilde{E}_n | \partial_\alpha \tilde{E}_0 \rangle$$

Thus we find from (450)

$$\Delta n(\alpha) = \frac{1}{2L} \int_0^T dt \sum_{n \neq 0} \Big(i \langle \partial_t \tilde{E}_0 | \underbrace{\tilde{E}_n \rangle \langle \tilde{E}_n}_{=1} | \partial_\alpha \tilde{E}_0 \rangle + \text{c.c.} \Big)$$

(456) $$\Delta n(\alpha) = \frac{1}{2L} \int_0^T dt \; i \left(\left\langle \frac{\partial \tilde{E}_0}{\partial t} \middle| \frac{\partial \tilde{E}_0}{\partial \alpha} \right\rangle - \left\langle \frac{\partial \tilde{E}_0}{\partial \alpha} \middle| \frac{\partial \tilde{E}_0}{\partial t} \right\rangle \right)$$

Now it is reasonable to assume that Δn should not depend on the choice of α in $-\frac{\pi}{L} \leq \alpha \leq \frac{\pi}{L}$ and we can average

$$\overline{\Delta n} = \frac{i}{4\pi L} \int_{-\pi/L}^{\pi/L} d\alpha \int_0^T dt \left[\left\langle \frac{\partial \tilde{E}_0}{\partial t} \middle| \frac{\partial \hat{E}_0}{\partial \alpha} \right\rangle - \left\langle \frac{\partial \hat{E}_0}{\partial \alpha} \middle| \frac{\partial \hat{E}_0}{\partial t} \right\rangle \right]$$

rescaling $\beta \equiv \alpha L$ finally yields an expression that is very similar to eq. (73) in the non-interacting case

$$\overline{\Delta n} = \frac{1}{4\pi} \int_{-\pi}^{\pi} d\beta \int_0^T dt \; i \left[\langle \partial_t \hat{E}_0 | \partial_\beta \tilde{E}_0 \rangle - \langle \partial_\beta \hat{E}_0 | \partial_t \tilde{E}_0 \rangle \right] \tag{457}$$

We thus define the

many-body Zak phase (in many-body ground state)

$$\phi_{ZAK} = \mathrm{Im} \int_{-\pi}^{\pi} d\beta \; \left\langle \tilde{E}_0 \middle| \frac{\partial \tilde{E}_0}{\partial \beta} \right\rangle \tag{458}$$

Its winding ν upon a periodic cycle in parameter space is a topological invariant

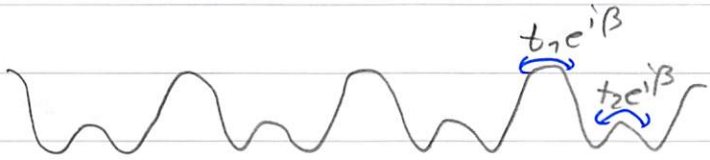
$$\nu \equiv \frac{1}{2\pi} \oint d\lambda \, \frac{\partial \phi_{ZAK}(\lambda)}{\partial \lambda} \tag{459}$$

(D) Symmetry-protected topology of SCBHM

finite U no Particle hole symmetry! What protects topology in this 1D system? ($\Delta = 0$)

$$\tilde{H} = T H T^{-1} = -t_1 e^{i\beta} \sum_{j\,\text{even}} a^\dagger_{j+1} a_j - t_2 e^{i\beta} \sum_{j\,\text{odd}} a^\dagger_{j+1} a_j + \text{h.c.}$$

(460)
$$+ \frac{U}{2} \sum_j a_j^{\dagger 2} a_j^2$$



Symmetry

(461) $$[\tilde{H}, \hat{S}] = 0$$

(462) $$\boxed{\hat{S} = IK}$$

inversion I + complex conjugation K

So

(463) $$|E_0(\beta)\rangle = e^{i\chi_\beta} \hat{S} |E_0(\beta)\rangle$$

$$A(\beta) \equiv i\langle E_0 | \partial_\beta E_0 \rangle = i\langle SE_0 | \partial_\beta | SE_0 \rangle - \partial_\beta \chi_\beta$$

$$= i\langle KE_0 | \partial_\beta | KE_0 \rangle - \partial_\beta \chi_\beta = i\langle E_0^* | \partial_\beta E_0^* \rangle - \partial_\beta \chi_\beta$$

$$= i\langle \partial_\beta E_0 | E_0 \rangle - \partial_\beta \chi_\beta$$

$$\text{Im}(A(\beta)) = -\text{Im}(A(\beta)) - \partial_\beta \chi_\beta$$

$$\phi_{ZAK} = \int_{-\pi}^{\pi} d\beta \; Im\left(A(\beta)\right) = \frac{1}{2}\int_{-\pi}^{\pi} d\beta \; \partial_\beta \chi_\beta = \frac{1}{2}\left(\chi_\pi - \chi_{-\pi}\right)$$

Since $\chi_\pi = \chi_{-\pi} \mod 2\pi$

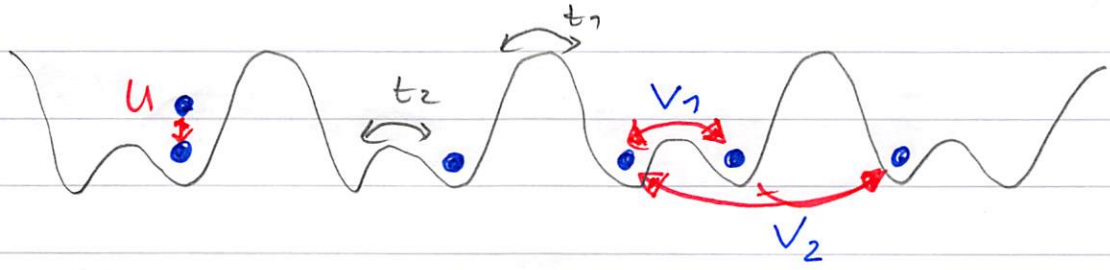
(464) $\Rightarrow$ $\boxed{\phi_{ZAK} = 0, \pi}$ $\mathbb{Z}_2$ topolog. invariant

11.2. The extended SLBHM: fractional charges

For the FQHE we have seen that interactions in topological models can give rise to new gapped phases with fractional charges. This can also happen in the simple 1D Superlattice model if we add non-local interactions

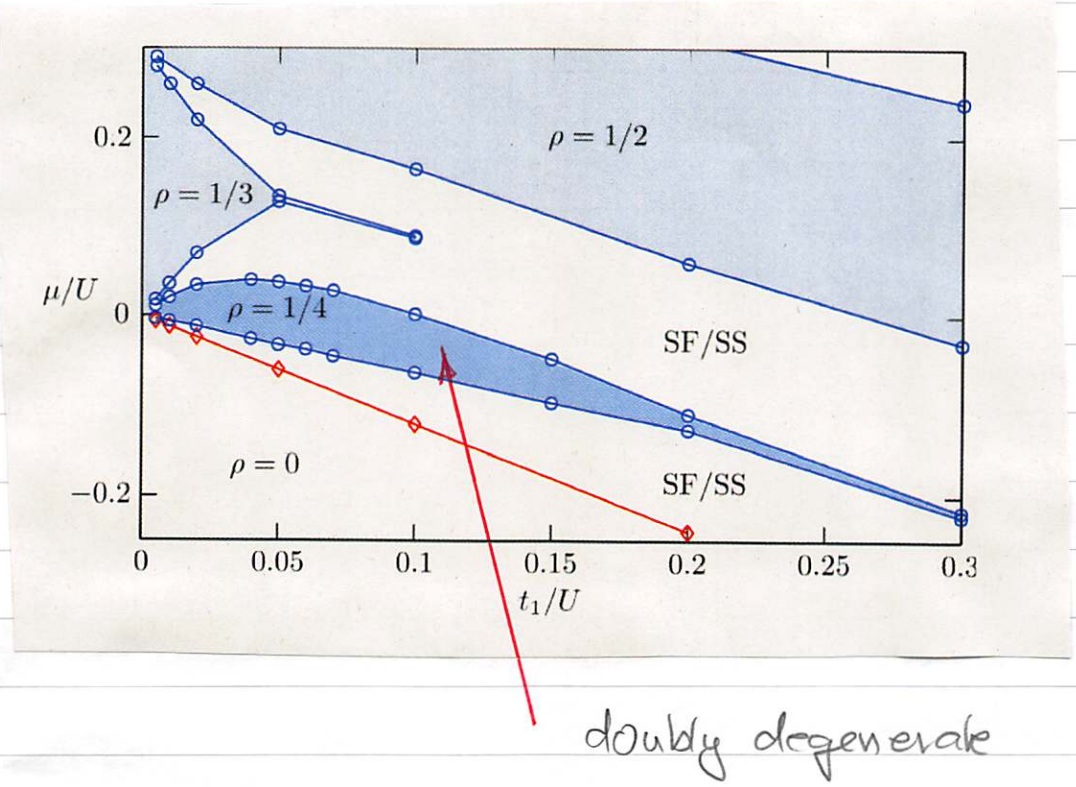
$$H = -t_1 \sum_{j\,even} \left(a_{j+1}^\dagger a_j + h.a.\right) - t_2 \sum_{j\,odd} \left(a_{j+1}^\dagger a_j + h.a.\right)$$
$$+ \Delta \sum_j (-1)^j a_j^\dagger a_j + \frac{U}{2} \sum_j a_j^{\dagger 2} a_j^2$$
$$+ V_1 \sum_{\langle ij\rangle} a_i^\dagger a_i a_j^\dagger a_j + V_2 \sum_{\langle\langle ij\rangle\rangle} a_i^\dagger a_i a_j^\dagger a_j \qquad (465)$$

NN interaction NNN interaction



(A) topological invariant and fractional bulk transport

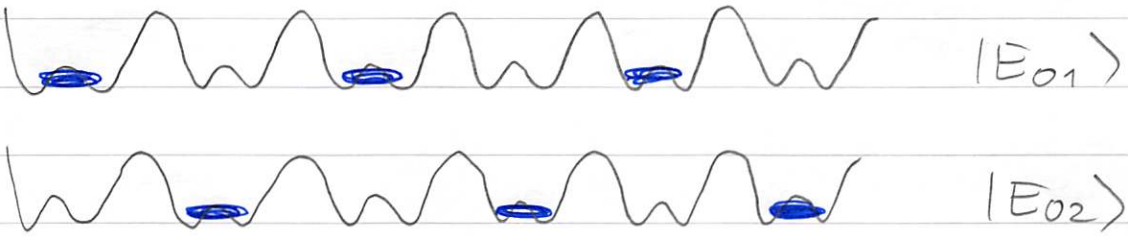
phase diagram of $H - \mu N$



doubly degenerate

- atomic limit $t_2 \gg V_1, V_2 \gg t_1$

two density waves

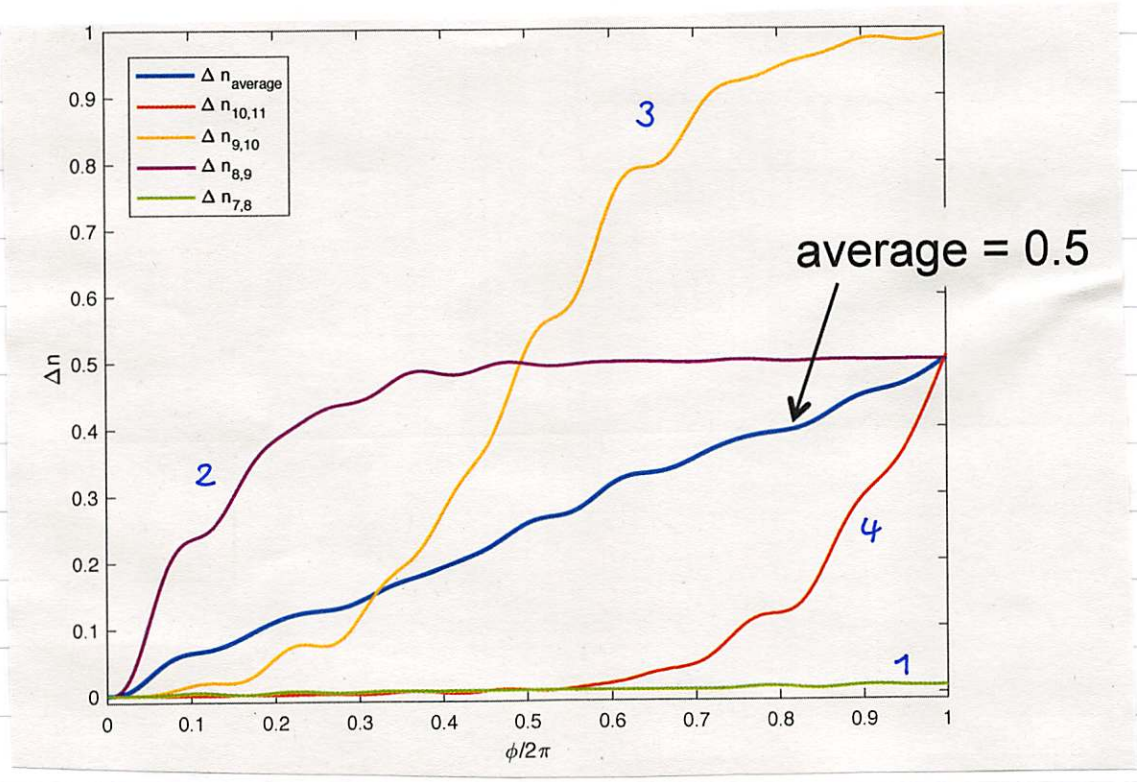


> gapped states with fractional filling are degenerate and spontaneously break a discrete symmetry here lattice translation by one unit cell

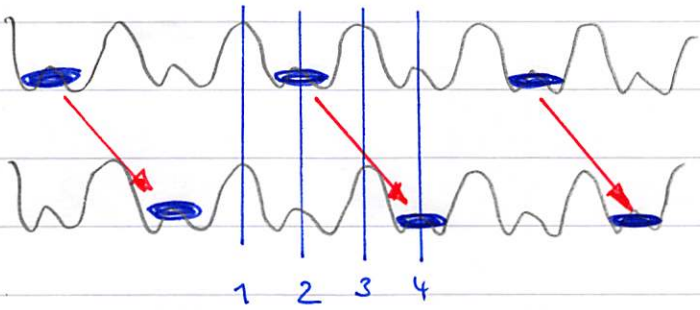
single cycle Thouless pump

(466) $|E_{01}\rangle \longrightarrow e^{i\Delta\varphi} |E_{02}\rangle$

p - degree of degeneracy $\Rightarrow$ p cycles to return to initial state modulo phase



atomic limit



topological invariant (integer) for p-fold degenerate, gapped GS

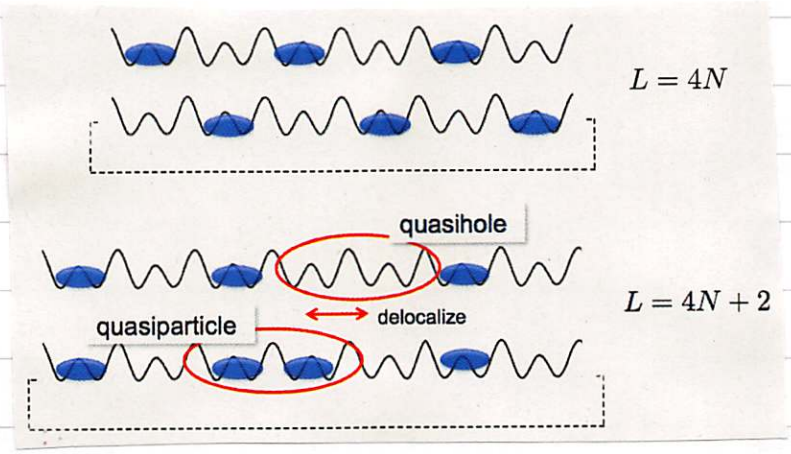
(467) $$\nu = \frac{1}{2\pi} \int_0^{2\pi p} d\theta \, \mathrm{Im} \left\langle E_{0\mu} \middle| \frac{\partial E_{0\mu}}{\partial \theta} \right\rangle \qquad \mu = 1, 2, \ldots, p$$

fractional transport (in one cycle)

(468) $$\Delta n = \frac{1}{p} \nu$$

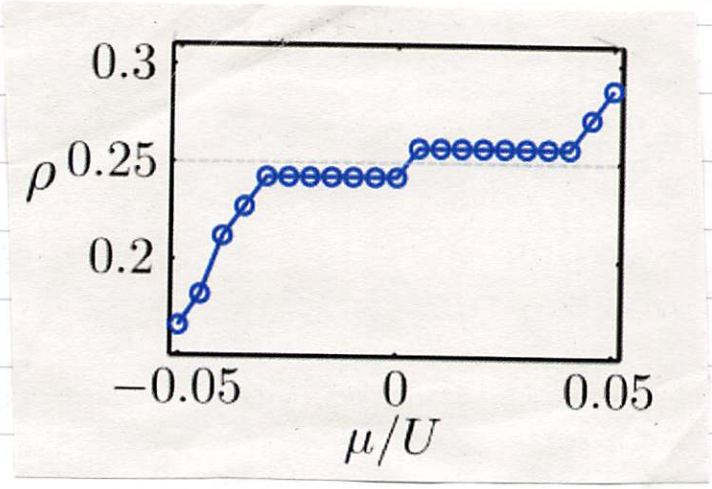
(B) topological excitations

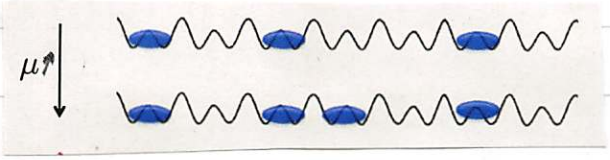
(unit-cell) periodic boundary conditions may be not commensurate with fractional filling



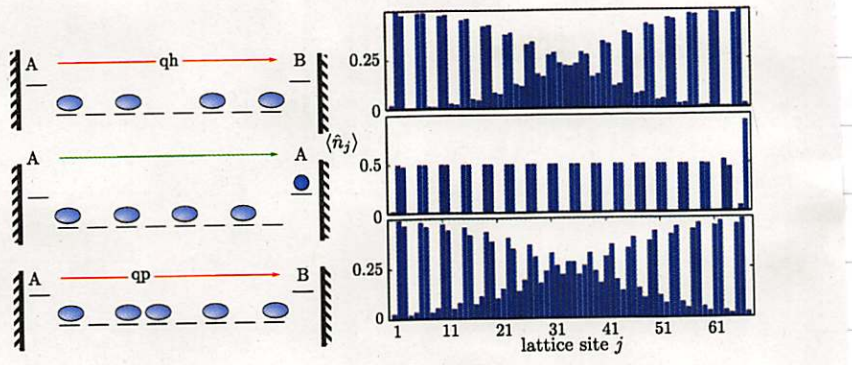
there are domain walls between the two degenerate GS realizations which are for $p = 2$

quasiholes or quasiparticles





behave similarly to qh and qp in FQH effect and can be trapped by a local potential



11.3 Non-Abelian topological invariants

With the extended SL-BHM we have seen that interaction-induced topological states can have degenerate ground states. This is a generic feature of all topological systems with fractional charges. We thus need to discuss the generalization of topological invariants such as the single- or many-body Zak phase to systems with degeneracies which will also provide a verification of eq. (467)

(A) From Berry phase to Wilson loops

An adiabatic (cyclic) variation of a non-degenerate state $|\phi(\lambda)\rangle$ will give rise to a geometric phase, the Berry phase (Zak phase for Bloch functions)

(469) $$\theta = \int d\lambda \, \langle \phi(\lambda) | \partial_\lambda \phi(\lambda) \rangle$$

This is connected with the U(1) gauge freedom of

the wavefunction. Apart from a dynamical phase one has

(470) $|\psi(0)\rangle = |\phi(\lambda=0)\rangle \qquad |\psi(t)\rangle = e^{i\theta(t)} |\phi(\lambda(t))\rangle$

Wilson and Zee (PRL 52, 2111 (1984)) have generalized this concept to the case of an n-fold degenerate sub-space which has a $U(n)$ gauge freedom. Thus apart from dynamical phase

(471) $|\psi(0)\rangle = |\phi_\alpha(\lambda=0)\rangle \rightarrow |\psi(t)\rangle = \sum_\beta \underline{\underline{U}}_{\beta\alpha}(\lambda(t)) |\phi_\beta(\lambda(t))\rangle$

(472) $$\underline{\underline{U}}(t) = T \exp\left\{ i \int_0^t d\tau\, \dot{\lambda}\, \underline{\underline{A}}(\lambda(t)) \right\}$$

$\underline{\underline{A}}(\lambda)$ is now the <u>non-Abelian</u> (i.e. matrix valued) <u>Berry connection</u>

(473) $$\boxed{A^{\alpha\beta} = i \langle \phi_\alpha(\lambda) | \frac{\partial}{\partial \lambda} | \phi_\beta(\lambda) \rangle}$$

If we consider a closed loop in parameter space the exponential of the Berry phase

(474) $e^{i\Delta\theta} \qquad \Delta\theta = \oint d\lambda \langle \phi(\lambda) | \partial_\lambda | \phi(\lambda) \rangle$

is replaced by the so-called <u>Wilson loop</u>

(475) $$\boxed{\underline{\underline{W}} = P \exp\left\{ i \oint d\lambda\, \underline{\underline{A}}(\lambda) \right\}}$$

$\underline{\underline{W}}$ is a unitary $n \times n$ matrix,

(476) $$\underline{\underline{W}}^{+}\,\underline{\underline{W}} = \underline{\underline{1}}$$

The invariance of the (closed loop) Berry phase under gauge transformations (modulo 2π) translates into a $U(n)$ gauge invariance of the Wilson loop

(477) $$\underline{\underline{W}} \to \underline{\underline{W}}' = U^{+}\,\underline{\underline{W}}\,U$$

$\underline{\underline{W}}' \neq \underline{\underline{W}}$ but this represents the freedom of choice in the n-dimensional basis. Gauge invariant quantities are $\det \underline{\underline{W}}$, $\mathrm{tr}\,\underline{\underline{W}}$ and the eigenvalues of $\underline{\underline{W}}$

(B) Multi-band Chern number and Wilson loop

In the case of a non-degenerate many-body ground state we have found (eq. 457)

(478) $$\Delta\bar{n} = \frac{1}{2\pi}\int_0^T dt\,\frac{\partial}{\partial t}\,\mathrm{Im}\int_{-\pi}^{\pi} d\beta\,\langle E_0 | \partial_\beta E_0\rangle$$

$$= \frac{1}{2\pi}\int_0^T dt\,\frac{\partial}{\partial t}\int_{-\pi}^{\pi} d\beta\,A(\beta)$$

Where A is the many-body Berry connection. If there are several (partially) degenerate states, then the derivation in 11.1 (c) must be revised. We have to choose a set of states

$$|E_{0M}\rangle$$

for which

(479) $$\langle E_{\alpha\nu} | \frac{\partial H}{\partial \beta} | E_{\alpha\mu} \rangle = \langle E_{\alpha\nu} | \partial_\beta E_{\alpha\mu} \rangle \sim \delta_{\nu\mu}$$

is diagonal (analog to degenerate perturbation theory). This is in principle possible as the Berry connection matrix

(480) $$A_{\nu\mu} = i \langle E_{\alpha\nu} | \partial_\beta E_{\alpha\mu} \rangle$$

is hermitian. The transport in one of these eigenbands is however not necessarily quantized since after changing β from 0 to 2π (or $-\pi$ to π) the state may not return to itself but can return to one of the other states (note $|E_{\alpha\nu}\rangle = |E_{\alpha\nu}(\beta)\rangle$). However, if we consider the <u>total transport</u> in all (partially) degenerate bands

(481) $$\Delta\bar{n} = \frac{1}{2\pi} \int_0^T dt \frac{\partial}{\partial t} \sum_\mu \int_{-\pi}^{\pi} d\beta \, i \langle E_{\alpha\mu} | \partial_\beta E_{\alpha\mu} \rangle$$

the situation is different as then

(482) $$\Delta\bar{n} = \frac{1}{2\pi} \int dt \frac{\partial}{\partial t} \int_{-\pi}^{\pi} d\beta \; \mathrm{Tr}\{A_{\nu\mu}\}$$

and the trace is invariant under all unitary rotations within the degenerate sub-space. Changing β from $-\pi$ to π leaves the degenate sub-space invariant.

In analogy to the non-degenerate case, eqs. (457)–(459), we now introduce the non-Abelian generalization of the Zak-phase winding number or Chern number.

For $\lambda = \lambda(t)$

$$C = \nu = \frac{1}{2\pi} \oint d\lambda \frac{\partial}{\partial \lambda} \int_{-\pi}^{\pi} d\beta \, \mathrm{Tr}\{\underline{\underline{A}}(\beta)\} \tag{483}$$

This can be expressed in terms of the Wilson loop, as

$$\underline{\underline{W}} = P \exp\left\{ i \int_{-\pi}^{\pi} d\beta \, \underline{\underline{A}}(\beta) \right\}$$

$$\Rightarrow \mathrm{Tr} \int_{-\pi}^{\pi} d\beta \, \underline{\underline{A}}(\beta) = \mathrm{Im} \, \mathrm{Tr} \log(\underline{\underline{W}})$$

using

$$\mathrm{Tr} \log(\underline{\underline{W}}) = \log(\det \underline{\underline{W}}) \tag{484}$$

eq. (483) can be expressed in terms of the determinant of the Wilson loop, i.e. the product of the eigenvalues of $\underline{\underline{W}}$

$$C = \frac{1}{2\pi} \oint d\lambda \frac{\partial}{\partial \lambda} \mathrm{Im} \log(\det \underline{\underline{W}}) \qquad \lambda = \lambda(t) \tag{485}$$

for a 1D system with λ-periodic (time-periodic) Hamiltonian. One recognizes the gauge invariance of C since $\det \underline{\underline{W}}$ is gauge invariant while $\underline{\underline{W}}$ itself is not

(c) Physical interpretation of Wilson loop: non-interacting systems

To obtain a physical interpretation of the Wilson loop, we go back to non-interacting systems. In the non-degenerate case we have found the King-Smith Vanderbildt relation between the Zak phase and the center of mass of Wannier functions (eq (60), eq. (248))

$$\langle w_n | \hat{x} | w_n \rangle = \frac{1}{2\pi} \int_{BZ} dk \, A(k) = \frac{i}{2\pi} \int_{BZ} dk \, \langle u_n(k) | \partial_k u_n(k) \rangle \tag{486}$$

where $A(k)$ is now the single-particle Berry connection in band n. In the case of a p-fold degeneracy we have

$$|u_n(k)\rangle \to |u_n^\alpha(k)\rangle \qquad \alpha = 1, 2, \dots, p$$

(487)

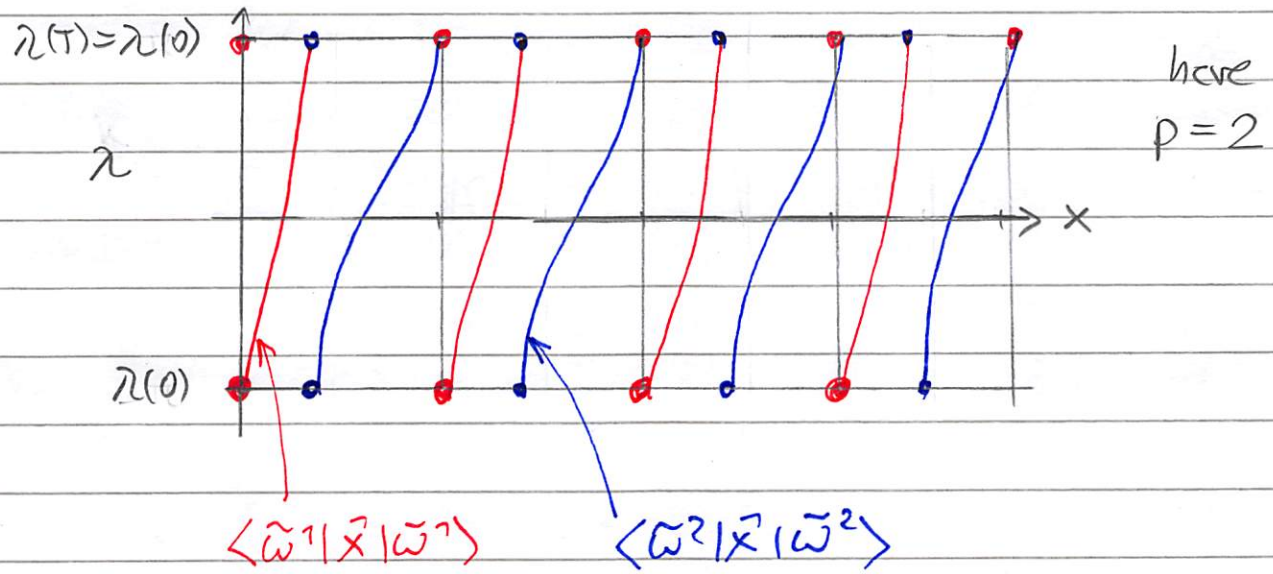
$$A(k) \to A_{\alpha\beta}(k) = i \langle u_n^\alpha(k) | \partial_k u_n^\beta(k) \rangle$$

Using a basis $|\tilde{u}_n^\alpha(k)\rangle$ or $|\tilde{w}_n^\alpha\rangle$ which diagonalizes $\underline{\underline{A}}$ we see that for the center of the α'th eigen-Wannier function holds

$$\langle \tilde{w}_n^\alpha | \hat{x} | \tilde{w}_n^\alpha \rangle = \frac{i}{2\pi} \int_{BZ} dk \, \langle \tilde{u}_n^\alpha(k) | \partial_k \tilde{u}_n^\alpha(k) \rangle \tag{488}$$

If we now change the 1D band hamiltonian $H(k)$ which is periodic in k in one period, the Wannier-

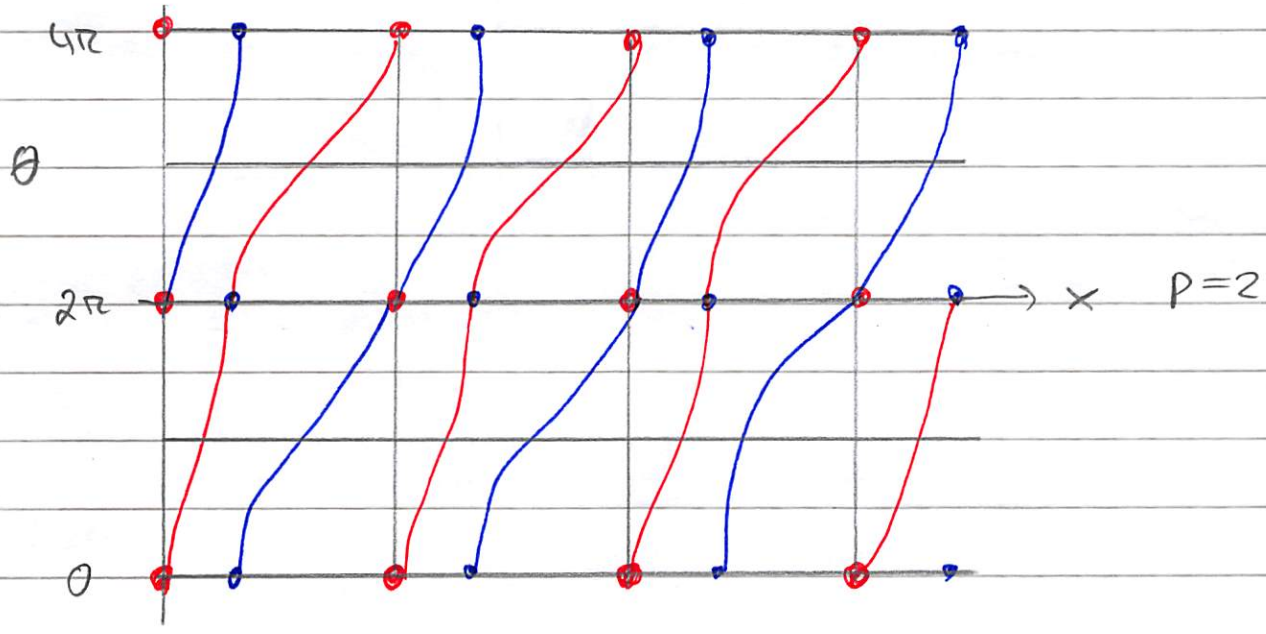
centers (488) do not have to return to themselves but will <u>exchange places</u>. This is the generic case unless there are accidential degeneracies in $A_{\alpha\beta}$.



<u>Rem:</u> that Wannier centers exchange places is the only other option than the trivial one, since the manifold of centers is mapped to itself in one period! Crossings do not occur unless there are accidential degeneracies of the eigenvalues of $\underline{\underline{A}}$

The above example also shows that after p cycles the Wannier centers do come back to themselves, thus the winding after a <u>p-fold parameter loop</u> in λ is the same for all eigen-Wannierfunctions ($\lambda \to \theta$)

$$\int_0^{2\pi p} d\theta \frac{\partial}{\partial\theta} \langle \tilde{\omega}^\alpha | \bar{x} | \tilde{\omega}^\alpha \rangle = \int_0^{2\pi p} d\theta \frac{\partial}{\partial\theta} \langle \tilde{\omega}^\beta | \bar{x} | \tilde{\omega}^\beta \rangle \quad \forall\, \alpha,\beta \tag{489}$$



Thus $(x \to \theta)$

$$C = \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{\partial}{\partial \theta} \int_{-\pi}^{\pi} d\beta \, Tr\{\underline{\underline{A}}(\beta)\} = \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{\partial}{\partial \theta} \int_{-\pi}^{\pi} d\beta \sum_\alpha A^{\alpha\alpha}(\beta)$$

(490)
$$\boxed{C = \frac{1}{2\pi} \int_0^{2\pi p} d\theta \frac{\partial}{\partial \theta} \int_{-\pi}^{\pi} d\beta \, A^{\alpha\alpha}(\beta)} \quad \text{for any } \alpha$$

which verifies eq. (467) and shows that there is integer transport after p cycles.

(D) Wilson loops and many-body polarization

In the non-degenerate case we have argued that the discussion in terms of Wannier center can be generalized to interacting systems by looking at Resta's many body formulation of the polarization ($\hat{=}$ Wannier center) for a filled band n

$$\langle \omega_n | \hat{x} | \omega_n \rangle \iff P_n = \frac{1}{2\pi} \mathrm{Im} \ln \left\langle \exp\left(i \frac{2\pi}{L} \hat{x}\right) \right\rangle$$

and we had

$$C_n = \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{\partial}{\partial \theta} \int_{BZ} dk \langle u_n(k) | \partial_\theta u_n(k) \rangle = \int_0^{2\pi} d\theta \frac{\partial}{\partial \theta} P_n \tag{491}$$

We can now generalize this to the case of degeneracies using (490)

$$C = \int_0^{2\pi p} d\theta \frac{\partial}{\partial \theta} P(\theta) = \frac{1}{2\pi} \int_0^{2\pi p} d\theta \frac{\partial}{\partial \theta} \mathrm{Im} \ln \left\langle e^{\frac{2\pi i}{L} \hat{x}} \right\rangle \tag{492}$$

where $\langle \dots \rangle = \mathrm{Tr}\{\rho \dots\}$ with ρ being any superposition of the degenerate many-body ground states.