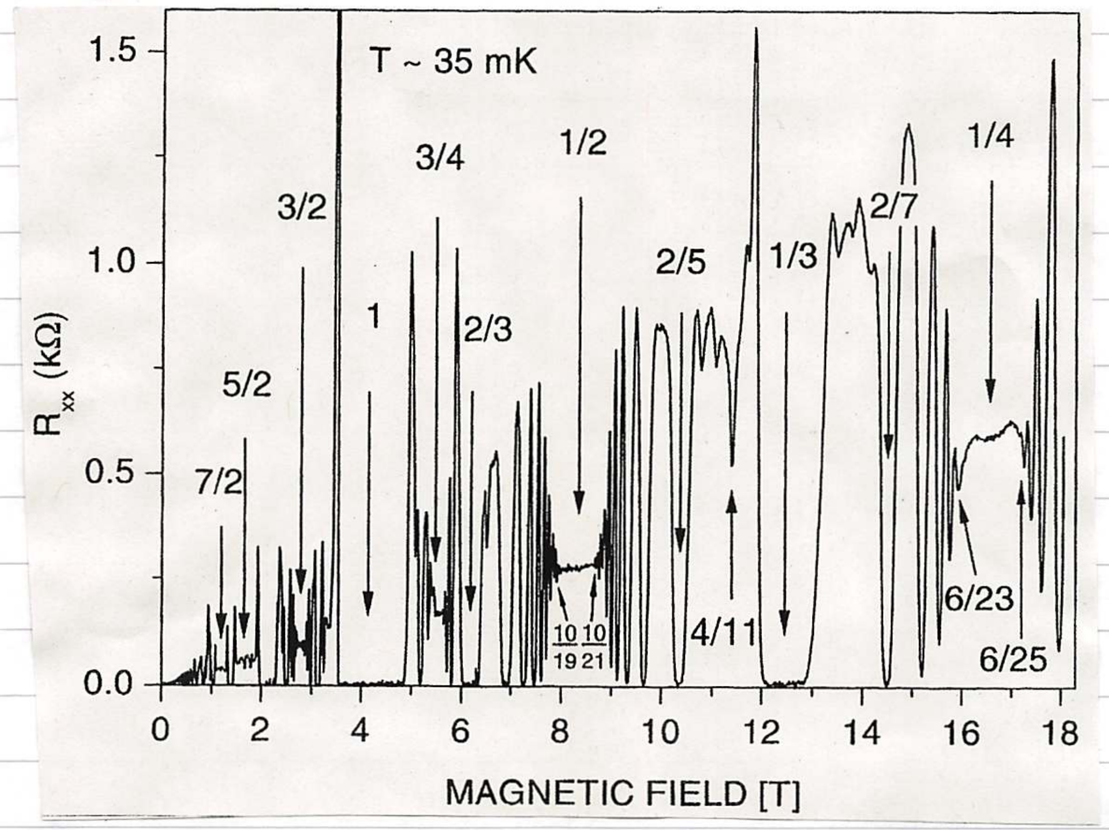
10. Fractional Quantum Hall Effect and composite fermions

Shortly after the discovery of the IQHE, Tsui, Störmer, and Gossard observed quantised Hall plateaus

$$\sigma_H = f \frac{e^2}{h}$$

at larger magnetic fields which corresponded to fractional fillings f mostly less than one



Most fractional plateaus had the structure

$$f = \frac{n}{2n+1} \tag{340}$$

These can only be understood taking into account interactions

10.1. Haldane's pseudopotentials

for sufficiently strong magnetic fields it is sufficient to

(i) consider only one spin component (Zeeman splitting)

(ii) consider only LLL

2-particle problem in 2D in external magnetic field

(341) $$H = H_1 + H_2 + V = H_0 + V$$

(342) $$H_i = \frac{1}{2m}\left(\vec{p}_i + \frac{e}{c}\vec{A}(\vec{r}_i)\right)^2 \qquad V = V(|r_1 - r_2|)$$

complex coordinates $\quad x_i, y_i \to z_i, z_i^*$

COM and relative coordinates

(343) $$Z = \frac{1}{2}(z_1 + z_2) \qquad z = z_1 - z_2$$

(344) $$H = \frac{1}{2}\left[-4\ell_R^2 \frac{\partial^2}{\partial Z \partial Z^*} + \frac{1}{4\ell_R^2} Z Z^* - Z\frac{\partial}{\partial Z} + Z^*\frac{\partial}{\partial Z^*}\right]$$

COM motion

$$+ \frac{1}{2}\left[-4\ell_r^2 \frac{\partial^2}{\partial z \partial z^*} + \frac{1}{4\ell_r^2} z z^* - z\frac{\partial}{\partial z} + z^*\frac{\partial}{\partial z^*}\right] + V(z, z^*)$$

relative motion

where

(345) $$\ell_R = \frac{1}{\sqrt{2}}\ell_m \qquad \ell_r = \sqrt{2}\,\ell_m$$

We can separate COM and relative motion (as usual)

COM	$\hat{a}_R, \hat{a}_R^+ \; ; \; \hat{b}_R, \hat{b}_R^+$	see chapter 2
relative	$\hat{a}, \hat{a}^+ \; ; \; \hat{b}, \hat{b}^+$	

LLL $$\boxed{\hat{a}_R^+\hat{a}_R\,|\psi\rangle = 0 \qquad \hat{a}^+\hat{a}\,|\psi\rangle = 0}$$

$\Rightarrow$ complete set of states in LLL for two particles

(346) $$|M,m\rangle = \frac{(b_R^+)^M\,(b^+)^m}{\sqrt{M!\,m!}}\,|0,0\rangle$$

- two-particle wavefunctions in LLL

(347) $$\psi^{LLL}_{M,m}(z,Z) \sim Z^M z^m \exp\left\{-\frac{zz^*}{4\ell_r^2} - \frac{ZZ^*}{4\ell_R^2}\right\}$$

We assume that the interaction V is weak enough, such that there are no transitions to higher LLL, so we only need to consider the projection to the LLL

$$PVP = \sum_{mM}\sum_{m'M'} |M'm'\rangle\langle M',m'|V|M,m\rangle\langle M,m|$$

since $V = V(|z|) \Rightarrow$

$$\langle M', m' | V | M, m \rangle = \delta_{MM'} \delta_{mm'} \langle m | V | m \rangle$$

so $$PVP = \sum_{m,M} |M,m\rangle\langle M,m| \, V_m$$

(348) $$\boxed{V_m = \langle m | V | m \rangle}$$ Haldane's pseudopotentials

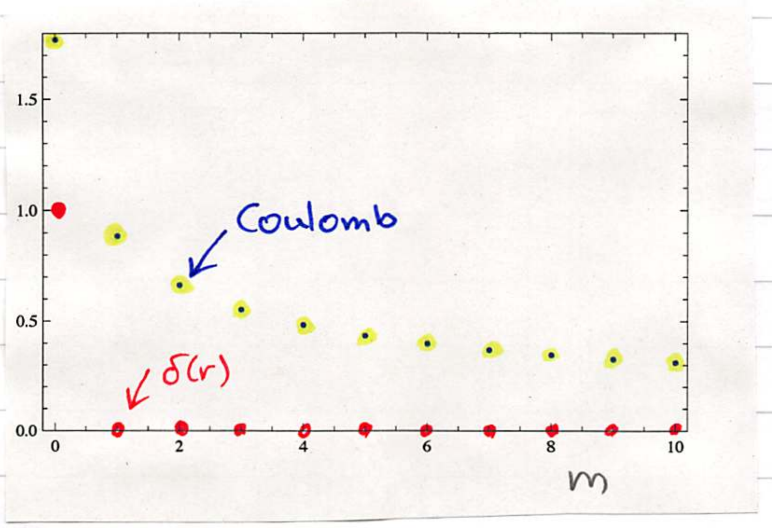
for Coulomb interaction

(349) $$\boxed{V_m^C = \frac{1}{2 l_m} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)}}$$ $V(r) = \frac{1}{r}$

for point interaction

(350) $$\boxed{V_0 = 1 \qquad V_2 = V_3 = V_4 = \cdots = 0}$$ $V(r) = \delta(r)$

V_m



bosons: only even m

fermions: only odd m

10.2. Laughlin's wavefunction

The single-particle wavefunctions in the LLL with angular momentum (in z direction) ℓ are ($\ell_m \overset{!}{=} 1$)

(351) $$\eta_\ell(z) = (2\pi\, 2^\ell \ell!)^{-1/2}\, z^\ell\, e^{-\frac{1}{4}|z|^2}$$

The many-body wavefunction thus must have the structure

(352) $$\Psi = F_A[\{z_i\}] \exp\left\{-\frac{1}{4}\sum_j |z_j|^2\right\}$$

F_A must be antisymmetric (for fermions) and only a function of the differences between coordinates. One can make some educated guesses

(i) motivated by the success of so called Jastrow-type wavefunctions in the description of superfluid He we set

$$F_A[\{z_i\}] = \prod_{j<k} f(z_j - z_k)$$

(ii) Ψ should be eigenfunction of total angular momentum

$\Rightarrow$ $\prod_{j<k} f(z_j - z_k)$ polynomial of z_j of order L

rotation in xy-plane by angle θ: $z_j \to z_j e^{i\theta}$
if ψ eigenfunction of $\hat{L}_z$ $\psi \to \psi e^{-iL\theta}$,
i.e $f(z_j - z_n)$ must have fixed angular momentum

(iii) f must be antisymmetric (symmetric for bosons)

$$f(-z) = f(z)$$

$$\Rightarrow \quad \boxed{f(z) = z^m} \qquad m \begin{cases} \text{odd} & \text{fermions} \\ \text{even} & \text{bosons} \end{cases}$$

This gives as an Ansatz

$$(353) \quad \boxed{\psi_{\frac{1}{m}} = N \prod_{j<k} (z_j - z_k)^m \exp\left\{-\frac{1}{4}\sum_j |z_j|^2\right\}}$$

<u>Laughlin wavefunction</u>

One finds that $\psi_{1/m}$ results in a <u>uniform density</u> distribution with <u>fractional filling</u>

$$\boxed{\nu = \frac{1}{m}}$$

One finds that for Coulomb interactions the wavefunctions $\psi_{1/3}$ and $\psi_{1/5}$ (i.e. $m = 3, 5$) are exact ground states with filling $\nu = 1/m$

<u>but:</u> $\exists$ other fractions at which there are Hall plateaus !

10.3 Jain's composite fermions

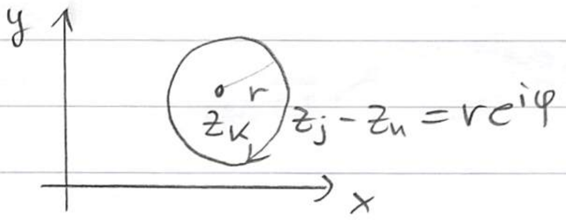
let's have a look again at Laughlin wavefunction ($l_m = 1$)

(354) $$\Psi_{1/2p+1} = \mathcal{N} \prod_{j<k} (z_j - z_k)^{2p} \underbrace{(z_j - z_k)\, e^{-\frac{1}{4}\sum_l |z_l|^2}}_{\text{like in IQHE}}$$

$$\nu = \frac{1}{2p+1}$$

the prefactor looks like a vortex solution for the j th particle around the position of the k th particle

$$(z_j - z_k)^{2p}$$



Vortex of strength $2p$
at position of every other particle

interpretation: electron in LLL with $2p$ vortices (flux quanta) attached to them

$\Rightarrow$ **composite fermions**

bare F $\quad$ ^{2}CF $\quad$ ^{4}CF $\quad$ ^{6}CF

$\boxed{\text{CF = bound state of electron and an even number of flux quanta attached to them}}$

If the original particles are bosons then one can construct

$$(3\mathrm{JJ}) \quad \Psi_{1/2p} = \mathcal{N} \prod_{j<k} (z_j - z_k)^{2p-1} (z_j - z_k) e^{-\frac{1}{4}\sum_e |z_e|^2}$$

These states describe fractional quantum Hall states at even filling fractions

$$\nu = 1/2p$$

and can be considered composite fermions with an odd number of flux quanta

$$\boxed{\text{CF} = \text{bound state of \underline{boson} with an odd number of flux quanta attached to them}}$$

<u>Note!</u> $\Psi_{1/2p+1}$ and $\Psi_{1/2p}$ describe <u>non-interacting</u> composite fermions

(A) Why are non-interacting composite fermions good candidates for (approximate) ground-states

- composite wavefunctions fulfill all symmetry requirements and are eigenfunctions of the kinetic energy H_0 ✓
- What is the effect of the interaction?

$$PVP = \sum_{m,M} V_m \, |M,m\rangle\langle M,m|$$

$\Rightarrow$

remember $L_z \sim -b^+b$

$$\langle 2\psi_{1/2p+1} | PVP | 2\psi_{1/(2p+1)} \rangle$$

(356) $$\sim \sum_m V_m \left(\sum_{n=0}^{m} c_n \int d^2z \; z^{*n} z^{2p+1} e^{-|z|^2/2} \right)^2$$

carry out angular integration $z = re^{i\varphi}$

$$\int_0^{2\pi} d\varphi \; e^{i\varphi(2p+1-n)} \sim \delta_{n,2p+1}$$

i.e. only terms with $m \geq 2p+1$ contribute in (356)

all pseudopotentials V_m

(357) $$\boxed{m < 2p+1 \qquad \text{are screened}}$$

i.e. do not give contribution to the energy

coupling of $2p$ flux quanta to an electron lead to formation of a ${}^{2p}CF$ which eliminates all pseudopotential contributions with $m \leq 2p$

(B) FQHE as integer QHE von CF's: idea

(i) electrons at filling ν at magnetic field B

(358) $$\nu = \frac{\varsigma \Phi_0}{|B|} \qquad \vec{B}: \uparrow\uparrow\uparrow$$

(ii) write electron as CF plus flux quanta in opposite direction

$$\underset{F}{o} \longrightarrow \underset{{}^{2p}CF}{\overset{\uparrow\uparrow}{o}} + \underset{2p \text{ flux quanta}}{\downarrow\downarrow}$$

(iii) distribute the 2p flux quanta homogeneously over whole system $\Rightarrow$ effective magnetic field

(359) $$\boxed{B^* = B - 2p\varsigma\Phi_0}$$

This corresponds to a filling ν^* in the reduced magnetic field of

$$\nu^* = \frac{\varsigma\Phi_0}{|B^*|} = \frac{\varsigma\Phi_0}{|B - 2p\varsigma\Phi_0|}$$

$$\nu^* = \frac{1}{1/\nu - 2p} \quad \text{or} \quad \nu^* = \frac{1}{2p - 1/\nu}$$

if $\nu^* = n \in \mathbb{N} \Rightarrow$

(360) $$\boxed{\nu = \frac{n}{2pn \pm 1}}$$ Jain's principle sequence

$\Rightarrow$ FQHE $\hat{=}$ IQHE for composite fermions at reduced effective magnetic field

10.4 Field theory of composite fermions

(A) bare composite particles

We want to construct a theory for composite fermions which arise from bare fermions by attachment of flux quanta.

bare fermion field $\hat{\psi}(z)$ $\qquad z = x + iy$

(361) $$\{\hat{\psi}(z), \hat{\psi}^{+}(z')\} = \delta^{(2)}(z-z')$$

define a bare composite particle

(362) $$\boxed{\hat{\phi}_m(z) \equiv e^{-im\hat{\theta}(z)}\,\hat{\psi}(z)}$$

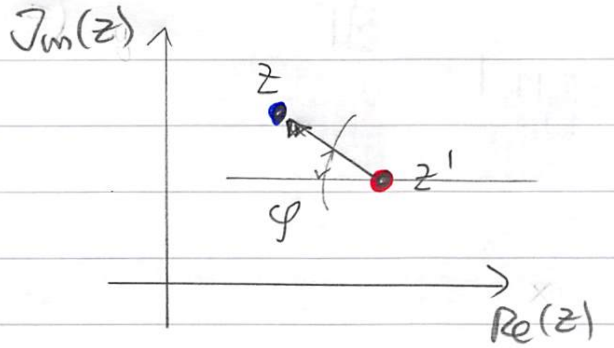
where the angle operator $\hat{\theta}(z)$ is defined as

(363) $$\boxed{\hat{\theta}(z) \equiv \int d^2z' \,\varphi(z-z')\,\hat{n}(z')}$$

Here

(364) $$\hat{n}(z) = \hat{\psi}^{+}(z)\hat{\psi}(z) = \hat{\phi}^{+}_m(z)\hat{\phi}_m(z)$$

is the density operator and φ is the azimuthal angle between z and z'



now use also "x" and "y" as names of complex coordinates

using (365)

$$[\hat{\Theta}(z), \hat{\Psi}(x)] = \int d^2z' \, \varphi(z-z') \underbrace{[\hat{n}(z'), \hat{\Psi}(x)]}_{= -\delta(z'-x)\hat{\Psi}(x)}$$

$$= -\varphi(z-x)\hat{\Psi}(x)$$

$\Rightarrow$

(365) $$e^{im\hat{\Theta}(y)}\hat{\Psi}(x)e^{-im\hat{\Theta}(y)} = e^{-im\varphi(y-x)}\hat{\Psi}(y)$$

Thus

$$\hat{\Phi}_m(x)\hat{\Phi}_m(y) = e^{-im\hat{\Theta}(x)}\hat{\Psi}(x)\,e^{-im\hat{\Theta}(y)}\hat{\Psi}(y)$$

$$= e^{-im\varphi(y-x)}e^{-im\hat{\Theta}(x)}e^{-im\hat{\Theta}(y)}\hat{\Psi}(x)\hat{\Psi}(y)$$

and

$$\hat{\Phi}_m(y)\hat{\Phi}_m(x) = e^{-im\varphi(x-y)}e^{-im\hat{\Theta}(y)}e^{-im\hat{\Theta}(x)}\hat{\Psi}(y)\hat{\Psi}(x)$$

since $[\hat{\Theta}(x), \hat{\Theta}(y)] = 0 \Rightarrow$

(366) $$\hat{\Phi}_m(x)\hat{\Phi}_m(y) = (-1)^{m+1}\hat{\Phi}_m(y)\hat{\Phi}_m(x)$$

Thus

(367)
$$\boxed{[\hat{\Phi}_m(x), \hat{\Phi}_m(y)] = 0 \qquad m = \text{odd}}$$
$$\boxed{\{\hat{\Phi}_m(x), \hat{\Phi}_m(y)\} = 0 \qquad m = \text{even}}$$

I.e. composite particles constructed from a bare fermion have <u>bosonic</u> (m=odd) or <u>fermionic</u> (m=even) transmutation statistics

note that composite bosons (m: odd) constructed from bare fermions still fulfill the hard-core constraint

(368) $$\hat{\phi}_m^2(x) = 0 \quad \text{since} \quad \hat{\psi}(x)^2 = 0$$

One can repeat the above calculations to find

(369)
$$\boxed{\{\hat{\phi}_m(x), \hat{\phi}_m^+(y)\} = \delta^{(2)}(x-y) \qquad m = \text{even}}$$
$$\boxed{[\hat{\phi}_m(x), \hat{\phi}_m^+(y)] = \delta^{(2)}(x-y) \qquad m = \text{odd}}$$

- relation between wave-functions

N-particle wavefunction for many body state $|\psi\rangle$

(370)
$$\Phi_m(x_1,..,x_N) \equiv \langle 0| \hat{\phi}_m(x_1) \cdots \hat{\phi}_m(x_N) |\psi\rangle \qquad \text{composite}$$
$$\Psi(x_1,..,x_N) \equiv \langle 0| \hat{\psi}(x_1) \cdots \hat{\psi}(x_N) |\psi\rangle \qquad \text{bare}$$

or

(371)
$$|\psi\rangle = \int d^{2N}x \; \Phi_m(\underline{x}) \, \phi_m^+(x_1) \cdots \phi_m^+(x_N) |0\rangle$$
$$= \int d^{2N}_x \; \Psi(\underline{x}) \, \psi^+(x_1) \cdots \psi^+(x_N) |0\rangle$$

$$\hat{\phi}^+_m(y)\hat{\phi}^+_m(x) = e^{im\varphi(y-x)}\,\hat{\psi}^+(y)\hat{\psi}^+(x)\,e^{im\hat{\Theta}(y)}e^{im\hat{\Theta}(x)}$$

so $\quad \hat{\phi}^+_m(y)\,\hat{\phi}^+_m(x)|0\rangle = e^{im\varphi(y-x)}\,\hat{\psi}^+(y)\,\hat{\psi}^+(x)\,|0\rangle$

thus

$$\hat{\phi}^+_m(x_1)\cdots\hat{\phi}^+_m(x_N)|0\rangle = e^{im\sum_{\ell<j}\varphi(x_\ell-x_j)}\,\hat{\psi}^+(x_1)\cdots\hat{\psi}^+(x_N)|0\rangle$$

which gives the relation between the wavefunctions

(372) $$\boxed{\Psi(x_1,\dots,x_N) = e^{im\sum_{\ell<j}\varphi(z_\ell-z_j)}\,\Phi_m(x_1\dots x_N)}$$

(B) effective magnetic field and Chern-Simons field

The Hamiltonian of electrons in the LLL, eq. (109), reads in 2nd quantization (dimensionless units)

$$H = \frac{1}{2m}\int d^2z\,\hat{\psi}^+(z)\left[\underbrace{\left(-i\frac{\partial}{\partial x}-\frac{y}{2}\right)}_{\equiv P_x}^2 + \underbrace{\left(-i\frac{\partial}{\partial y}+\frac{x}{2}\right)}_{\equiv P_y}^2\right]\hat{\psi}(z)$$

(373)

$$= \frac{1}{2m}\int d^2z\,\hat{\psi}^+(z)\left[P_x^2+P_y^2\right]\hat{\psi}(z)$$

$$\int d^2z\,\hat{\psi}^+(z)\left[(P_x+iP_y)(P_x-iP_y)\right]\hat{\psi}(z)$$

$$+\frac{1}{2}N\hbar\omega_c$$

where we have reintroduced units.

note that $[P_x, P_y] \neq 0$. Using eq. (362) we can express this in terms of composite particles

(374) $$H = \frac{1}{2m} \int d^2z\, \hat{\Phi}_m^{+}(z) \left(\tilde{P}_x + i\tilde{P}_y\right)\left(\tilde{P}_x - i\tilde{P}_y\right) \hat{\Phi}_m(z) + \frac{1}{2} N\hbar\omega_c$$

where the "covariant" momenta are

(375) $$\tilde{P}_\mu = P_\mu + e\, C_\mu = -i\hbar \frac{\partial}{\partial x_\mu} + e A_\mu^{eff}$$

Here $(\mu = x, y)$ $(z = x + iy)$

(376) $$\boxed{C_\mu(z) = \frac{\hbar m}{e} \partial_\mu \hat{\Theta}(z) = \frac{\hbar m}{e} \int d^2z'\, \partial_\mu \varphi(z - z')\, \hat{n}(z')}$$

is the vector potential of the Chern-Simon field

and the effective total vector potential reads

(377) $$\boxed{A_\mu^{eff} = A_\mu + C_\mu}$$

Non-interacting fermions in the LLL are identical to composite particles with an additional Chern-Simon field (provided a composite-particle state exists that also fulfills the LLL condition)

The Chern Simon field is given by

(378)
$$\boxed{\begin{aligned}\varepsilon_{jk}\partial_j C_k(z) &= \frac{\hbar m}{e}\varepsilon_{jk}\partial_j\partial_k\Theta(z)\\ &= m\,\Phi_0\,\delta^{(2)}(z)\end{aligned}}$$

The Chern Simon field of Φ_m-composite's corresponds to m Dirac flux quanta at the position of the composite particle. One easily finds that composite particles in an external magnetic field $B_\perp$ experience an effective magnetic field

(379)
$$\boxed{B_{eff}(z) = B_\perp - m\,\Phi_0\,\hat{n}(z)}$$

Composite fermions ($m = 2p$; i.e. even) with filling

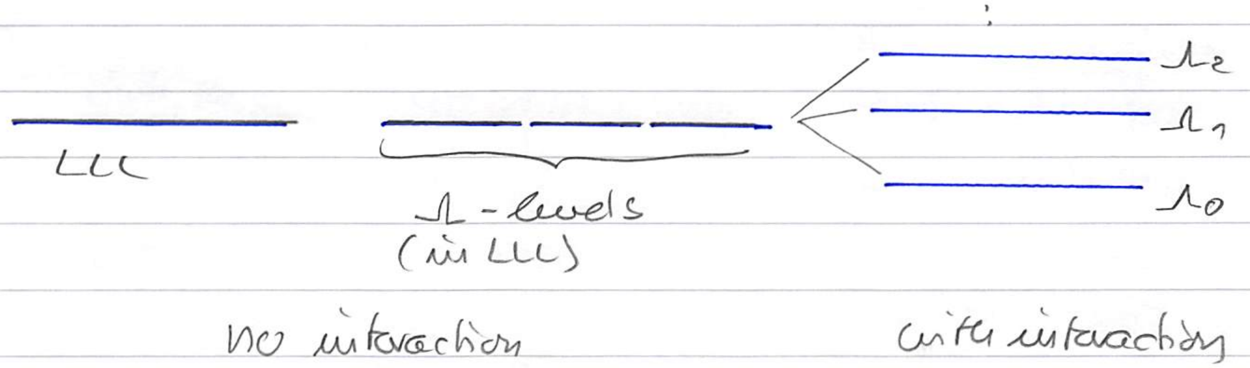
(380)
$$\nu = \frac{n\,\Phi_0}{B_\perp} = \frac{1}{2p+1}$$

fill all states of the lowest Λ-level with effective filling

(381)
$$\nu^* = \frac{n\,\Phi_0}{B_{eff}} = \frac{n\,\Phi_0}{B_\perp - 2p\,\Phi_0 n} = \frac{1}{\frac{1}{\nu} - 2p} = 1$$

States with filling $\nu > \frac{1}{2p+1}$ occupy higher Λ levels. Without interaction the higher Λ-levels can be however energetically degenerate.

With interactions the lowest Λ-level wavefunction screens all small pseudopotentials, while states with components in the higher Λ levels have interaction energy and are thus split



(C) dressed composite fermions

Bare composite wavefunctions have singularities. To avoid them we slightly modify the phase operator (363). This modification is different for composite bosons and composite fermions. We here discuss CF's in the LLL with $m = 2p$ (even)

(382) $$\boxed{\hat{\Psi}_{2p}(z) \equiv e^{-i2p\hat{\Theta}(z) - \hat{a}_{cs}(z)}\, \hat{\Psi}(z)}$$

with a scalar field

(383) $$\boxed{\hat{a}_{cs}(z) \equiv 2p \int d^2z' \ln\left(\frac{|z-z'|}{2\ell_0}\right)\hat{n}(z') - \frac{eB^{cs}}{4\hbar}|z|^2}$$

Where B_{cs} is the (homogenized) Chern Simon field

(384) $$B^{cs} = 2p\,\Phi_0\,\rho$$

The transmutation properties are just defined by $\hat{\Theta}(z)$ and thus do not change.

For $\nu^* = 1$ i.e. $\nu = \frac{1}{2p+1}$ the many-particle wave-functions read

(385) $$\Phi_{2p}(z_1,..,z_N) = \mathcal{N} \prod_{j<e} (z_j - z_e) \exp\left(-\sum_j \frac{|z_j|^2}{2 l_{eff}^2}\right)$$
$$= \mathcal{N} \prod_{j<e} (z_j - z_e) \exp\left(-\frac{eB_{eff}}{4\hbar} \sum_j |z_j|^2\right)$$

with the effective magnetic length

(386) $$l_{eff} = \sqrt{\frac{\hbar c}{eB_{eff}}}$$

and using the relation between $\hat{\Psi}_{2p}(z)$ and $\hat{\Psi}(z)$ following the arguments on page 151-152 we find for the wavefunction of the original fermions

(387) $$\boxed{\begin{aligned} \Psi(z_1,..,z_N) &= \Phi_{2p}(z_1.. z_N) \prod_{j<e} (z_j - z_e)^{2p} \\ &\quad \cdot \exp\left(-\frac{eB_{cs}}{4\hbar} \sum_j |z_j|^2\right) \end{aligned}}$$

where $B_{eff} + B_{cs} = B_\perp$ and thus

(388) $$\Psi(z_1,..,z_N) = \mathcal{N} \prod_{j<e} (z_j - z_e)^{2p+1} \exp\left(-\frac{1}{2}\sum_j \frac{|z_j|^2}{l_0^2}\right)$$

which are Laughlin wavefunctions for the FQHE

(D) general CF-diagonalization: Λ-levels

Eq. (387) can be generalized to a general construction scheme

$$\nu = \frac{\nu^*}{2p\nu^* \pm 1}$$

given $\boxed{\nu}$

$\downarrow$

find $\boxed{\nu^*}$ — choose $2p$ such that ν^* maximum

$\downarrow$

construct basis of composites $\boxed{\Phi^{\alpha}_{\nu^*}}$ — α: index for different EZ of integer QHE at ν^*

$\downarrow$

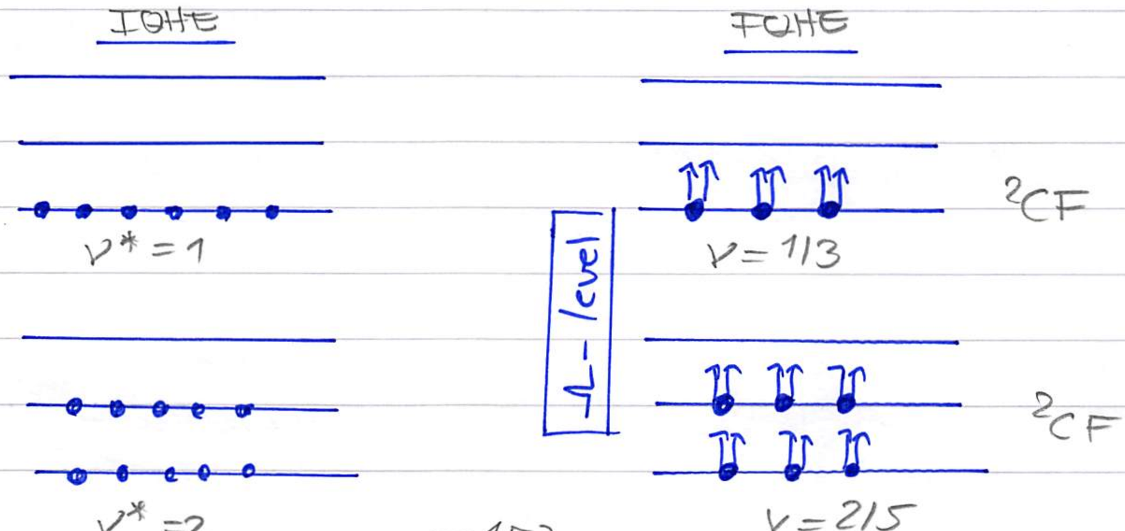
$$\boxed{{}^{2p}\psi^{\alpha}_{\nu} = P_{LLL} \prod_{j<k} (z_j - z_k)^{2p}\, \Phi^{\alpha}_{\nu^*}}$$ basis of bare fermions

$\downarrow$

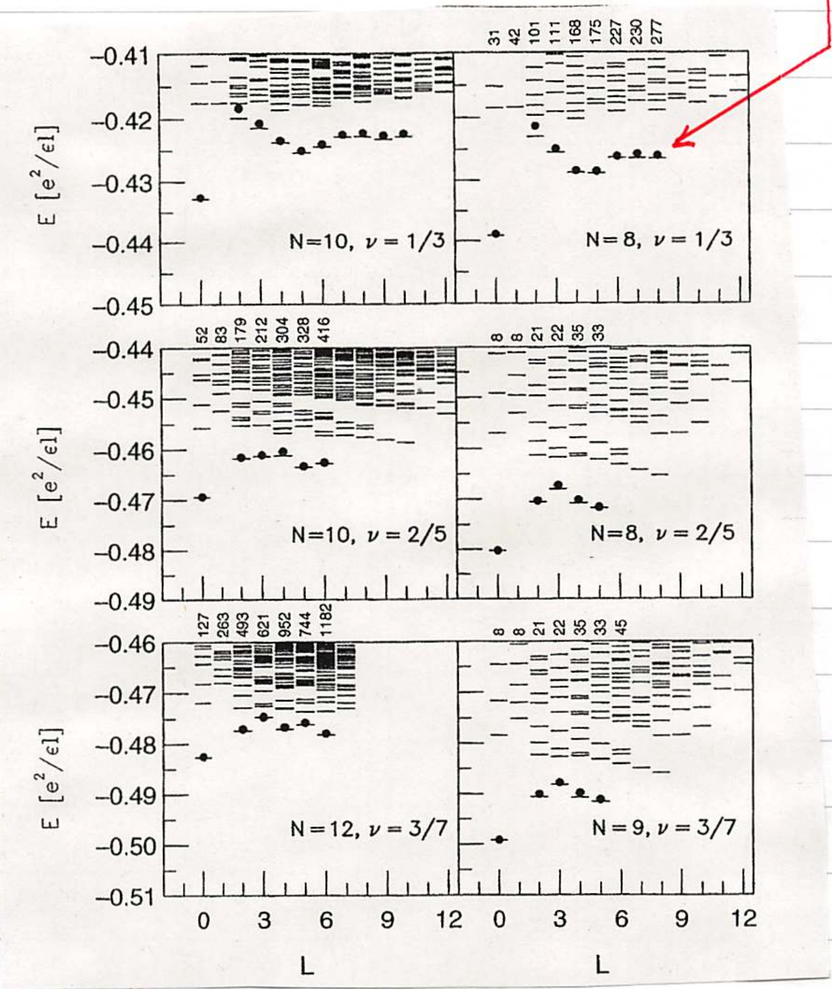
$\boxed{\text{orthogonalize}}$

$\downarrow$

$\boxed{\text{diagonalize full Hamiltonian with interactions}}$



comparison full diagonalization and CF – theory



differences between CF – theory on 1-levels and FQHE

(i) energy of particle hole pairs is <u>not</u> $(\nu^* = n)$

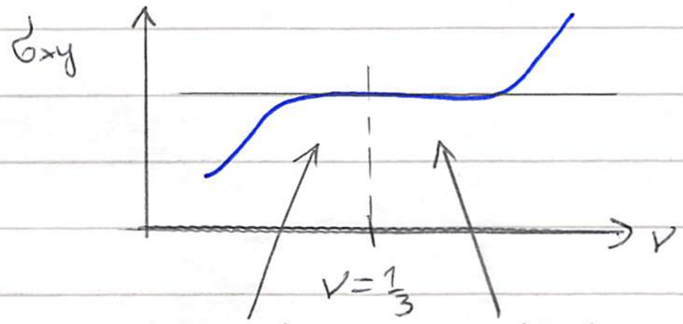
$$\hbar\omega_c^* = \frac{\hbar e B_{eff}}{mc} = \frac{\hbar e B}{mc}\,\frac{1}{2pn \pm 1}$$

(ii) Hall conductivity is still

$$\sigma_H = \nu \frac{e^2}{h} \quad \text{and } \underline{\text{not}} \quad \sigma_H^{false} = \nu^* \frac{e^2}{h}$$

10.5 Quasihole and quasi-particle excitations

We now want to discuss the elementary excitations of an incompressible FQH state which become relevant if we move away from the strict fractional filling by changing the magnetic field

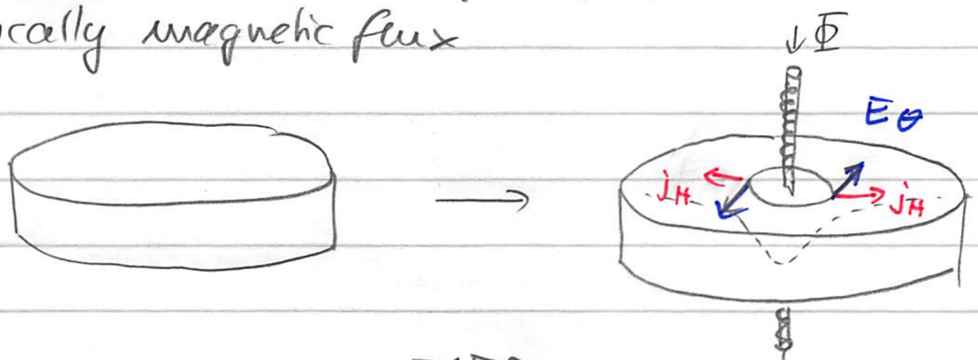


→ quasi-holes created → quasi-particles created

Both, qh and qp are pinned to disorder / impurity potentials and do not contribute to the transport which then explains the presence of the QH-plateaus.

(A) Fundamentals of quasiparticles

Lets start with a thought experiment of inserting locally magnetic flux



particles are pushed outwards

after insertion of one flux quantum Φ_0 the angular momentum with respect to the origin is increased (reduced) by one unit,

(389) $$\lambda(z) = z^{\ell} \longrightarrow \lambda'(z) = e^{iq\vartheta} z^{\ell} \qquad q = \pm 1$$

ϑ is azimutal angle around origin.
If flux insertion is adiabatic $\Rightarrow$ wave function follows adiabatically.

(390) $$\psi \rightarrow \psi' = e^{\pm i\vartheta} \psi$$

is a singular gauge transformation and leaves H unchanged except at the origin $\Rightarrow$ state at the end of the process should be an eigenstate of H if it was one at the beginning

$\Rightarrow$ elementary excitations

$q = 1$	$q = -1$
quasi hole	quasi-particle

- problem: $\psi' = e^{\pm i\vartheta}\psi$ is no longer in LLL

Laughlin: only change polynomial part of wave function

$q = +1$ $$\lambda(z) = z^{\ell} \longrightarrow \lambda_{qh}(z) = z\,\lambda(z) \sim e^{i\vartheta} z^{\ell}$$

$q = -1$ $$\lambda(z) = z^{\ell} \longrightarrow \lambda_{qe}(z) = \frac{\partial}{\partial z}\lambda(z) \sim e^{-i\vartheta} z^{\ell}$$

(391) $$\Psi(z_1,\dots z_N) \to \Psi_{qh}(z_1,\dots z_N) = \prod_{r=1}^{N} b_r^+ \Psi(z_1,\dots z_N)$$

(392) $$\Psi(z_1,\dots z_N) \to \Psi_{qp}(z_1,\dots,z_N) = \prod_{r=1}^{N} b_r \Psi(z_1,\dots z_N)$$

where we used the property of the angular momentum ladder operators

(393) $$b_r = \frac{1}{\sqrt{2}}\left(z_r^* + \frac{\partial}{\partial z_r}\right) \qquad b_r^+ = \frac{1}{\sqrt{2}}\left(z_r - \frac{\partial}{\partial z_r^*}\right)$$

(Note: different length unit as compared to (113))

(394) $$b_r \Psi(\underline{z}) = \sqrt{\tfrac{1}{2}}\left(\frac{\partial}{\partial z_r}\lambda(z)\right)\cdot e^{-\sum_s |z_s|^2}$$

(395) $$b_r^+ \Psi(\underline{z}) = \sqrt{2}\, z_r\, \lambda(z)\cdot e^{-\sum_s |z_s|^2}$$

for $\Psi(\underline{z}) = \lambda(z)\, e^{-\sum_s |z_s|^2}$

<u>quasi-hole wave function</u>

(396) $$\boxed{\Psi_{qh} = \prod_{r=1}^{N} z_r \Psi_{LN}}$$

<u>quasi-particle wave function</u>

more complicated $\frac{\partial}{\partial z}$ acts only on polynomial part

(397) $$\boxed{\Psi_{qe} = \frac{\prod_r \frac{\partial}{\partial z_r} \prod_{r<s}(z_r - z_s)^m}{\prod_{r<s}(z_r - z_s)^m}\, \Psi_{LN}}$$

remark: $\nexists$ qe-excitation for the integer QH state as qe-generation corresponds to removal of angular momentum. Then electrons would be pulled inward but then the electron would have to go to the next LL and the state will have to be discarded

charge of a quasiparticle

flux insertion by $\pm \Phi_0 \Rightarrow$ electric field

(398) $$E_\theta = \frac{d\Phi}{dt}\frac{1}{2\pi r} \quad \Rightarrow \quad j_r \rightarrow \frac{e\rho_0}{B_\perp} E_\theta$$

for large r

total change of charge

$$q^* = 2\pi r \int dt\, j_r = \frac{e\rho_0}{B_\perp}\int dt \frac{d\Phi}{dt} = \pm \frac{e\rho_0}{B_\perp}\Phi_0$$

(399) $\Rightarrow$ $$\boxed{q^* = \pm \nu e}$$

quasiparticle excitations on an incompressible state of filling ν have an effective (fractional) charge $\pm\nu e$

loss / addition of bare particle

$$\underset{F}{\bullet} = \underset{^2CF}{\overset{\uparrow\uparrow}{\bullet}} + \downarrow\downarrow$$

$\Rightarrow$ loss/addition of composite fermion plus local flux insertion i.e. creation of 2 quasi-holes

(B) Anyons

We will see that qh and qe have unconventional exchange properties, they are so-called anyons

3D only two types of particles — bosons / fermions

(i) bosons have integer spin $s = 0, 1, 2, 3 \dots$
fermions have half-integer spin $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$

(ii) exchange statistics

$$\psi(x_2, x_1) = \begin{cases} \psi(x_1, x_2) & \text{bosons} \\ -\psi(x_1, x_2) & \text{fermions} \end{cases}$$

(iii) Pauli exclusion for fermions

spin in 3D is (half)-integer quantized, enforced by non-Abelian spin algebra

(400) $$[S_i, S_j] = i\hbar\, \varepsilon_{ijk} S_k$$

2D only S_z → no algebra ; quantization not enforced

(i) ⇒ $\boxed{\text{particle in 2D can have fractional spin}}$

how about the exchange statistics? Exchanging two identical particles can only affect the phase of the wavefunction

(401) $$\psi(\vec{r}_2, \vec{r}_1) = e^{i\alpha\pi}\, \psi(\vec{r}_1, \vec{r}_2)$$

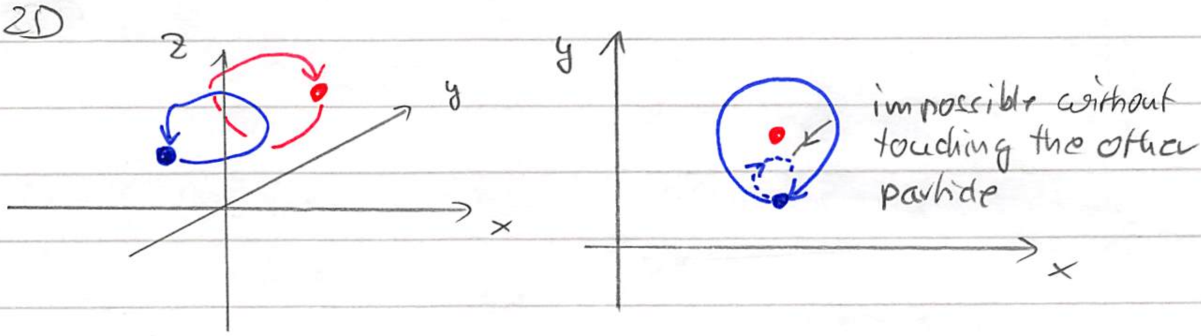
α: statistics parameter only defined modulo 2

$\alpha = ?$ exchanging positions twice leads to

$$\psi(\bar{r}_1, \bar{r}_2) = e^{i2\alpha\pi}\, \psi(\bar{r}_1, \bar{r}_2) \Rightarrow e^{i\alpha\pi} \overset{?}{=} \pm 1$$

$\alpha = 0$ bosons $\qquad$ $\alpha = 1$ fermions

but argument for $e^{i2\alpha\pi} = 1$ comes from continuous deformation of exchange path which is not possible in 2D



$\Rightarrow$ $\boxed{\text{particles in 2D can have any exchange statistics (anyons)}}$

quantum mechanics of anyons and Chern Simon field

consider composite boson field $\hat{\phi}(z)$

(402) $$\left[\hat{\phi}(z), \hat{\phi}^\dagger(z')\right] = \delta^{(2)}(z - z')$$

than define

anyon field $\hat{\psi}(z)$

(403) $$\boxed{\hat{\psi}(z,t) = e^{i\alpha\hat{\Theta}(z,t)}\,\hat{\phi}(z,t)}$$

where $\hat{\Theta}(z,t)$ is again the angle operator (363)

(404) $$\hat{\Theta}(z) \equiv \int d^2z'\, \varphi(z-z')\,\hat{n}(z')$$

Following similar arguments as in Sec. 10.4 A one finds the exchange statistics

(405) $$\boxed{\begin{aligned}\hat{\psi}(y)\hat{\psi}(x) &= e^{i\alpha\pi}\,\hat{\psi}(x)\hat{\psi}(y)\\ \hat{\psi}^+(y)\hat{\psi}(x) &= e^{i\alpha\pi}\,\hat{\psi}(x)\hat{\psi}(y) + \delta^{(2)}(x-y)\end{aligned}}$$

Note that different from Sec. 10.4 α is not necessarily an integer!

Following the same lines as in Sec. 10.4 we can rewrite the Hamiltonian of electrons in the LLL in terms of the composite boson field $\hat{\phi}_{m=2p+1}$

(406) $$H \longrightarrow \frac{1}{2m}\int d^2z\, \hat{\phi}_m^+(z)\,(\hat{\tilde{P}}_x + i\hat{\tilde{P}}_y)(\hat{\tilde{P}}_x - i\hat{\tilde{P}}_y)\,\phi_m(z) + \frac{1}{2}N\hbar\omega_c$$

where

(407) $$\tilde{P}_\mu = -i\hbar\frac{\partial}{\partial x_\mu} + eA_\mu^{eff} \qquad A_\mu^{eff} = A_\mu + C_\mu$$

Here C_μ is the Chern-Simons field associated with the composite boson ${}^{2p}CB$.

When the filling of the electrons is exactly $\nu = \frac{1}{2p+1}$ we have a FQH state and the effective field seen by the composite bosons vanishes

$$A_\mu^{eff} = 0 \qquad \text{for} \quad \nu = \frac{1}{2p+1} \tag{408}$$

If the filling slightly deviates from $\frac{1}{2p+1}$ the Chern Simon field is not completely eliminated and the energy is minimized by the creation of quasiholes $(\nu < \frac{1}{2p+1})$ or quasiparticles $(\nu > \frac{1}{2p+1})$.

To create N_{qh} quasiholes we perform a singular gauge transformation

$$\hat{\phi}_m \longrightarrow e^{if}\hat{\phi}_m \qquad f = \sum_{r=1}^{N_{qh}} \hat{\Theta}(z - z_r) \tag{409}$$

for antivortex i.e. quasi-particle $f \to -f$

z_r are the positions of the quasi-holes. Then

$$C_\mu \rightarrow C_\mu' = C_\mu - \frac{\hbar}{e}\partial_\mu f \tag{410}$$

such that

$$A_\mu^{eff} = \underbrace{A_\mu + C_\mu}_{\neq 0} - \frac{\hbar}{c}\partial_\mu f = 0 \quad \text{(on average)} \tag{411}$$

Instead of (378) we now have

$$(412) \qquad \varepsilon_{jk} \partial_j C_k(z) = m \Phi_0 \, n(z) + \Phi_0 \, n_{qh}(z)$$

with $n_{qh}(z)$ being the density of quasiholes at positions $z_r(t)$

$$(413) \qquad n_{qh}(z) = \sum_{r=1}^{N_{qh}} \delta^{(2)}(z - z_r(t))$$

To derive an effective field-theoretical model of qh's we start from the Lagrange density corresponding to Hamiltonian (406)

$$(414) \qquad \mathcal{L} = \phi_m^+ \left(i\hbar \partial_t - eC_t - eA_t^{ext}\right)\phi_m - \mathcal{H} + \mathcal{L}_{CS}$$

where we have added the Lagrange density of the "free" Chern-Simon field

$$(415) \qquad \mathcal{L}_{CS} = -\frac{e^2}{4\pi m\hbar} \varepsilon^{\mu\nu\lambda} C_\mu \partial_\lambda C_\lambda$$

Now performing the singular gauge trafo (409) one finds

$$(416) \qquad \mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} + \Delta\mathcal{L}_{qh}$$

Integrating $\Delta\mathcal{L}_{qh}$ over the 2D space gives the Lagrangian

$$(417) \qquad \int d^2z \, \Delta\mathcal{L}_{qh} = \frac{1}{m} \sum_{r=1}^{N_{qh}} C_\mu \frac{dz_r^\mu}{dt} + \frac{\hbar}{m} \sum_{r \neq s}^{N_{qh}} \frac{d}{dt} \varphi(z_r - z_s)$$

Now we add for convenience a kinetic energy of the quasiholes introducing a mass M_{qh}. We note that in the LLL the kinetic energy contribution is quenched anyway and thus we can get away with this without specifying M_{qh}.

(418) $$L_{qh} = \sum_{r=1}^{N_{qh}} \left[\frac{M_{qh}}{2} \frac{dz_r^k}{dt}\frac{dz_r^k}{dt} + \frac{1}{m} C_\mu \frac{dz_r^\mu}{dt} \right] + \frac{\hbar}{m} \sum_{r<s} \frac{d}{dt} \varphi(z_r - z_s)$$

This then gives the effective Hamiltonian of quasi-holes

(419) $$\boxed{H_{qh} = \frac{1}{2M_{qh}} \sum_{r=1}^{N_{qh}} \left(p_r^k - \frac{e}{m} A_k(z_r) - e C_k^{(1)}(z_r) \right)^2}$$

(summation over k)

with

(420) $$\boxed{C_k^{(1)}(z) = -\frac{\hbar}{me} \sum_s \partial_k \varphi(z - z_s)}$$

which then can be quantized in the standard way.

(421) $$\varepsilon_{jk} \partial_j C_k^{(1)} = -\frac{1}{m} \Phi_0 n_{vor}$$

showing the fractional charge $q^* = -\frac{1}{m} e$

In order to see that quasi-holes have anyonic exchange statistics we calculate the exchange phase from the Lagrangian

(422) $$L = L_0 + \frac{\hbar}{m} \sum_{r<s} \frac{d}{dt} \varphi(x_r - x_s)$$

Classical action between t_i and t_f

(423) $$S = \int_{t_i}^{t_f} dt\, L = \int_{t_i}^{t_f} dt\, L_0 + \frac{\hbar}{m} \sum_{r<s} \left(\varphi(x_r^f - x_s^f) - \varphi(x_r^i - x_s^i) \right)$$

In Feynman path integral formulation of wavefunctions we pick up a phase

$$e^{iS/\hbar} = e^{i \frac{1}{m} \Delta\varphi} \exp\left\{ \frac{i}{\hbar} \int_{t_i}^{t_f} dt\, L_0 \right\}$$

where

(424) $$\Delta\varphi = \varphi(x_r^f - x_s^f) - \varphi(x_r^i - x_s^i)$$

So if we exchange two particles we obtain

(425) $$e^{iS/\hbar} = \underbrace{e^{i \frac{1}{m} \pi}}_{\text{exchange phase}} \underbrace{\exp\left\{ \frac{i}{\hbar} \int_{t_i}^{t_f} dt\, L_0 \right\}}_{\text{dynamical phase}}$$

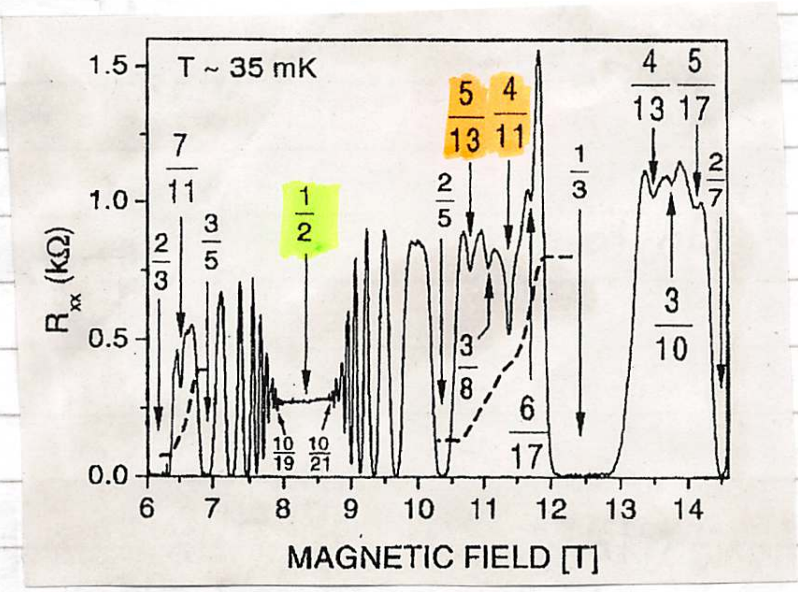
which proves that the solutions of H_α behave like anyons

10.6 interacting composite fermions

we have seen that the CF theory explains the incompressible states at fillings

$$\nu = \frac{n}{2pn \pm 1} \qquad (426)$$

but there are also Hall plateaus at other fillings



e.g. at fillings $\nu = \frac{5}{13}$ or $\nu = \frac{4}{11}$ which do not fit to eq. (417)

$\Rightarrow$ interaction between CF !

(A) higher FQH series

Pseudopotentials are not completely screened. So one can discuss the fractionalization of CF

e.g. electrons with filling $\quad \frac{1}{3} < \nu = \frac{\nu^*}{2\nu^*+1} < \frac{2}{5}$

are mapped to ^2CF with effective fillings

$$1 < \nu^* = 1 + \frac{f}{2f+1} < 2$$

$\uparrow$ full LLL

$\Rightarrow$ FQHE for ^2CF with fillings

$$f=1 \Rightarrow \nu^* = 1+\frac{1}{3} \Rightarrow \underline{\underline{\nu = \frac{4}{11}}}$$

$$f=2 \Rightarrow \nu^* = 1+\frac{2}{3} \Rightarrow \underline{\underline{\nu = \frac{5}{13}}}$$

(B) absence of FQH state at $\nu = \frac{1}{2}$

At $\nu = \frac{1}{2}$ there is no dip in $R_{\parallel}$ but the state is still interesting

(427) $$\nu = \frac{1}{2} = \lim_{n\to\infty} \frac{n}{2n\pm 1} \qquad p=2$$

i.e. ^2CF with filling

(428) $$n = \frac{g}{B_{eff}\,\Phi_0} = \infty \Rightarrow \boxed{B_{eff} = 0}$$

These are free ^2CF fermions without magnetic field. The state can be considered as Fermi sea of free ^2CF without magnetic field

(c) The $\nu = 5/2$ state: CF-pairing

Willett et al (PRL 59, 1776 (1987)) have seen a FQH plateau at $\nu = 5/2$. Since the Zeeman splitting is here small one has to consider both spin components

$$\nu = \frac{5}{2} = \underset{\substack{\text{lowest LL} \\ \text{for 2 spins}}}{2} + \underset{\substack{\text{2nd LL} \\ \text{1 spin}}}{\frac{1}{2}}$$

$\Rightarrow$ Fermi sea in 2nd LL? No!

Moore & Reed (Nucl. Phys. B 360, 362 (1991)) have shown that a good candidate for the $\nu = 5/2$ FQH state is a p-wave paired state of spin-polarized electrons in the 2nd LL

(429) $$\boxed{\Psi^{Pf}_{1/2} = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{j<k} (z_j - z_k)^2 \exp\left[-\frac{1}{4}\sum_{u} |z_u|^2\right]}$$

(famous) "Pfaffian" state

remark: p-wave pairing requires attractive interaction

$\rightarrow$ CF in 2nd LLL have attractive interaction

$\nu = 5/2$ state still open question