1. Topology of Bloch hamiltonians: an introduction

In the following we will consider lattice Hamiltonians with discrete translational invariance. Here the lattice quasimomentum is conserved and we will investigate the homotopy classes of mappings

(40) $$\boxed{\vec{k} \longrightarrow h(\vec{k})}$$ no external parameter

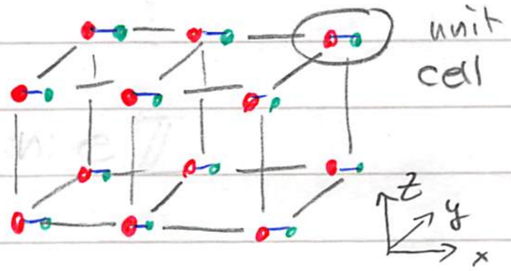
or

(41) $$\boxed{\vec{k}, \vec{\lambda} \longrightarrow h(\vec{k}, \vec{\lambda})}$$ with external parameter

1.1 Bloch bands: a reminder

crystals have discrete translational invariance

- lattice vector



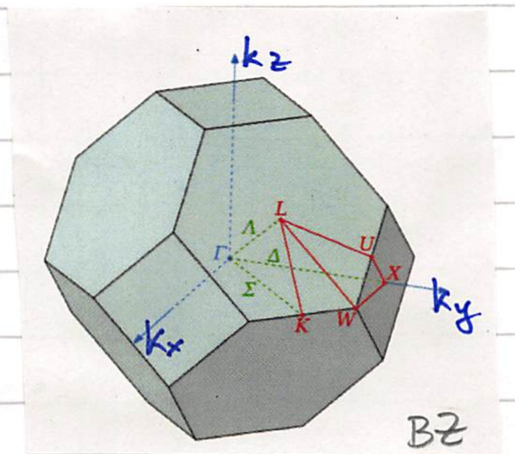
(42) $$\vec{R} = \sum_{i=1}^{3} n_i \vec{a}_i$$

$\vec{a}_i$ basis vectors, $n_i \in \mathbb{Z}$

- dual lattice: Bravais lattice

(43) $$\vec{G} = \sum_{i=1}^{3} m_i \vec{b}_i$$

(44) $$\vec{a}_i \cdot \vec{b}_j = 2\pi \delta_{ij}$$



Brillouin zone (BZ)

$$|\vec{k}| < |\vec{k} - \vec{G}| \quad \forall \vec{G}$$

Bloch theorem $\quad T_{\vec{R}}$: lattice translation

since $T_R^{-1} H T_R = H \Rightarrow$ common eigenfunctions

(45) $$\boxed{\langle r | \psi_{nk} \rangle = e^{i\vec{k}\cdot\vec{r}} \langle r | u_{nk} \rangle}$$ n - Bloch index

with $\vec{k} \in 1BZ$ and

(46) $$\langle \vec{r} + \vec{R} | u_{nk} \rangle = \langle \vec{r} | u_{nk} \rangle$$ i.e. $\vec{R}$ - periodic

(single-particle) Hamiltonian

(47) $$H = \bigotimes_{k \in 1BZ} h(\vec{k})$$

$h(\vec{k})$ Bloch Hamiltonian; periodic in reciprocal space

(48) $$h(\vec{k} + \vec{b}_j) = h(\vec{k}) \quad \forall \vec{k} \in BZ$$

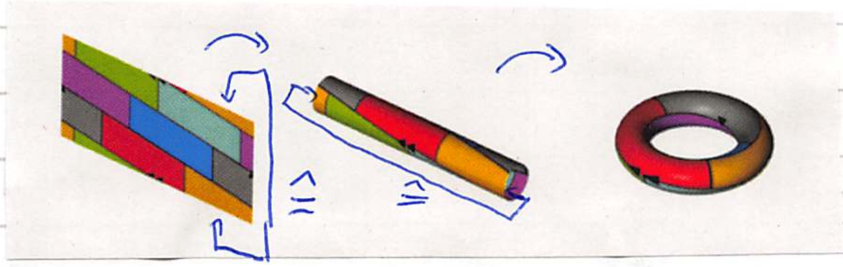
Schrödinger equation

(49) $$h(\vec{k}) | u_{n\vec{k}} \rangle = E_n(\vec{k}) | u_{n\vec{k}} \rangle$$

Bloch states only defined modulo gauge trafo

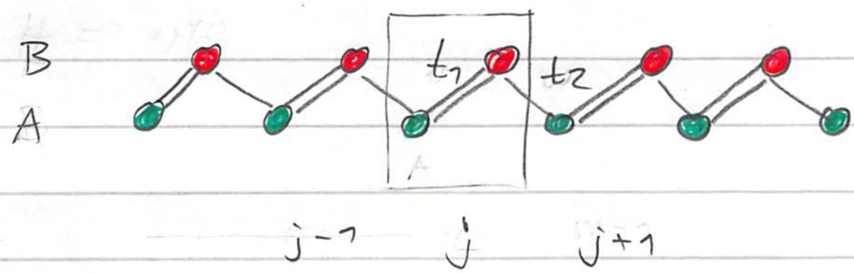
(50) $$| u_{n\vec{k}} \rangle \longrightarrow e^{i\alpha_{\vec{k}}} | u_{n\vec{k}} \rangle$$

Brillouin zone is a torus T^3_{BZ}



1.2. One-dimensional Bloch hamiltonians with topology: the Rice Mele model

Let us consider a simple 1d model with unit cell 2 and thus in general with 2 bands



M. J. Rice, E. J. Mele
Phys. Rev. Lett. **49** 1455 (1982)

$$H = -t_1 \sum_j \left(|j,B\rangle\langle j,A| + h.c. \right)$$

$$- t_2 \sum_j \left(|j+1,A\rangle\langle j,B| + h.c. \right) \tag{51a}$$

$$+ \Delta \sum_j \left(|j,A\rangle\langle j,A| - |j,B\rangle\langle j,B| \right)$$

in second quantization H reads

$$H = -t_1 \sum_j \left(c_{Bj}^{+} c_{Aj} + h.a.\right) - t_2 \sum_j \left(c_{Aj+1}^{+} c_{Bj} + h.a.\right)$$

(51b)

$$+ \Delta \sum_j \left(c_{Aj}^{+} c_{Aj} - c_{Bj}^{+} c_{Bj}\right)$$

Bloch hamiltonian $\qquad L$: # of unit cells $\qquad$ discrete F-trafo

(52) $$\tilde{c}_{Ak} = \frac{1}{L} \sum_{j=1}^{L} e^{\frac{2\pi i}{L} kj} c_{Aj}$$

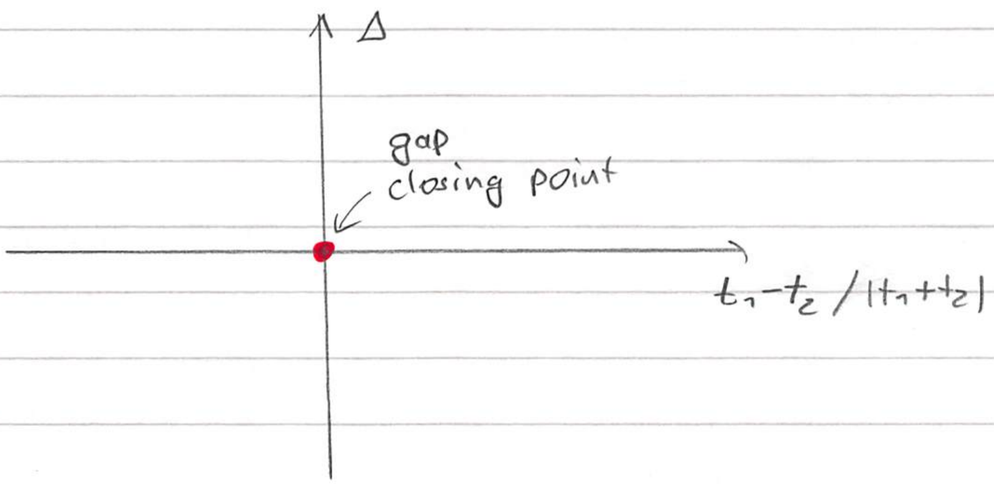
- lattice momentum now: $\frac{2\pi}{L} k \qquad k$ integer

(53) $$H = \sum_{k \in BZ} \begin{pmatrix} \tilde{c}_{Ak} \\ \tilde{c}_{Bk} \end{pmatrix}^{+} \underbrace{\begin{pmatrix} \Delta & -t_1 - t_2 e^{-ik} \\ -t_1 - t_2 e^{ik} & -\Delta \end{pmatrix}}_{\text{Bloch hamiltonian}} \begin{pmatrix} \tilde{c}_{Ak} \\ \tilde{c}_{Bk} \end{pmatrix}$$

2 energy bands

(54) $$\boxed{\mathcal{E}_{\pm}(k) = \pm \sqrt{\Delta^2 + \left(t_1^2 + t_2^2 + 2 t_1 t_2 \cos k\right)} = \pm \mathcal{E}(k)}$$

gap closes for $\Delta = 0$ and $t_1 = t_2$ at $\cos k = -1$



diagonalization of Bloch hamiltonian $\qquad U(k)$: 2×2 matrix

$$(55) \qquad U^\dagger(k) \begin{pmatrix} \Delta & -t_1-t_2e^{-ik} \\ -t_1-t_2e^{ik} & -\Delta \end{pmatrix} U(k) = \begin{pmatrix} \mathcal{E}(k) & 0 \\ 0 & -\mathcal{E}(k) \end{pmatrix}$$

$$(56) \qquad \begin{pmatrix} \tilde{c}_{+k} \\ \tilde{c}_{-k} \end{pmatrix} = U^\dagger(k) \begin{pmatrix} \tilde{c}_{Ak} \\ \tilde{c}_{Bk} \end{pmatrix}$$

from Bloch basis to Wannier basis

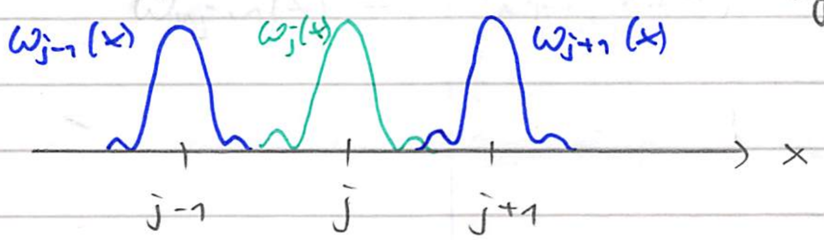
going back from k-space to position space ($k = \frac{2\pi}{L}q$)

$$(57) \qquad |w_{nj}\rangle = \frac{1}{\sqrt{L}} \sum_k e^{-\frac{2\pi i j q}{L}} e^{i\alpha_k} |u_{nk}\rangle$$

α_k is some appropriately chosen gauge (freedom!)

$$(58) \qquad \langle w_{nj} | w_{n'j'} \rangle = \delta_{jj'}\delta_{nn'}$$

$$w_{nj+1}(x+a) = w_{nj}(x)$$ shift in real space by lattice constant



an important quantity is the center of mass of a Wannier function with respect to origin of unit cell

$$\langle w_{nj} | \hat{x} - ja | w_{nj} \rangle$$

It is sufficient to consider $j=0$ and include gauge into definition of Bloch states; $|w_{nj=0}\rangle = |w_n\rangle$

$$\hat{x}\,|w_n\rangle = \frac{1}{2\pi}\int_{BZ} dk \underbrace{\sum_j j}_{\hat{x}} \; e^{ikj}\,|u_{nk}\rangle$$

with

$$j e^{ikj} = -i\partial_k e^{ikj}$$

$$\hat{x}\,|w_n\rangle = \frac{1}{2\pi}\sum_j \int_{BZ} dk\; e^{ikj}\,\partial_k |u_{nk}\rangle$$

i.e.

$$(60) \qquad \langle w_n|\hat{x}|w_n\rangle = \frac{i}{2\pi}\int_{BZ} dk\;\langle u_{nk}|\partial_k|u_{nk}\rangle$$

and thus

$$(61) \qquad \boxed{\langle w_{nj}|\hat{x}|w_{nj}\rangle = \frac{i}{2\pi}\int_{BZ} dk\;\langle u_{nk}|\partial_k u_{nk}\rangle + j}$$

the center of mass of the Wannier function is given by the Berry phase of Bloch functions, called

Zak phase

$$(62) \qquad \boxed{\varphi_n^{zak} \equiv i\int_{BZ} dk\;\langle u_{nk}|\partial_k u_{nk}\rangle}$$

What happens if we adiabatically change the parameter

$$\vec{\lambda} = \begin{pmatrix} t_1 - t_2 \\ \Delta \end{pmatrix} \qquad \vec{\lambda} \in \mathbb{R}^2$$

in a closed loop in parameter space? Then the center of the Wannier functions move

$$\Delta X = \oint d\lambda \frac{\partial \langle \hat{x} \rangle}{\partial \lambda} = \frac{1}{2\pi} \oint d\lambda \frac{\partial \varphi_n^{Zak}}{\partial \lambda}$$

$$= \frac{i}{2\pi} \int dk \int d\lambda \frac{\partial}{\partial \lambda} \langle u_{nk} | \partial_k u_{nk} \rangle$$

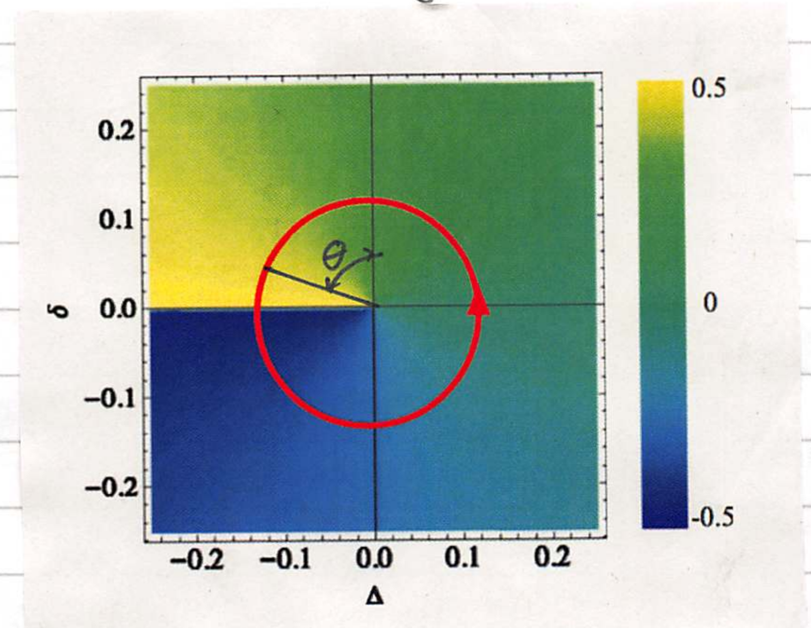
$$(63) \qquad \boxed{\Delta X = \frac{1}{2\pi} \int dk \int d\lambda \, i \underbrace{\left[\langle \partial_\lambda u_{nk} | \partial_k u_{nk} \rangle - \langle \partial_k u_{nk} | \partial_\lambda u_{nk} \rangle \right]}_{\text{Berry curvature in } k, \lambda \text{ space}}}$$

$(a = 1)$

shift of the center of mass of Wannier functions upon a closed loop in $\vec{\lambda}$ space is given by an integer-valued topological invariant, the winding of the Zak phase

$$(64) \qquad \boxed{\Delta X = \nu = \frac{1}{2\pi} \oint d\lambda \frac{\partial \varphi^{ZAK}}{\partial \lambda} \in \mathbb{Z}}$$

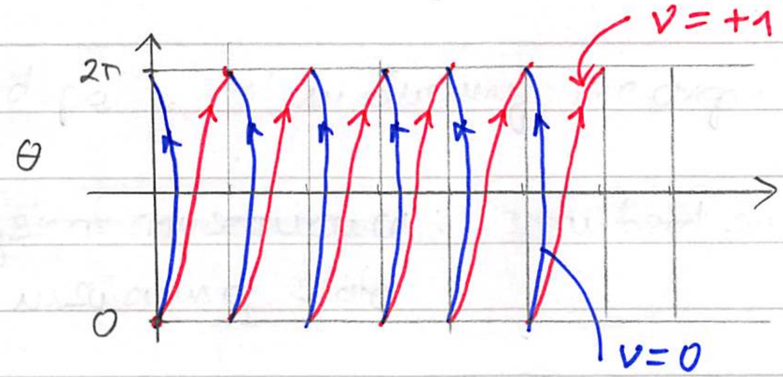
calculating ~~a~~ the Zak phase (Berry connection) for the Rice-Mele model gives



$\delta = t_1 - t_2$

$\dfrac{\phi_{ZAK}}{2\pi}$

closed path including the origin $\underline{\nu = \pm 1}$



movement of Wannier center in

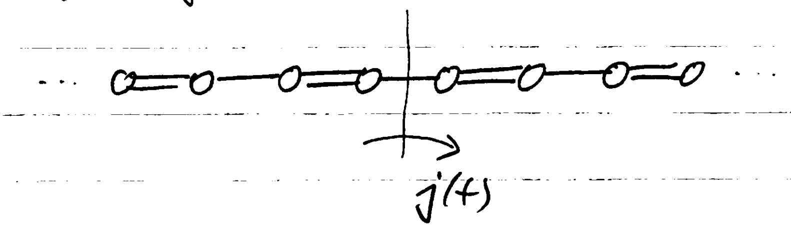
closed path excluding the origin $\underline{\nu = 0}$

physical consequence: quantized transport in the insulating bulk; Thouless pump

consider a many-body system with a completely filled lower band; i.e. each bloch state $|u_-(k)\rangle$ is occupied by one fermion

$\Rightarrow$ expect intuitively
quantized transport by ν particles per period

To see this let us calculate the adiabatic current $j(t)$ through any cut in the lattice and the transported charge



(65) $$\Delta n = \int_0^T dt\, j(t)$$

Since in the adiabatic limit $T \to \infty$ we expect $j \sim \frac{1}{T}$.
To calculate j we need first-order nonadiabatic corrections. If at $t=0$ $|\psi(t=0)\rangle = e^{ikx} |u_{nk}(t=0)\rangle$ then at t:

(66) $$|\psi_n(t)\rangle = e^{ikx} \left[|u_{nk}\rangle - i\hbar \sum_{n' \neq n} \frac{|u_{n'k}\rangle \langle u_{n'k}| \partial_t u_{nk}\rangle}{E_n(k) - E_{n'}(k)} \right]$$

velocity operator

(67) $$\hat{v} = \frac{d}{dt} \hat{x} = \frac{i}{\hbar} [\hat{H}, \hat{x}]$$

in momentum representation $\hat{x} \to i \frac{\partial}{\partial q}$

(68) $$\hat{v}(q) = \frac{1}{\hbar} \frac{\partial H(q,t)}{\partial q}$$

I.e. in the adiabatic state (66) we find

$$v_n(k) = \langle \psi_n(t) | \hat{v} | \psi_n(t) \rangle = \frac{1}{\hbar} \frac{\partial E_n(k)}{\partial k}$$

(69)

$$- i \sum_{n' \neq n} \frac{\langle u_{nk} | \partial H / \partial k | u_{n'k} \rangle \langle u_{n'k} | \partial u_{nk} / \partial t \rangle}{E_n(k) - E_{n'}(k)} - c.c.$$

using $\langle u_{nk} | \frac{\partial H}{\partial k} | u_{n'k} \rangle = (E_n - E_{n'}) \langle \frac{\partial u_{nk}}{\partial k} | u_{n'k} \rangle$

yields

(70) $$v_n(k) = \frac{1}{\hbar} \frac{\partial E_n(k)}{\partial k} - i \left[\langle \frac{\partial u_{nk}}{\partial k} | \frac{\partial u_{nk}}{\partial t} \rangle - \langle \frac{\partial u_{nk}}{\partial t} | \frac{\partial u_{nk}}{\partial k} \rangle \right]$$

i.e. — Berry curvature

(71) $$v_n(k) = \frac{1}{\hbar} \frac{\partial E_n(k)}{\partial k} + \Omega_{nk}(t)$$

If all states in the Brillouin zone of the n'th Bloch band are occupied the total particle current is thus

(72) $$j_n = \underbrace{\frac{1}{\hbar} \int_{BZ} \frac{dk}{2\pi} \frac{\partial E_n(k)}{\partial k}}_{=0} + \int_{BZ} \frac{dk}{2\pi} \Omega_{nk}(t)$$

Thus we find indeed quantized bulk transport

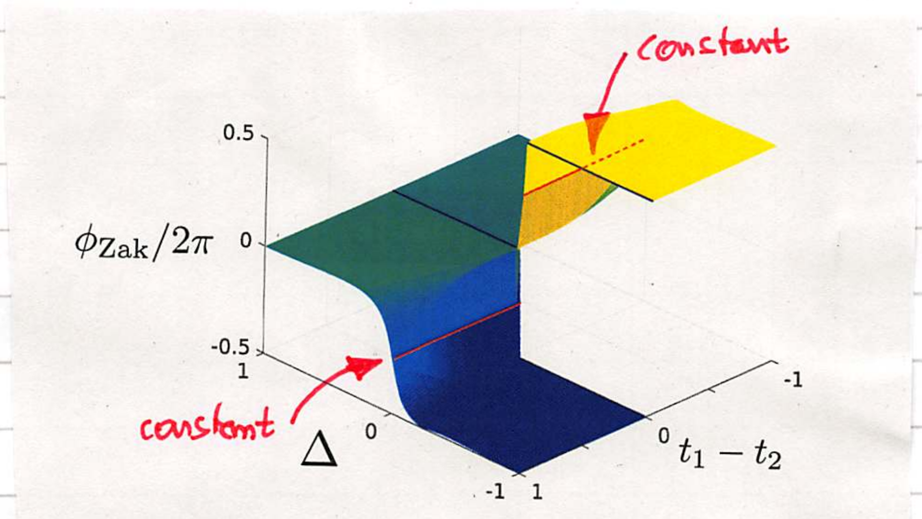
(73) $$\boxed{\Delta n = \frac{1}{2\pi} \int_{BZ} dk \int_0^T dt \; i \left[\langle \partial_t u_{nk} | \partial_k u_{nk} \rangle - \langle \partial_k u_{nk} | \partial_t u_{nk} \rangle \right]}$$

(74) $$\boxed{\Delta n = \nu = \Delta\phi_{Zak}/2\pi}$$

winding of Zak phase of single-particle Bloch functions implies an integer quantized, adiabatic particle transport in the insulating state of a Bloch band completely filled with (noninteracting) fermions.
D.J. Thouless Phys. Rev. B 27, 6083 (1983) Thouless pump

1.3 The Su-Schrieffer Heeger (SSH) model

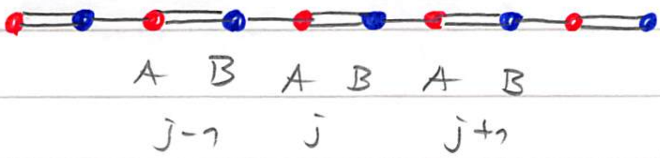
Lets have a look again at the Zak phase of the Rice-Mele model



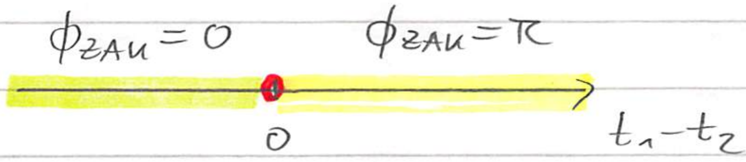
Zak phase is constant for $\Delta = 0$ and jumps by π between $t_1 < t_2$ and $t_2 > t_1$

RMM with $\Delta = 0$: Su-Schrieffer Heeger model

(75) $H = -t_1 \sum_j \left(c^\dagger_{Bj} c_{Aj} + h.c. \right) - t_2 \sum_j \left(c^\dagger_{Aj+1} c_{Bj} + h.c. \right)$



- Bulk properties are unchanged if $t_1 \leftrightarrow t_2$. However we see that the Zak phase is different by π



- note that Zak phase is gauge dependent, but Zak phase difference on the two sides is not
We will see that the SSH hamiltonian has two topologically different ground states characterized by a

$\underline{\mathbb{Z}_2 \text{ topological invariant}} \begin{cases} \phi_{ZAK} = 0 \\ \phi_{ZAK} = \pi \end{cases}$

Bloch hamiltonian of SSH

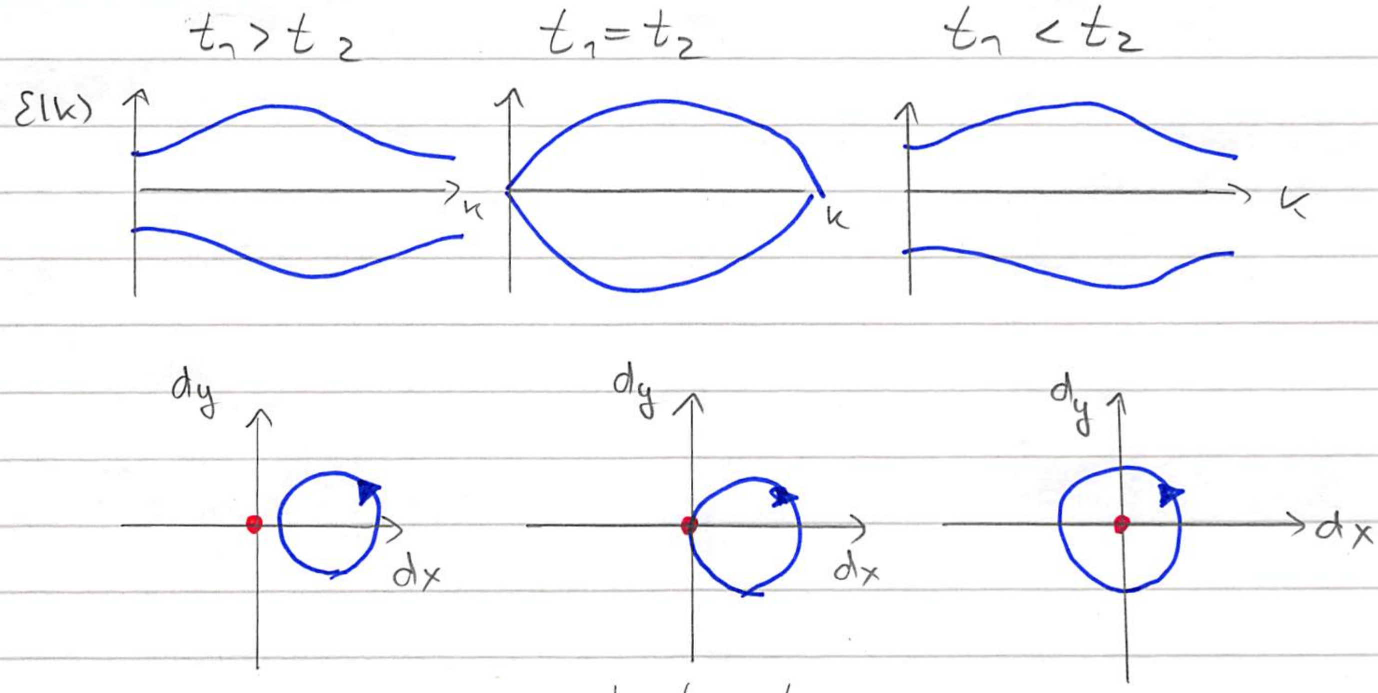
(76) $$h(k) = \begin{pmatrix} 0 & -t_1 - t_2 e^{-ik} \\ -t_1 - t_2 e^{ik} & 0 \end{pmatrix} = d_x(k)\, \hat{\sigma}_x + d_y(k)\, \hat{\sigma}_y$$

(77)
$$d_x(k) = -t_1 - t_2 \cos k$$
$$d_y(k) = -t_2 \sin k$$

since $\hat{\sigma}_z$ is missing, the hamiltonian $h(k)$ can be fully described by a unit vector $\vec{n}(k)$ which lies in the x-y plane

$$h(k) = \varepsilon(k)\, \vec{n}(k) \cdot \vec{\sigma} \tag{78}$$

When going through the Brillouin zone (periodic!) $h(k)$ can either wind around the origin or not



topolog. phase transition

- in the Rice Mele model there is also a z-component

$$d_z(k) = \Delta$$

here $\vec{n}(k)$ is <u>not</u> confined to the x-y plane the reason for the confinement of $\vec{n}(k)$ in the SSH model is a generalized symmetry

unitary symmetries

if $\quad U H U^{\dagger} = H \qquad U = e^{iX} \quad U^{\dagger} = U^{-1}$

$\Rightarrow \quad [X, H] = 0 \qquad X$ is constant of motion

generalized symmetry (chiral symmetry)

(79) $$\Gamma H \Gamma^{\dagger} = -H \qquad \Gamma^{\dagger} = \Gamma^{-1}$$

here

(80) $$\boxed{\Gamma = \hat{\sigma}_z = \Gamma^{\dagger}} \qquad \Gamma^2 = 1$$

$$\begin{aligned}\hat{\sigma}_z h(k) \hat{\sigma}_z = \; & d_x(k) \hat{\sigma}_z \hat{\sigma}_x \hat{\sigma}_z \\ & + d_y(k) \hat{\sigma}_z \hat{\sigma}_y \hat{\sigma}_z \\ & + d_z(k) \hat{\sigma}_z \hat{\sigma}_z \hat{\sigma}_z \\ = \; & -d_x(k) \hat{\sigma}_x - d_y(k) \hat{\sigma}_y + d_z(k) \hat{\sigma}_z\end{aligned}$$

so

(81) $$\boxed{\hat{\sigma}_z h_{SSH}(k) \hat{\sigma}_z = -h_{SSH}(k)} \Leftrightarrow d_z(k) = 0$$

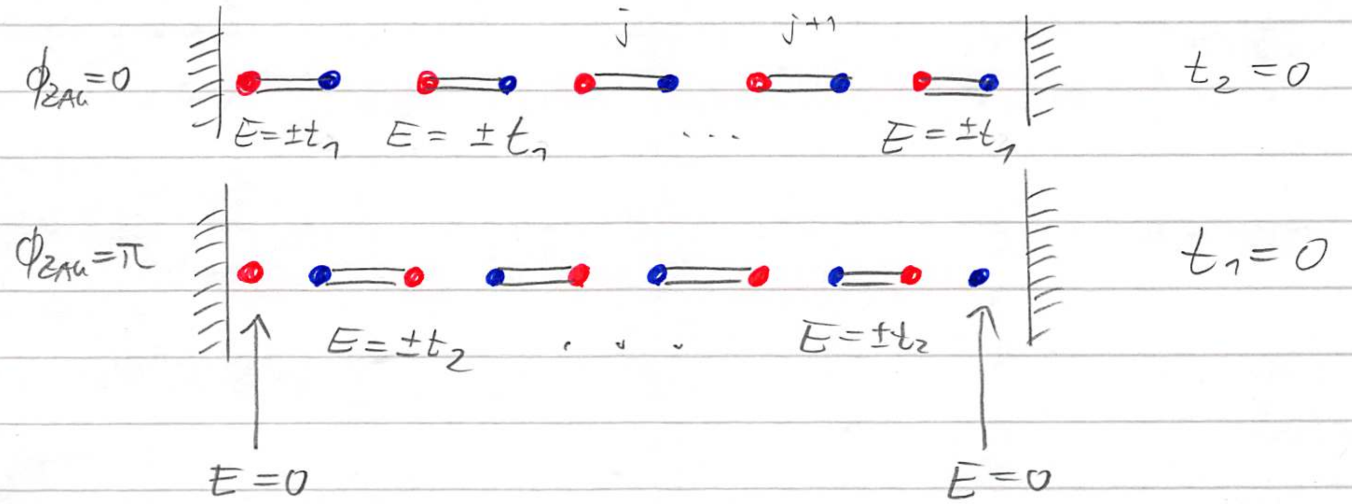
$\Rightarrow$ topological property ($d_z(k) = 0$) is symmetry induced

symmetry protected topology

of Hamiltonian itself (not of set of Hamiltonians as in case of RMM)

physical consequence: edge states

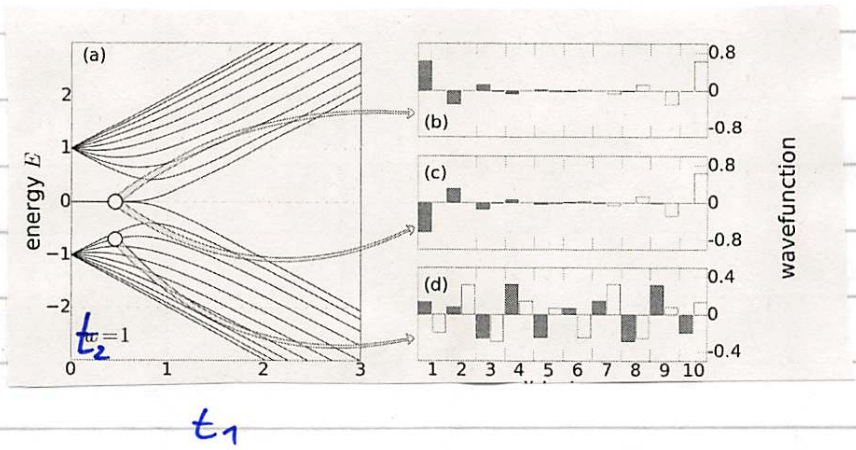
- consider finite lattice, first in fully dimerized limit



in the case with $\phi_{ZAK}=\pi$ two additional states with energy $E=0$ localized at the edges

edge states

- away from dimerized limit $\phi_{ZAK}=\pi$



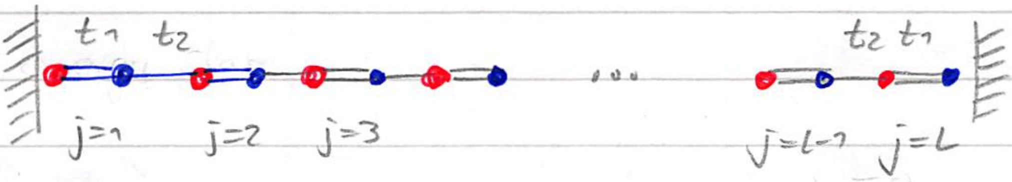
two mid-gap states also away from dimerized limit

- localization length of edge state increases with t_1

- as soon as edge state wavefunctions overlap in a finite lattice they hybridize and their energy splits

▶ <u>finite lattice with L unit cells</u>

$$(82) \quad H_L = -t_1 \sum_{j=1}^{L} \left(|j,B\rangle\langle j,A| + h.c. \right) - t_2 \sum_{j=1}^{L-1} \left(|j+1,A\rangle\langle j,B| + h.c. \right)$$



$$(83) \quad |\phi_0\rangle = \sum_{j=1}^{L} \alpha_j |j,A\rangle + \beta_j |j,B\rangle$$

$$H_L |\phi_0\rangle = 0 \qquad \text{zero energy state}$$

$$(84) \quad t_1 \alpha_j + t_2 \alpha_{j+1} = 0 \qquad j = 1, \dots, L-1$$

$$(85) \quad t_2 \beta_j + t_1 \beta_{j+1} = 0$$

$$t_1 \alpha_L = 0 \qquad t_1 \beta_1 = 0 \qquad \text{(boundaries)}$$

from (84), (85) $\Rightarrow$ amplitudes with alternating signs

$$(86) \quad \boxed{\begin{aligned} \alpha_j &= \left(-\frac{t_1}{t_2}\right)^{j-1} \alpha_1 \\ \beta_j &= \left(-\frac{t_1}{t_2}\right)^{L-(j+1)} \beta_L \end{aligned}} \qquad \begin{aligned} & j = 2, \dots, L \\ & j = 1, \dots, L-1 \end{aligned}$$

but since $\beta_1 = \alpha_L = 0 \Rightarrow \nexists$ zero energy solution
In any finite system the exponentially small, but nonvanishing hybridization of the two edges leads to a small energy splitting

$\Rightarrow$ eq. (86) describes approximate solutions for $L \to \infty$

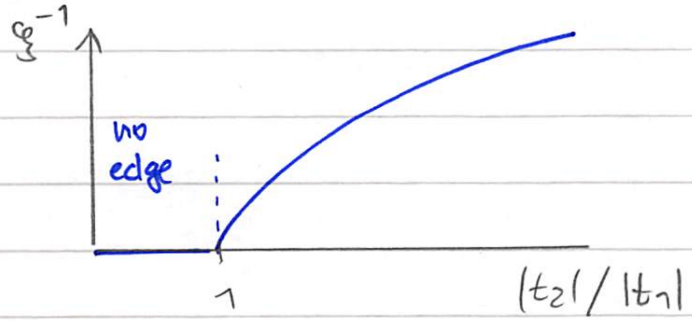
(87) $$\boxed{|L\rangle = \sum_{j=1}^{L} \alpha_j |j A\rangle} \qquad \boxed{|R\rangle = \sum_{j=1}^{L} \beta_j |j, B\rangle}$$

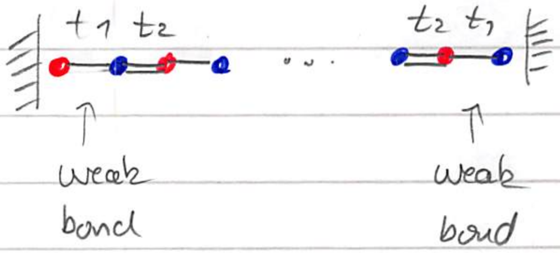
only A sites occupied — only B sites occupied

(88) $$|\alpha_L| = |\alpha_1| \, e^{-(L-1)/\xi} \qquad |\beta_1| = |\beta_L| e^{-(L-1)/\xi}$$

(89) $$\boxed{\xi^{-1} = \log\left|\frac{t_2}{t_1}\right|}$$

localization length for $|t_2| > |t_1|$





1.4 A two-dimensional Bloch hamiltonian with topological properties: The Qi-Wu-Zhang (QWZ) Model

In Sec. 1.3 we have discussed a one-dimensional Bloch hamiltonian with an additional periodic parameter

$$\vec{\lambda}(t) = \begin{pmatrix} t_1 - t_2 \\ \Delta \end{pmatrix}$$

which had non-trivial topological properties when encircling the origin $\vec{r}=0$, e.g.

$$t_1 = -(1+\cos(\Omega t)) \tag{90}$$
$$t_2 = -1$$
$$\Delta = -\sin(\Omega t)$$

$$h_{RM}(k,t) = \left[(1+\cos(\Omega t)) + \cos k\right] \sigma_x + \sin k\, \sigma_y + \sin \Omega t\, \sigma_z \tag{91}$$

Now lets go to 2d system and replace the periodic functions $\cos(\Omega t)$ and $\sin(\Omega t)$ by $\cos k_y$, $\sin k_y$. After an additional (unimportant) rotation in space we find the <u>QWZ Hamiltonian</u> in 2d

$$\boxed{h_{QWZ}(k_x,k_y) = \sin k_x\, \sigma_x + \sin k_y\, \sigma_y + \left[u+\cos k_x + \cos k_y\right]\sigma_z} \tag{92}$$

i.e.

$$\vec{d}(k_x,k_y) = \begin{pmatrix} \sin k_x \\ \sin k_y \\ u+\cos k_x+\cos k_y \end{pmatrix} \tag{93}$$

since $h_{QWZ} = \vec{d}\cdot\vec{\sigma}$

$$h_{QWZ}^2 = |\vec{d}|^2 \cdot \mathbb{1} \tag{94}$$

Thus the energy eigenvalues are the two square roots of $|\vec{d}|^2$

$$\boxed{E_\pm(k_x,k_y) = \pm\sqrt{\sin^2 k_x + \sin^2 k_y + (u + \cos k_x + \cos k_y)^2}} \qquad (95)$$

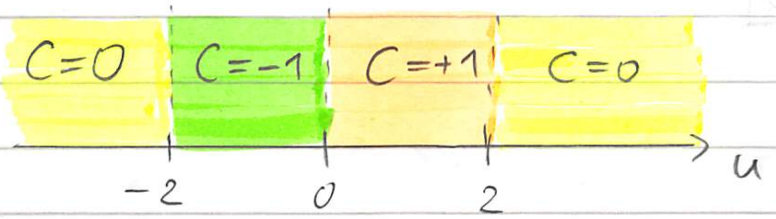
gap closes for $u = 0, \pm 2$

$u=-2$	$k_x = k_y = 0$	Γ point
$u=0$	$k_x = 0 \quad k_y = \pi$ and $k_x = \pi \quad k_y = 0$	two X points
$u=2$	$k_x = k_y = \pi$	M point

From the eigenstates $|u_\pm(k_x,k_y)\rangle = |u_\pm(\vec{k})\rangle$ one can calculate the Chern number

$$(96) \qquad C_\pm = \frac{1}{2\pi}\int_{BZ} dk_x \int_{BZ} dk_y \, i\left[\langle\partial_{k_x} u_\pm | \partial_{k_y} u_\pm\rangle - \langle\partial_{k_y} u_\pm | \partial_{k_x} u_\pm\rangle\right]$$

and one finds as function of u $\quad C_\pm = \mp C$



real-space Hamiltonian

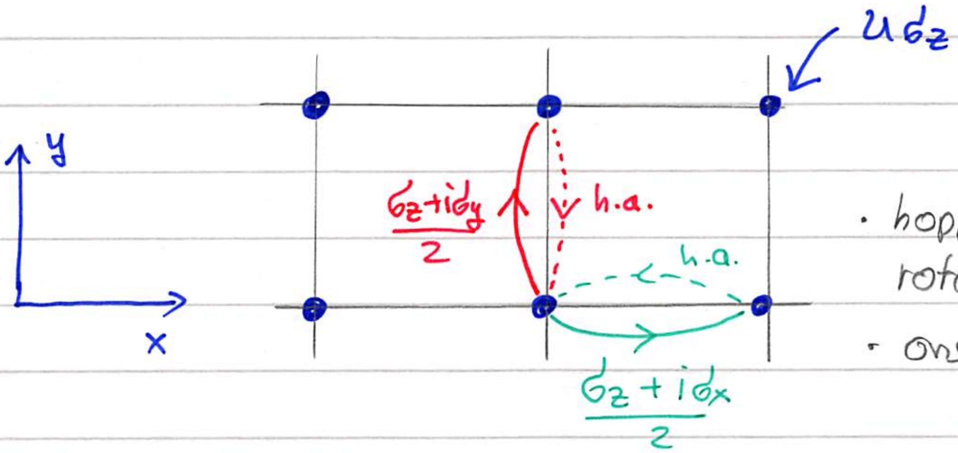
In order to see the physical meaning of the momentum-space Bloch hamiltonian we perform an inverse Fourier-transform of $h_{QWZ}(\vec{k})$

This gives a band hamiltonian of two spin components

$$H = \sum_{j=1}^{L_x-1} \sum_{l=1}^{L_y} \left(|j+1,l\rangle\langle j,l| \frac{\sigma_z + i\sigma_x}{2} + h.a. \right)$$

$$(97) \qquad + \sum_{j=1}^{L_x} \sum_{l=1}^{L_y-1} \left(|j,l+1\rangle\langle j,l| \frac{\sigma_z + i\sigma_y}{2} + h.a. \right)$$

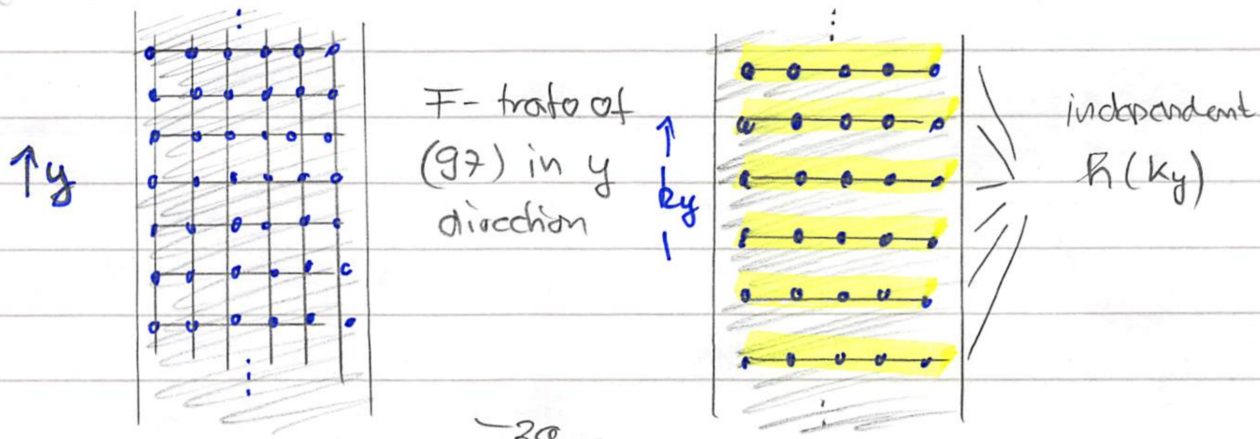
$$+ u \sum_{j=1}^{L_x} \sum_{j=1}^{L_y} |j,l\rangle\langle j,l| \, \sigma_z$$



- hopping with spin rotation
- onsite energy $\pm u$

edge states

Now the edge is one dimensional and this gives rise to additional interesting phenomena. Consider half infinite stripe with finite extension in x-direction

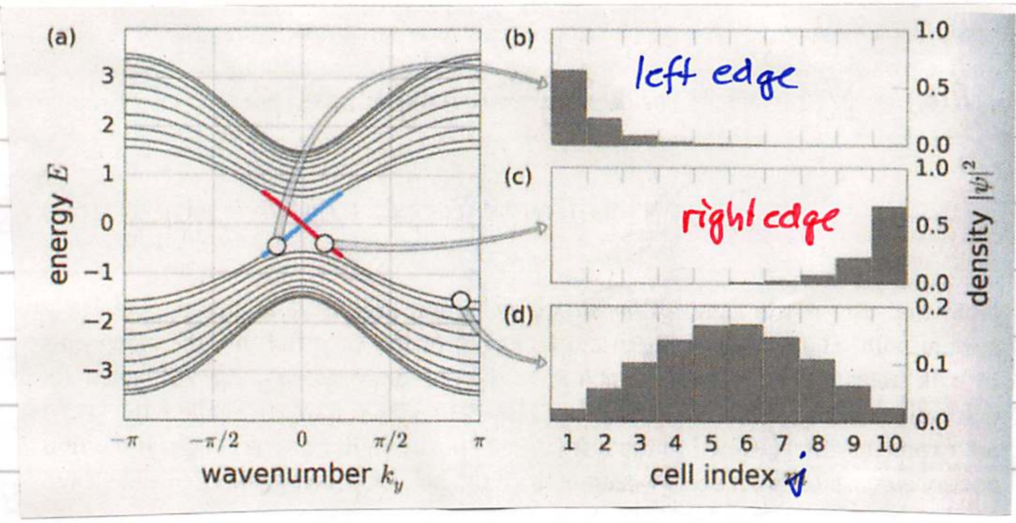


$$h(k_y) = \sum_{j=1}^{L_x-1} \left(|j+1\rangle\langle j| \, \frac{\hat{\sigma}_z + i\hat{\sigma}_x}{2} + h.a. \right) \tag{98}$$
$$+ \sum_{j=1}^{L_x} |j\rangle\langle j| \left(\cos k_y \hat{\sigma}_z + \sin k_y \hat{\sigma}_y + u \hat{\sigma}_z \right)$$

for every value of k_y we find an edge state with energy in the gap and localized either on the left or the right side of the stripe ($j=1$ or $j=L_x$)

the corresponding energies $\mathcal{E}_{L/R}(k_y)$ are a smooth function of k_y and correspond to a 1d band structure

$\Rightarrow$ 1d edge modes



the edge modes have a dispersion with fixed sign!

$$\frac{\partial \mathcal{E}_L(k_y)}{\partial k_y} > 0 \qquad \frac{\partial \mathcal{E}_R(k_y)}{\partial k_y} < 0 \tag{99}$$

thus they can carry a current, but only in one direction

chiral edge modes

The existence of the edge states / edge modes can be explained as follows:

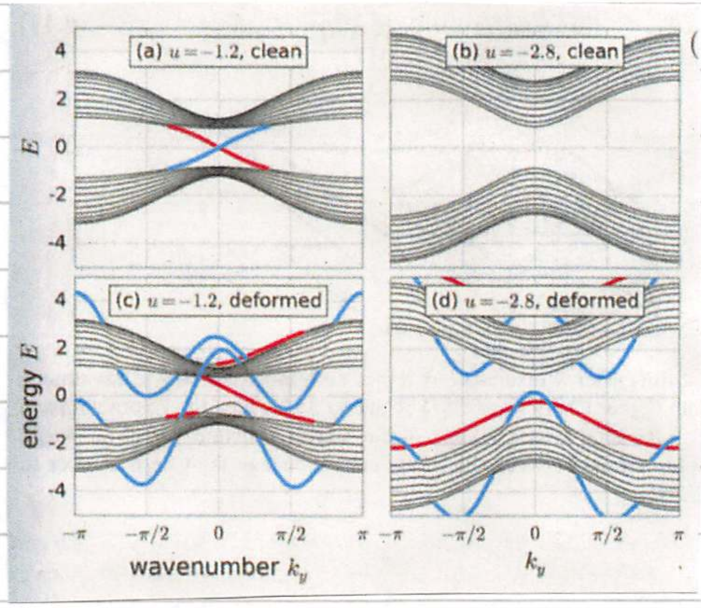
- if bulk to the left and right of interface characterized by two different topological integers (invariants) there have to be states at the interface that close the gap. Topological integers cannot change smoothly. They can only change by closing the energy gap.

From this follows a robustness of edge states / modes: perturbations that do not lead to a closing of energy gap (or break certain symmetries that are required to have a nontrivial topology) edge states should remain present.

For example lets add a next-nearest neighbor hopping at the edge in the y-direction to the Qi-Wu-Zhang model. This corresponds to an extra term to $h(k_y)$, eq. (98)

(100)
$$\tilde{h}(k_y) = h(k_y) + |j=1\rangle\langle j=1| \left(\mu^{(1)} + h_2^{(1)} \cos(2k_y)\right) + |j=L\rangle\langle j=L| \left(\mu^{(2)} + h_2^{(2)} \cos(2k_y)\right)$$

$C=-1$ $\qquad\qquad$ $C=0$



presence of edge modes survives!

Furthermore since the edge-modes are chiral a localized impurity cannot lead to a resistance, since an electron which scatters off the impurity can only go into bulk states or scatter forward or backward along the edge. Scattering into the bulk is suppressed by the energy gap and the fact that the lower band is fully occupied. Backscattering along the edge is however forbidden since the edge is chiral and has only forward propagating states

$\Rightarrow$ robustness of edge current against disorder