$\Rightarrow$ FQHE $\hat{=}$ IQHE for composite fermions in a reduced magnetic field

9.4* Chern-Simons field theory of composite fermions

(A) bare composite particles

Our goal is to construct a field theory of composite fermions which arise from bare fermions by attachment of flux quanta

- bare fermion field $\hat{\psi}(z)$ $\qquad z = x + iy$

(382) $$\{\hat{\psi}(z), \hat{\psi}^+(z')\} = \delta^{(2)}(z-z')$$

- define bare composite particle

(383) $$\boxed{\hat{\phi}_m(z) \equiv e^{-im\hat{\Theta}(z)}\,\hat{\psi}(z)}$$

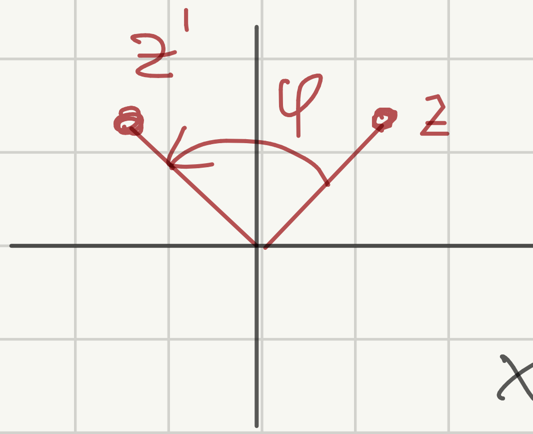
where the angle operator $\hat{\Theta}(z)$ is defined as

(384) $$\boxed{\hat{\Theta}(z) = \int d^2z'\ \varphi(z-z')\ \hat{\psi}^+(z')\hat{\psi}(z')}$$

Note that

$$\hat{\psi}^\dagger(z)\,\hat{\psi}(z) = \hat{\phi}_m^\dagger(z)\,\hat{\phi}_m(z) = \hat{n}(z)$$

- $\varphi(z-z')$ is the angle between z and z' in the complex plane



Using

x, y also used as complex coordinates

(385)
$$[\hat{\Theta}(z), \hat{\psi}(x)] = \int d^2z'\, \varphi(z-z')\,\underbrace{[\hat{n}(z'), \hat{\psi}(x)]}_{-\delta(x-z')\,\hat{\psi}(x)}$$
$$= -\varphi(z-x)\,\hat{\psi}(x)$$

$\Rightarrow$

(386)
$$e^{im\hat{\Theta}(y)}\,\hat{\psi}(x)\,e^{-im\hat{\Theta}(y)} = e^{-im\varphi(y-x)}\,\hat{\psi}(y)$$

With this we can determine the commutation relations of the $\hat{\phi}_m$

$$[\hat{\phi}_m(x), \hat{\phi}_m(y)] = e^{-im\hat{\Theta}(x)}\,\hat{\psi}(x)\,e^{-im\hat{\Theta}(y)}\,\hat{\psi}(y)$$
$$- e^{-im\hat{\Theta}(y)}\,\hat{\psi}(y)\,e^{-im\hat{\Theta}(x)}\,\hat{\psi}(x)$$
$$= e^{-im\varphi(y-x)}\,e^{-im\hat{\Theta}(x)}\,e^{-im\hat{\Theta}(y)}\,\hat{\psi}(x)\,\hat{\psi}(y)$$
$$- e^{-im\varphi(x-y)}\,e^{-im\hat{\Theta}(y)}\,e^{-im\hat{\Theta}(x)}\,\hat{\psi}(y)\,\hat{\psi}(x)$$

with $[\hat{\Theta}(x), \hat{\Theta}(y)] = 0$

$\Rightarrow$

(387) $$\hat{\phi}_m(x)\hat{\phi}_m(y) = (-1)^{m+1}\hat{\phi}_m(y)\hat{\phi}_m(x)$$

Thus

(388) $$\boxed{\begin{array}{lll} m: \text{odd} & [\hat{\phi}_m(x), \hat{\phi}_m(y)] = 0 & \text{bosons} \\ m: \text{even} & \{\hat{\phi}_m(x), \hat{\phi}_m(y)\} = 0 & \text{fermions} \end{array}}$$

Note that composite bosons (m: odd) constructed from bare fermions still fulfill the hard-core constraint

(389) $$\hat{\phi}_m^2(x) = 0 \quad \text{since} \quad \hat{\Psi}^2(x) = 0$$

One can repeat the same calculations to find

(390) $$\boxed{\begin{array}{ll} [\hat{\phi}_m(x), \hat{\phi}_m^+(y)] = \delta^{(2)}(x-y) & m: \text{odd} \\ \{\hat{\phi}_m(x), \hat{\phi}_m^+(y)\} = \delta^{(2)}(x-y) & m: \text{even} \end{array}}$$

- relation between wave-functions

N-particle wave-function for many-body state $|\Psi\rangle$

$$\Phi_m(x_1,\dots,x_N) \equiv \langle 0 | \hat{\Phi}_m(x_1) \cdots \hat{\Phi}_m(x_N) | \psi \rangle$$

(391) composite

$$\psi(x_1,\dots,x_N) = \langle 0 | \hat{\psi}(x_1) \cdots \hat{\psi}(x_N) | \psi \rangle$$

bare

or

$$|\psi\rangle = \int d^{2N}x \; \Phi_m(x_1,\dots,x_N) \, \hat{\Phi}^+_m(x_1) \cdots \hat{\Phi}^+_m(x_N) |0\rangle$$

(392)
$$= \int d^{2N}x \; \psi(x_1,\dots,x_N) \, \hat{\psi}^+(x_1) \cdots \hat{\psi}^+(x_N) |0\rangle$$

Since

$$\hat{\Phi}^+_m(y) \hat{\Phi}^+_m(x) = e^{im\varphi(y-x)} \hat{\psi}^+(y) \hat{\psi}^+(x) \, e^{im\hat{\theta}(y)} e^{im\hat{\theta}(x)}$$

$$\Rightarrow \hat{\Phi}^+_m(y) \hat{\Phi}^+_m(x) |0\rangle = e^{im\varphi(y-x)} \hat{\psi}^+(y) \hat{\psi}^+(x) |0\rangle$$

thus

$$\hat{\Phi}^+_m(x_1) \cdots \hat{\Phi}^+_m(x_N) |0\rangle = e^{im\sum_{e<j} \varphi(x_e - x_j)} \cdot \hat{\psi}^+(x_1) \cdots \hat{\psi}^+(x_N) |0\rangle$$

With this we find the relation between the N-particle wavefunctions from (392)

(393) $$\boxed{\Psi(z_1,\dots,z_N) = e^{im\sum_{l<j}\varphi(z_l - z_j)}\,\Phi_m(z_1,\dots,z_N)}$$

relation to bare composite wavefunction

(B) Effective magnetic field and Chern-Simons field

The Hamiltonian of fermions in the LLL in the symmetric gauge reads (in dimensionless units)

$$H_0 = \frac{1}{2m}\int d^2z\; \hat{\Psi}^+(z)\left[\underbrace{\left(-i\frac{\partial}{\partial x} - \frac{y}{2}\right)}_{\equiv P_x}^2 + \underbrace{\left(-i\frac{\partial}{\partial y} - \frac{x}{2}\right)}_{\equiv P_y}^2\right]\hat{\Psi}(z)$$

(394)
$$= \frac{1}{2m}\int d^2z\; \hat{\Psi}^+(z)\left[P_x^2 + P_y^2\right]\hat{\Psi}(z)$$

$$= \frac{1}{2m}\int d^2z\; \hat{\Psi}^+(z)\left[(P_x + iP_y)(P_x - iP_y)\right]\hat{\Psi}(z) + \frac{N}{2}\hbar\omega_c$$

Note that $[P_x, P_y] \neq 0$

We now express H_0 in terms of composite fields using eq. (383)

$$H = \frac{1}{2m}\int d^2z\, \hat{\Phi}^{+}_{m}(z)\left[(\tilde{P}_x + i\tilde{P}_y)(\tilde{P}_x - i\tilde{P}_y)\right]\hat{\Phi}_m(z) + \frac{N}{2}\hbar\omega_c \tag{395}$$

where the "covariant" momenta read $(\mu = x, y)$

$$\tilde{P}_\mu = P_\mu + eC_\mu = -i\hbar\frac{\partial}{\partial x_\mu} + eA^{eff}_\mu \tag{396}$$

Here ($\hbar$ reintroduced)

$$\boxed{C_\mu(z) = \frac{\hbar m}{e}\partial_\mu \hat{\Theta}(z) = \frac{\hbar m}{e}\int d^2z'\, \partial_\mu \varphi(z-z')\hat{n}(z')} \tag{397}$$

is the vector potential of the Chern-Simon field

and the effective total vector potential reads

$$\boxed{A^{eff}_\mu = A_\mu + C_\mu} \tag{398}$$

Non-interacting fermions in the LLL are identical to composite particles with an additional Chern-Simon field (provided a composite state exists that fulfills the LLL condition.

The Chern Simon field is then given by

(399)
$$\boxed{\begin{aligned} B_{cs} &= \varepsilon_{jk}\partial_j C_k(z) \\ &= \frac{\hbar m}{e}\varepsilon_{jk}\partial_j\partial_k \bar{\theta}(z) = m\Phi_0\delta^{(2)}(z) \end{aligned}}$$

I.e. the Chern Simon field of Φ_m-composites corresponds to m Dirac flux quanta at the position of the composite particle. One easily finds that composite particles in an external field B experiences an effective reduced magnetic field

(400)
$$\boxed{B_{eff}(z) = B - m\Phi_0\hat{n}(z)}$$

Composite fermions ($m = 2p$) with filling

(401)
$$\nu = \frac{\rho\Phi_0}{B} = \frac{1}{2p+1}$$

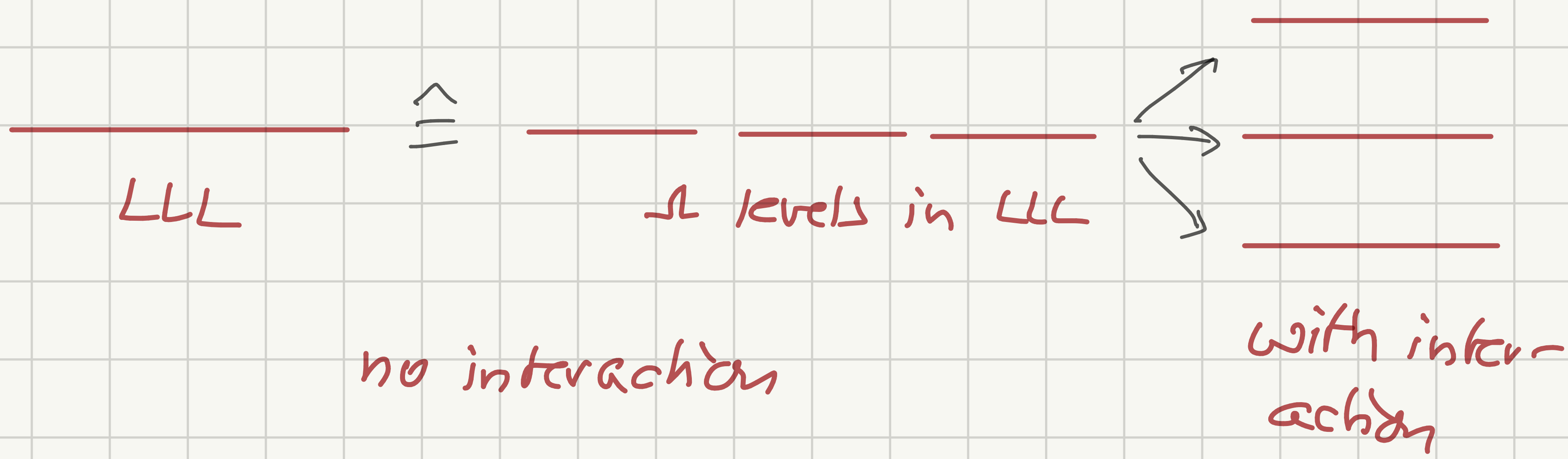
fill all states of the lowest effective Landau level or Λ-levels with effective filling

(402)
$$\nu^* = \frac{\rho\Phi_0}{B_{eff}} = \frac{\rho\Phi_0}{B-2p\Phi_0\rho} = \frac{1}{\frac{1}{\nu}-2p} = 1$$

IQHE!

For filling $\nu > \frac{1}{2p+1}$ higher Λ-level become occupied. Without interactions these higher Λ levels might be energetically degenerate with the lowest Λ-level

With interactions the Lowest Λ-level wavefunction screens all small pseudopotentials, while states with components in higher Λ-levels have interaction energy $\Rightarrow$ splitting



(C) Dressed composite fermions

Bare composite wavefunctions have singularities. To avoid them we slightly modify the phase operator. The modification is different for composite bosons and composite fermions. We here discuss CF's in the LLL with $m=2p$

$$\boxed{\Phi_{2p}(z) \longrightarrow \hat{\Psi}_{2p}(z) = e^{-i2p\,\hat{\theta}(z) - \hat{a}_{cs}(z)}\,\hat{\Psi}(z)}$$

(403)

where we added a scalar field

(404) $$\hat{a}_{cs}(z) \equiv 2p\int d^2z' \ln\left(\frac{|z-z'|}{2\ell_0}\right)\hat{n}(z') - \frac{e\bar{B}_{cs}}{4\hbar}|z|^2$$

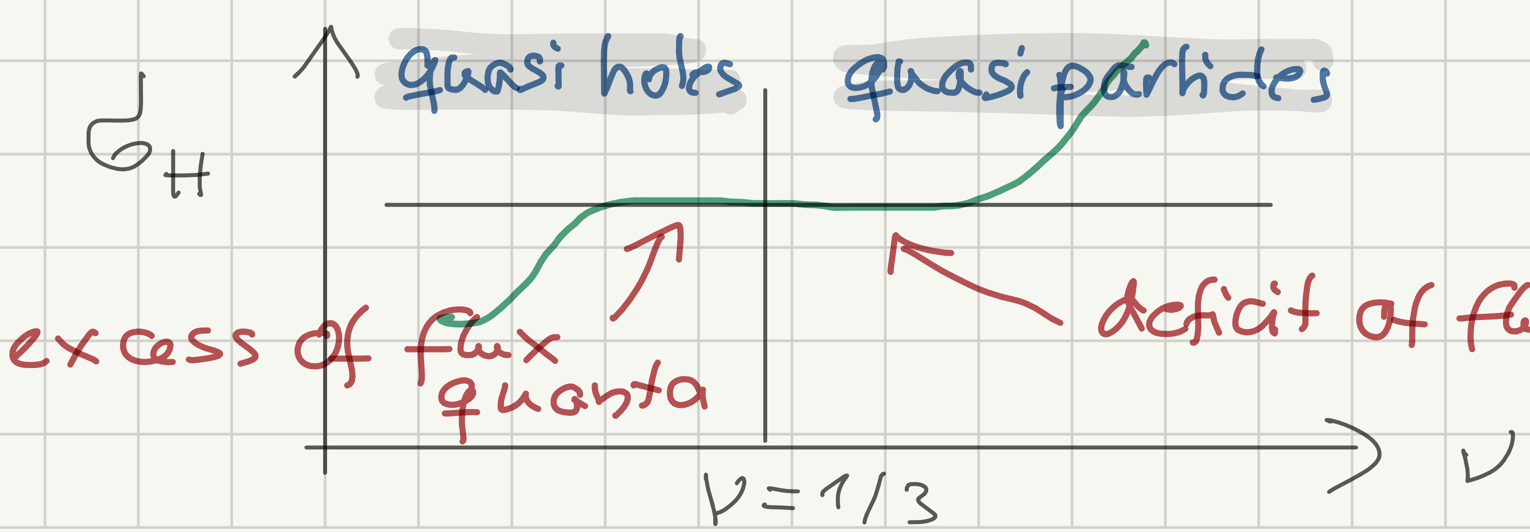
$\bar{B}_{cs}$ is the homogenized Chern-Simon field

(405) $$\bar{B}_{cs} = 2p\,\Phi_0\, S$$

The transmutation properties (commutation / anticommutation) are not modified

9.5 Elementary excitations: quasi-holes and quasi-particle states

FQH states are incompressible. Moving away from strict fractional filling by changing the magnetic field leads to elementary excitations which we want to discuss how

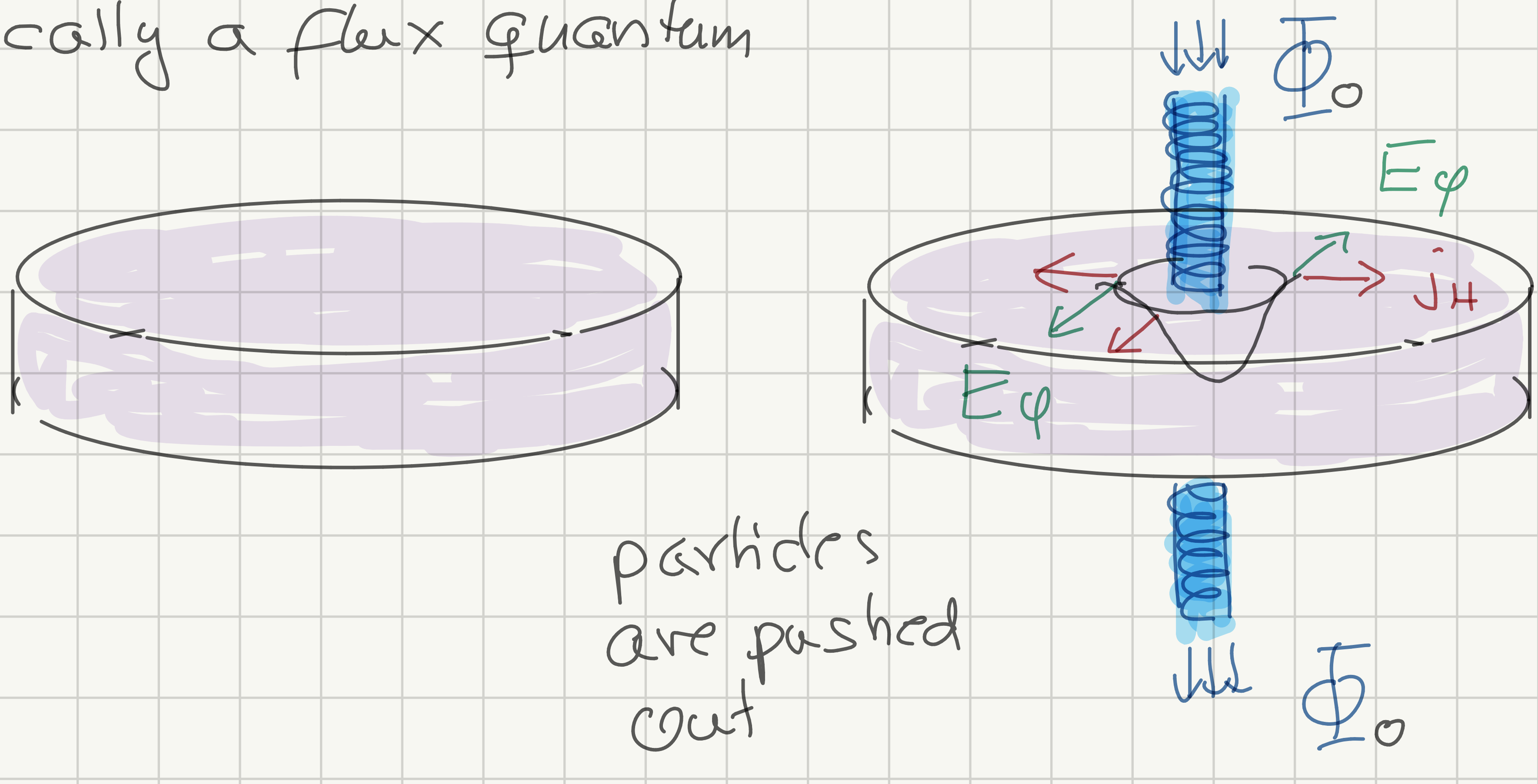


Both quasi-holes (qh) and quasi-particles (qp) are pinned to disorder / impurities and thus do not contribute to the transport which explains the presence of the QH plateaus

... so disorder is in fact important to see plateaus at all

(A) fundamentals of elementary excitations

let's start with a thought experiment by inserting locally a flux quantum

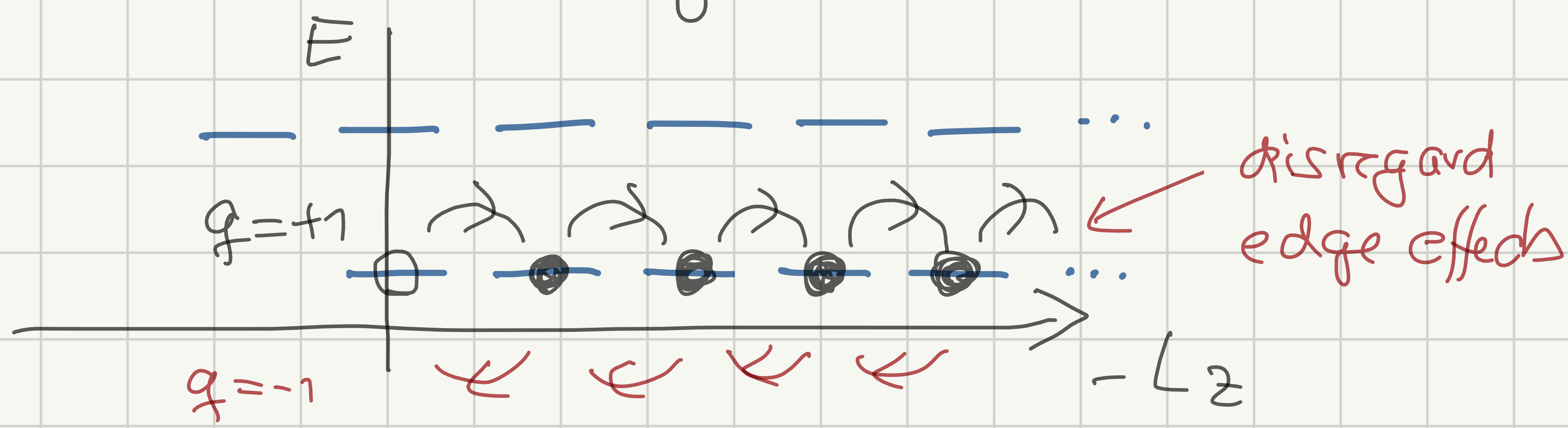


- after insertion of one flux quantum Φ_0 the angular momentum with respect to the origin is increased (reduced) by one unit

(406) $$\lambda(z) = z^{\ell} \longrightarrow \lambda'(z) = e^{iq\varphi} z^{\ell}$$

$$q = \pm 1$$

φ is the azimutal angle around the z axis



- If flux insertion is adiabatic the wavefunction follows adiabatically

(407) $$\Psi(z_1 \dots z_N) \longrightarrow \Psi'(z_1 \dots z_N) = e^{\pm i\varphi}\Psi$$

This is a (singular) gauge transformation that does not change the Hamiltonian (disregard edge effects!) except at origin where $e^{\pm i\varphi}$ is undefined. $\Rightarrow$ state after flux insertion should again be an eigenstate of H (see figure above)

$\Rightarrow$ elementary excitation

$q = +1$	$q = -1$
quasi-hole	quasi-particle

problem: $\Psi' = e^{\pm i\varphi}\Psi$ is no longer in LLL
need instead increase or reduce power of z

$q=+1 \quad \lambda(z)=z^{\ell} \longrightarrow \lambda_{qh}(z)=z\,\lambda(z)=re^{+i\varphi}z^{\ell}$

$q=-1 \quad \lambda(z)=z^{\ell} \longrightarrow \lambda_{qp}(z)=\frac{\partial}{\partial z}\lambda(z)=\frac{1}{r}e^{-i\varphi}z^{\ell}$

use angular-momentum ladder operators

(408) $$b_j^{+}=\frac{1}{\sqrt{2}}\left(z_j-\frac{\partial}{\partial z_j^{*}}\right) \qquad b_j=\frac{1}{\sqrt{2}}\left(z_j^{*}+\frac{\partial}{\partial z_j}\right)$$

(409) $$\boxed{\psi(z_1 \ldots z_N)\rightarrow \psi_{qh}(z_1 \ldots z_N)=\prod_{j=1}^{N} b_j^{+}\,\psi_{LN}(z_1 \ldots z_N)}$$

quasi-hole increases angular momentum of all particles by one unit

(410) $$\boxed{\psi(z_1 \ldots z_N)\rightarrow \psi_{qp}(z_1 \ldots z_N)=\prod_{j=1}^{N} b_j\,\psi_{LN}(z_1 \ldots z_N)}$$

quasi-particle decreases angular momentum of all particles by one unit

for

$$\psi_{LN}(\underline{z})=\lambda(\underline{z})\,e^{-\sum_j |z_j|^2}$$

different length unit used for simplicity

(411) $$b_\ell\,\psi_{LN}(\underline{z}) \sim \left(\frac{\partial}{\partial z_\ell}\lambda(\underline{z})\right)e^{-\sum_j |z_j|^2}$$

$$b_\ell^{+}\,\psi_{LN}(\underline{z}) \sim z\,\lambda(\underline{z})\,e^{-\sum_j |z_j|^2}$$

quasi-hole wavefunction

(4.12) $$\boxed{\psi_{qh} = \prod_{e=1}^{N} z_e \, \psi_{LN}}$$

quasi-particle wavefunction

more complicated

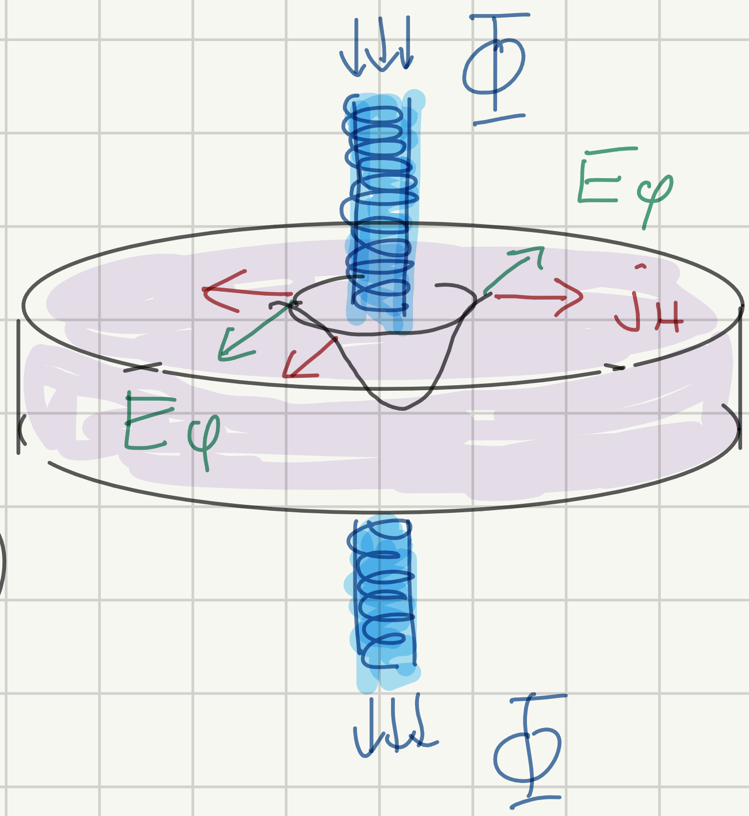
(4.13) $$\boxed{\psi_{qp} = \frac{\prod_{j=1}^{N} \frac{\partial}{\partial z_j} \prod_{j<s} (z_j - z_s)^m}{\prod_{j<s} (z_j - z_s)^m} \, \psi_{LN}}$$

remark: $\nexists$ qp excitation for integer QH state as qp generation corresponds to removal of angular momentum. Fermions would then be pushed to the next higher Landau level.

An interesting question is that of the effective charge of quasi-holes / quasiparticles

- flux insertion by $\pm \Phi_0$ $\Rightarrow$ electric field in time T (adiabatic)

(414) $$E_\varphi = \frac{d\Phi}{dt}\,\frac{1}{2\pi r}$$



$\Rightarrow$ (415) $$j_H = \frac{e\varrho}{B}\,E_\varphi$$

for large r $\quad (j = \sigma E)$

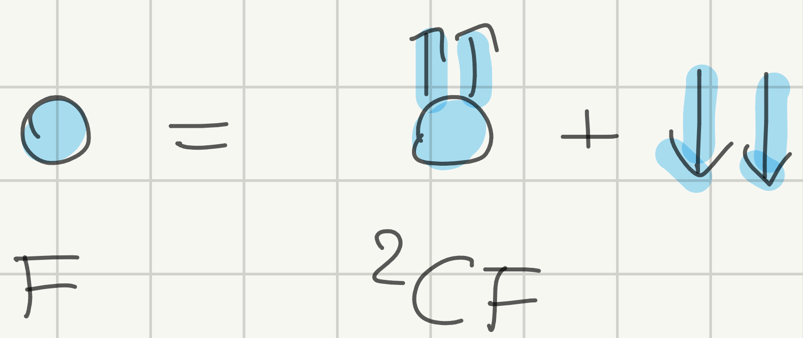
Total change of charge

$$q^* = 2\pi r \int_0^T dt\, j_H = \pm \frac{e\varrho}{B}\int_0^T dt\,\frac{d\Phi}{dt}$$

(416) $$\boxed{q^* = \pm \frac{e\varrho}{B}\,\Phi_0 = \pm \nu e}$$

quasi-hole / quasi-particle excitations on top of an incompressible FQH fluid with filling ν have a fractional charge

- What happens if we add / subtract a bare fermion from the system?



$\Rightarrow$ loss (addition) of composite fermion and creation of $2p$ local flux quanta i.e. creation of quasi-holes

- We will now argue that quasi-holes / particles have an unusual transmutation property: they are anyons

(B) Anyons

In the real 3D world there are only two types of particles

$\boxed{3D}$ — bosons / fermions

(i) spin: bosons: integer S; fermions: half-integer S

(ii) exchange statistics

$$\psi(x_2, x_1) = \begin{cases} \psi(x_1, x_2) & B \\ -\psi(x_1, x_2) & F \end{cases}$$

(iii) Pauli exclusion for fermions

The spin of bosons/fermions is enforced to be integer / half-integer by the non-Abelian spin algebra

(4.17) $$[S_i, S_j] = i\hbar \varepsilon_{ijk} S_k$$

This changes if we are in 2D

$\boxed{2D}$ only S_z no algebra!
quantization of spin not enforced

(i) $\Rightarrow$ $\boxed{\text{quasiparticles in 2D can have fractional spin}}$

How about exchange statistics? Exchanging two identical particles can only affect the phase of the wavefunction

(4.18) $$\psi(\vec{r}_2, \vec{r}_1) = e^{i\alpha\pi} \psi(\vec{r}_1, \vec{r}_2)$$

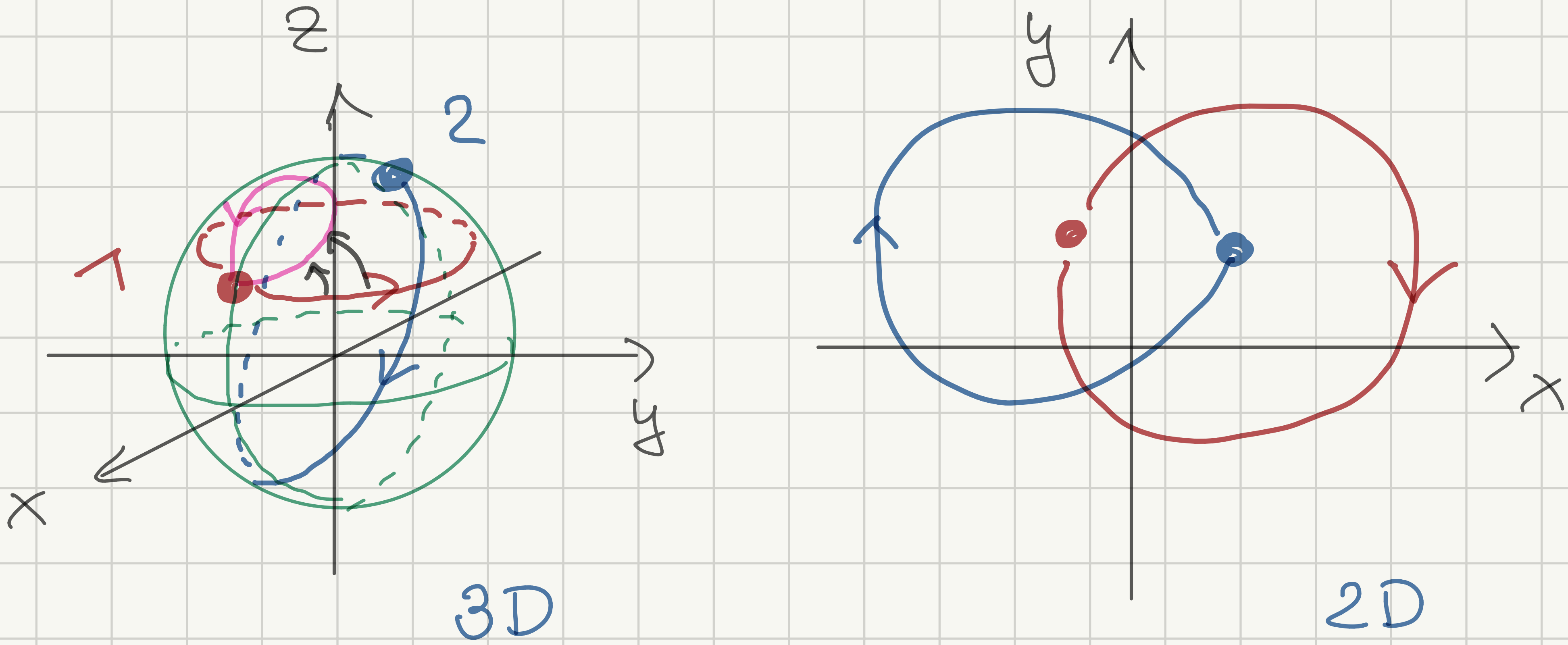
α : $\alpha \bmod 2$ statistics parameter

- $\alpha = ?$ Exchanging position twice leads to

$$\psi(\vec{r}_1, \vec{r}_2) = e^{i2\alpha\pi} \psi(\vec{r}_1, \vec{r}_2) \Rightarrow e^{2i\alpha\pi} = 1$$

$\alpha = 0$ bosons $\qquad$ $\alpha = 1$ fermions

But: argument for requiring $e^{2i\alpha\pi} = 1$ comes from ability of continuously deform exchange path which does not apply in 2D



$$\boxed{\text{quasi-particles in 2D can have any exchange statistics (anyons)}}$$

(C) Quantum mechanics of anyons

consider composite-boson field $\hat{\Phi}(z)$

(4.19) $$\left[\hat{\Phi}(z), \hat{\Phi}^{+}(z')\right] = \delta^{(2)}(z-z')$$

define a new field

$$(420) \quad \boxed{\hat{\psi}(z,t) = e^{i\alpha\hat{\Theta}(z,t)}\,\hat{\Phi}(z,t)}$$

anyon field

where $\hat{\Theta}(z)$ is again the angle operator

$$(421) \qquad \hat{\Theta}(z) = \int d^2z'\,\varphi(z-z')\,\underbrace{\hat{\Phi}^{\dagger}(z')\hat{\Phi}(z')}_{=\hat{n}(z')}$$

then following the same arguments as in Sec. 9.4 A

$\Rightarrow$

$$(422) \quad \boxed{\begin{aligned}\hat{\psi}(y)\,\hat{\psi}(x) &= e^{i\alpha\pi}\,\hat{\psi}(x)\,\hat{\psi}(y)\\ \hat{\psi}^{\dagger}(y)\,\hat{\psi}(x) &= e^{i\alpha\pi}\,\hat{\psi}(x)\,\hat{\psi}^{\dagger}(y) + \delta^{(2)}(x-y)\end{aligned}}$$

Note that different from Sec. 9.4 A α is not necessarily an integer!

9.6 Effective Hamiltonian of quasi-holes: Anyons and Chern-Simon field

Now we want to sketch the derivation of an effective anyon Hamiltonian of quasiholes in the FQH system.