4. Classification of topological systems of noninteracting fermions by generalized symmetries: 10-fold way

Topological features are robust against local perturbations. Thus the key difference between topological and nontopological systems must be rooted in some global, non-local properties. Generalized symmetries define such properties and we will discuss them in this chapter. They will be both of unitary and of anti-unitary character

4.1 Wigner theorem and invariance transformations

A linear transformation $|\psi\rangle \longrightarrow G|\psi\rangle$ is called an invariance transformation if for all $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ holds

(192) $$|\langle G\psi | G\phi\rangle| = |\langle \psi | \phi\rangle|$$

Wigner (1931)

> invariance transformations are either
>
> (193) **unitary** $\langle G\psi | G\phi\rangle = \langle \psi | \phi\rangle$
>
> or
>
> (194) **anti-unitary** $\langle G\psi | G\phi\rangle = \langle \psi | \phi\rangle^*$

Def: invariance transformations are called (generalized) Symmetry transformations of a quantum system if and only if

(195) $$\boxed{GHG^{-1} = H} \quad \text{or} \quad \boxed{[H, G] = 0}$$

- properties of anti-unitary operators

(i) if G is anti-unitary $\Rightarrow$ G^2 is unitary

(ii) every anti-unitary operator can be written as product of a unitary operator U and the complex conjugation K

(196) $$\boxed{G = UK} \qquad U^+ = U^{-1} \qquad Kf = f^*$$

(197) (iii) inverse $G^{-1} = KU^+$ $\qquad (K^2 = 1)$

(iv) transformation of operators

(198) $$A' = GAG^{-1} = UKAKU^{-1} = UA^*U^{-1}$$

since $KA = A^*K$

(v) transformation of matrix elements

$$A' = GAG^{-1} \qquad |\psi'\rangle = G|\psi\rangle$$

$$\Rightarrow \langle\psi'|A'|\phi'\rangle = \langle G\psi|GAG^{-1}|G\phi\rangle$$

(199) $$= \langle G\psi|GA\phi\rangle = \langle\psi|A\phi\rangle^*$$

4.2 time reversal of spinless particles

How can we define a "time reversal" operation?

$$T: \quad t \to -t$$

Consider quantum mechanics $\hat{x}, \hat{p}$

(200) $$T\hat{x}T^{-1} \overset{!}{=} \hat{x} \qquad T\hat{p}T^{-1} \overset{!}{=} -\hat{p}$$

$$\Rightarrow \hat{T}[\hat{x},\hat{p}]\hat{T}^{-1} = Ti\hbar T^{-1} \overset{!}{=} -[\hat{x},\hat{p}] = -i\hbar$$

$\Rightarrow$ T must be anti-unitary!

(201) $$\boxed{T = UK} \qquad TiT = -i$$

Furthermore it holds

(202) $$\boxed{T^2 = \pm 1}$$

<u>proof:</u> $T^2 = UKUK = UU^*K^2 = UU^*$

$$= U(U^T)^{-1} = A$$

since $T^2|\psi\rangle = e^{i\varphi}|\psi\rangle$ (for any $|\psi\rangle$) the operator

A must be diagonal, i.e. $A = A^T$

$$\Rightarrow \quad U = A U^T \quad \leadsto \quad U^T = U A^T = U A$$

$$\Rightarrow U = AUA \qquad AUA = A^2 U A^2 \qquad A^2 = 1 \qquad A = \pm 1$$

(203) $$\boxed{\begin{array}{lll} T^2 = 1 & \Rightarrow & U = U^T \text{ symmetric} \\ T^2 = -1 & \Rightarrow & U = -U^T \text{ antisymmetric} \end{array}}$$

(A) action of T in Fock space on a lattice

From similarity between boson annihilation operator and ladder operator in quantum mechanics

$$\hat{a}_e = \alpha \hat{x}_e + i \beta \hat{p}_e \qquad e \text{ position index}$$

$$\alpha, \beta \in \mathbb{R}$$

$\Rightarrow$

(204) $$\boxed{T \hat{a}_e T^{-1} = T(\alpha \hat{x}_e + i \beta \hat{p}_e) T^{-1} = \hat{a}_e}$$

time reversal of real-space operators

So here

(205) $$\boxed{U = \mathbb{1} \quad \text{and} \quad T = K} \qquad \boxed{T^2 = +1}$$

Now lets construct the action of T on momentum-

Space operators:

$$\hat{a}_\ell \overset{!}{=} T\hat{a}_\ell T^{-1} = T\frac{1}{\sqrt{L}}\sum_{\vec{k}} e^{i\vec{k}\vec{R}_\ell}\tilde{a}_{\vec{k}} T^{-1}$$

$$= \frac{1}{\sqrt{L}}\sum_{\vec{k}} e^{-i\vec{k}\vec{R}_\ell} T\tilde{a}_{\vec{k}}T^{-1} \overset{!}{=} \frac{1}{\sqrt{L}}\sum_{\vec{k}} e^{+i\vec{k}\vec{R}_\ell}\tilde{a}_{k}$$

Thus

(206) $$\boxed{T\tilde{a}_{\vec{k}}T^{-1} = \tilde{a}_{-\vec{k}}}$$

Lets make a sanity check $\hat{\vec{p}} = \sum_{\vec{k}} \hbar\vec{k}\,\tilde{a}^+_{\vec{k}}\tilde{a}_{\vec{k}}$

$$T\hat{\vec{p}}T^{-1} = \sum_{\vec{k}} \hbar\vec{k}\, T\tilde{a}^+_{\vec{k}}\tilde{a}_{\vec{k}}T^{-1}$$

$$= \sum_{\vec{k}} \hbar\vec{k}\;\tilde{a}^+_{-\vec{k}}\tilde{a}_{-\vec{k}} = -\sum_{\vec{k}} \hbar\vec{k}\,\tilde{a}^+_{\vec{k}}\tilde{a}_{\vec{k}}$$

$$= -\hat{\vec{p}} \quad \checkmark$$

- Under what conditions is a single-particle Hamiltonian

(207) $$H = \sum_{\vec{k}\mu\nu} \tilde{c}^+_{\vec{k}\mu}\, h_{\mu\nu}(\vec{k})\, \tilde{c}_{\vec{k}\nu}$$

time-reversal symmetric (TR invariant)? For this

$$THT^{-1} = \sum_{\vec{k}\mu\nu} T h_{\mu\nu}(\vec{k})T^{-1}\; T\tilde{c}^+_{\vec{k}\mu}\tilde{c}_{\vec{k}\nu}T^{-1}$$

$$= \sum_{\vec{k}\mu\nu} T h_{\mu\nu}(\vec{k}) T^{-1} \tilde{c}^{\dagger}_{-k,\mu} \tilde{c}_{-k\nu} \overset{!}{=} H$$

$$(208) \Longleftrightarrow \quad \boxed{T \underline{\underline{h}}(\vec{k}) T^{-1} \overset{!}{=} \underline{\underline{h}}(-\vec{k})}$$

Since for spinless particle $T = K$

$$(209) \quad \boxed{\underline{\underline{h}}^{*}(\vec{k}) \overset{!}{=} \underline{\underline{h}}(-\vec{k})}$$

TR invariance of single-particle Hamiltonian of spinless particles

(B) Vanishing Chern number for TR invariant Hamiltonians

- assume TR invariance

$$h(\vec{k})\,|\psi(\vec{k})\rangle = T^{-1} h(-\vec{k}) T\,|\psi(\vec{k})\rangle = \varepsilon_k |\psi(\vec{k})\rangle$$

i.e. if $|\psi(\vec{k})\rangle$ is eigenfunction of $h(\vec{k})$ then $T|\psi(\vec{k})\rangle$ is eigenfunction of $h(-\vec{k})$

- Bloch functions

$$u_{-\vec{k}} = T u_{\vec{k}} = u^{*}_{\vec{k}}$$

$\Rightarrow$ Berry curvature

$$F_{\ell m}(\vec{k}) = -i\left(\langle\partial_\ell u_{\vec{k}}|\partial_m u_{\vec{k}}\rangle - \langle\partial_m u_{\vec{k}}|\partial_\ell u_{\vec{k}}\rangle\right)$$

$$= -i\left(\langle\partial_\ell u^*_{-\vec{k}}|\partial_m u^*_{-\vec{k}}\rangle - \text{"}\ell \leftrightarrow m\text{"}\right)$$

$$= -i\left(\langle\partial_m u_{-\vec{k}}|\partial_\ell u_{-\vec{k}}\rangle - \text{"}\ell \leftrightarrow m\text{"}\right)$$

$$= -F_{\ell m}(-\vec{k})$$

integral of $F_{\ell m}$ over Brillouin zone

$$\int_{BZ} d\vec{k}\; F_{\ell m}(\vec{k}) = 0$$

(210) $$\boxed{C = 0 \quad \text{i.e. } \sigma_{xy} = 0}$$

There is no QHE for systems with TR symmetry
The Chern number vanishes

We will see in the course of this lecture that also systems with TR symmetry can be topologically nontrivial, but the topological invariant is not the Chern number.

These topological insulators with TR invariance require spin-full particles.

4.3 Time reversal for particles with spin

For particles with spin we have to think about how the T-reversal acts in the spin space. To this end we first look at the orbital angular momentum

$$\vec{L} = \vec{r} \times \vec{p} \quad \Rightarrow \quad T \vec{L} T^{-1} \stackrel{!}{=} -\vec{L}$$

Thus we impose

(211) $$\boxed{T \vec{S} T^{-1} = -\vec{S}}$$

The sign change can be achieved by rotation about arbitrary axis by π. Convention: y-axis

(212) $$\boxed{T = e^{-i\pi \hat{S}_y} K} \qquad \text{i.e. } U = e^{-i\pi \hat{S}_y}$$

$$T^2 = e^{-i\pi \hat{S}_y} K e^{-i\pi \hat{S}_y} K = e^{-i\pi \hat{S}_y} e^{i\pi \hat{S}_y^*}$$

since $\hat{S}_y^* = -\hat{S}_y$

(213) $$T^2 = e^{-2\pi i \hat{S}_y}$$

integer spin — half-integer spin

(214) $\boxed{T^2 = 1}$ — $\boxed{T^2 = -1}$

a (single) boson — a (single) fermion

Rem.: It is important that (214) holds for single particle states

time-reversal of lattice models of spinfull fermions

(for simplicity no sub-bands)

$$(215)\quad H = \sum_{\vec{k}\sigma\sigma'} \tilde{c}^{+}_{\vec{k}\sigma}\, h_{\sigma\sigma'}(\vec{k})\, \tilde{c}_{\vec{k}\sigma'} \qquad \sigma = \pm 1$$

Time reversal rotates spin $|\uparrow\rangle \leftrightarrow |\downarrow\rangle$, so

$$(216)\quad \boxed{\begin{aligned} T c_{j\uparrow} T^{-1} &\overset{!}{=} c_{j\downarrow} \\ \Rightarrow\quad T c_{j\downarrow} T^{-1} &= -c_{j\uparrow} \end{aligned}}$$

The minus sign in the second line seems strange. In fact where put a "$-$" in the upper or lower equation is a matter of convention, but one "$-$" must be there. It follows

$$T^2 \hat{c}_{j\uparrow} T^{-2} = -\hat{c}_{j\uparrow} \qquad ?$$

but $T^2 = -1$ so T^2 and T^{-2} given $+1$?
No! We need to remember that $\hat{c}_{j\uparrow}$ reduces the number of fermions by one

$$T^2 \hat{c}_{j\uparrow} \underbrace{T^{-2} |\psi\rangle} \qquad = T^2 \underbrace{\hat{c}_{j\uparrow} (-|\psi\rangle)}$$

e.g. state with odd number of fermions $\Rightarrow T^{-2} = -1$ — state with even number $\Rightarrow T^2 = +1$

$$= -\hat{c}_{j\uparrow}|\psi\rangle \qquad \square$$

compact form using Pauli matrix

$$-i\sigma^y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(217)
$$\boxed{\begin{aligned} T\,\hat{c}_{j\lambda}\,T^{-1} &= -i\sigma^y_{\lambda\lambda'}\,\hat{c}_{j\lambda'} \\ T\,\underline{\hat{c}}_j\,T^{-1} &= -i\,\underline{\underline{\sigma}}^y\,\underline{\hat{c}}_j \end{aligned}}$$

Time reversal of real-space fermion operators with spin

- transformation to momentum space (as for spin-less particle)

(218)
$$\boxed{T\,\underline{\tilde{c}}(\vec{k})\,T^{-1} = -i\,\underline{\underline{\sigma}}^y\,\underline{\tilde{c}}(-\vec{k})}$$

- Let's ask also here under what conditions a single particle Hamiltonian (here unit cell of 1 for simplicity) for spinfull fermions is TR invariant

(219)
$$H = \sum_{\vec{k}\lambda\lambda'} h^{\lambda\lambda'}(\vec{k})\,\tilde{c}^+_\lambda(\vec{k})\,\tilde{c}_{\lambda'}(\vec{k})$$

$$= \sum_{\vec{k}} \left(\underline{\tilde{c}}^+(\vec{k})\right)^T \underline{\underline{h}}(\vec{k})\,\underline{\tilde{c}}(\vec{k})$$

where $\underline{\tilde{c}}(\vec{k}) = \begin{pmatrix} \tilde{c}_\uparrow \\ \tilde{c}_\downarrow \end{pmatrix}$

TR invariance: $\boxed{THT^{-1} = H}$

$$H \overset{!}{=} THT^{-1} = \sum_{\vec{k}} T\underline{\tilde{c}}^{+}(\vec{k})T^{-1}\, T\underline{\underline{h}}(\vec{k})T^{-1}\, T\underline{\tilde{c}}(\vec{k})T^{-1}$$

$$= \sum_{\vec{k}} \left(-i\underline{\underline{\sigma}}^{y}\underline{\tilde{c}}^{+}(-\vec{k})\right)^{T} \underline{\underline{h}}^{*}(\vec{k})\left(-i\underline{\underline{\sigma}}^{y}\right)\underline{\tilde{c}}(-\vec{k})$$

$$= \sum_{\vec{k}} \left(\underline{\tilde{c}}^{+}(-\vec{k})\right)^{T} i\underline{\underline{\sigma}}^{yT}\underline{\underline{h}}^{*}(\vec{k})\; i\underline{\underline{\sigma}}^{y}\underline{\tilde{c}}(-\vec{k})$$

$$= \sum_{\vec{k}} \left(\underline{\tilde{c}}^{+}(\vec{k})\right)^{T} i\underline{\underline{\sigma}}^{yT}\underline{\underline{h}}^{*}(-\vec{k})\; i\underline{\underline{\sigma}}^{y}\underline{\tilde{c}}(\vec{k})$$

$$\overset{!}{=} \sum_{\vec{k}} \left(\underline{\tilde{c}}^{+}(\vec{k})\right)^{T} \underline{\underline{h}}(\vec{k})\,\underline{\tilde{c}}(\vec{k})$$

$\Rightarrow$

(220) $$\underline{\underline{h}}(\vec{k}) = i\underline{\underline{\sigma}}^{yT}\,\underline{\underline{h}}(-\vec{k})^{*}\, i\underline{\underline{\sigma}}^{y}$$

since $\underline{\underline{\sigma}}^{yT} = -\underline{\underline{\sigma}}^{y}$

(221) $$\boxed{\underline{\underline{h}}(\vec{k}) = \underline{\underline{\sigma}}^{y}\,\underline{\underline{h}}(-\vec{k})^{*}\,\underline{\underline{\sigma}}^{y}}$$

condition for TR invariant single-particle Hamiltonian for spin 1/2 particle

Kramers theorem

> For a TR invariant fermionic Hamiltonian every (single-particle) eigenstate is at least doubly degenerate

proof: $H|\psi\rangle = E|\psi\rangle \qquad [T,H]=0$

$\Rightarrow$ also $T|\psi\rangle$ is eigenstate of H with energy E but it may not be orthogonal to $|\psi\rangle$?

$$\langle\psi|T\psi\rangle = \sum_{mn} \psi_m^* U_{mn} K \psi_n$$

$$= \sum_{mn} \psi_m^* U_{mn} \psi_n^* \qquad \text{remember } T^2=-1 \Rightarrow U^T=-U$$

$$= \sum_{mn} \psi_m^* (-U_{nm}) \psi_n^* = -\sum_{mn} \psi_n^* U_{nm} \psi_m^*$$

$$= -\sum_{m,n} \psi_n^* U_{nm} K \psi_m = -\langle\psi|T\psi\rangle$$

$$\Rightarrow \qquad \langle\psi|T\psi\rangle = 0 \qquad \text{for } T^2=-1 \tag{2.22}$$

$\square$

4.4 Charge conjugation

We will now introduce a second fundamental transformation in fermion Fock space: charge conjugation. To motivate this operation let us consider the

BCS hamiltonian which is the foundation of the Bardeen - Cooper - Schrieffer theory of superconductivity where a fermion-fermion interaction is replaced by a mean-field term describing the pairing of fermions of opposite spin and momenta

(223) $$c_\uparrow^+(k)c_\uparrow(k)\,c_\downarrow^+(-k)c_\downarrow(-k) \to \langle c_\uparrow^+(k)c_\downarrow^+(-k)\rangle\, c_\uparrow(k)c_\downarrow(-k)$$

The resulting mean-field Hamiltonian, called Bogoliubov-de Genne Hamiltonian reads

(234) $$H = \frac{1}{2}\sum_{\vec{k}} \begin{pmatrix} c^+(k) \\ c(-k) \end{pmatrix}^T \underbrace{\begin{pmatrix} h_0 & \Delta \\ -\Delta & -h_0 \end{pmatrix}}_{= h_{BdG}(\vec{k})} \begin{pmatrix} c(k) \\ c^+(-k) \end{pmatrix}$$

Here $\Psi(k) = \begin{pmatrix} c(k) \\ c^+(-k) \end{pmatrix}$ is called Nambu spinor

One recognize the following property

(235) $$\text{if } \; c(k) \leftrightarrow c^+(-k) \;\hat{=}\; H \Leftrightarrow -H$$

So we define the charge conjugation in real space of spinless particle

(236) $$\boxed{\mathcal{C}\,\hat{c}_{\mu j}\,\mathcal{C}^{-1} = \sum_{\nu} U^*_{c\,\mu\nu}\,\hat{c}^+_{\nu j}}$$

where U is some unitary matrix in the space of internal degrees of freedom of the jth unit cell

The action in momentum space follows

$$C\hat{c}_\mu(k)C^{-1} = \frac{1}{\sqrt{L}}\sum_j e^{-jk}\, C\hat{c}_{\mu j}C^{-1}$$

$$= \frac{1}{\sqrt{L}}\sum_{j\nu} e^{-jk}(U_C^*)_{\mu\nu}\,\hat{c}^\dagger_{\nu j}$$

$$= \sum_\nu (U_C^*)_{\mu\nu}\,\hat{c}^\dagger_\nu(-k)$$

$$\Rightarrow \boxed{C\hat{c}_\mu(k)C^{-1} = \sum_\nu (U_C^*)_{\mu\nu}\,\hat{c}^\dagger_\nu(-k)} \tag{237}$$

- consider double application of C

$$C^2\hat{c}^\dagger_A C^{-2} = \sum_B (U_C)_{AB}\, C\hat{c}_B C^{-1}$$

$$= \sum_B (U_C U_C^*)_{AB}\,\hat{c}^\dagger_B$$

If we consider a single-particle state $|\psi\rangle = \hat{c}^\dagger_A|0\rangle$

$$C^2|\psi\rangle = U_C U_C^*|\psi\rangle \equiv A|\psi\rangle$$

and also

$$C^2|\psi\rangle \overset{!}{=} e^{i\varphi}|\psi\rangle$$

$$A = U_c U_c^* = U_c (U_c^T)^{-1} \text{ is diagonal} \quad A = A^T$$

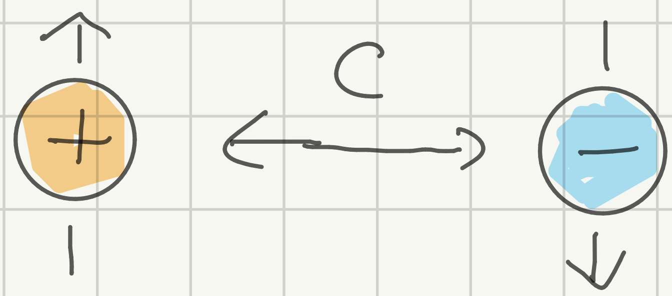
$$\Rightarrow U_c = A U_c^T \qquad U_c^T = U_c A^T = U_c A$$

$$\Rightarrow U_c = A U_c A \qquad \Rightarrow A = \pm 1$$

(238) $$\boxed{C^2 = \pm 1}$$

I.e. also the charge conjugation operator squares to ± 1.

- action of charge conjugation on spin



magnetic moment

(239) $$C \hat{\vec{\mu}} C^{-1} = -\hat{\vec{\mu}} \qquad \hat{\vec{\mu}} \sim \vec{S}$$

$\Rightarrow$ similar to time reversal of spin-full particle

real space

(240) $$\boxed{\begin{aligned} C \hat{c}_{j\uparrow} C^{-1} &= \hat{c}_{j\downarrow}^\dagger \\ C \hat{c}_{j\downarrow} C^{-1} &= -\hat{c}_{j\uparrow}^\dagger \end{aligned}}$$

plus some possible unitary transformation in the space of internal state

- momentum space

(241)
$$\boxed{\begin{aligned} C\,\tilde{c}_{\uparrow}(k)\,C^{-1} &= \tilde{c}^{+}_{\downarrow}(-k) \\ C\,\tilde{c}_{\downarrow}(k)\,C^{-1} &= -\tilde{c}^{+}_{\uparrow}(-k) \end{aligned}}$$

(242)
$$\boxed{C\,\underline{\tilde{c}}(k)\,C^{-1} = -i\sigma_y\,\underline{\tilde{c}}^{+}(-k)}$$

Invariance of a Hamiltonian under charge conjugation

(243)
$$\boxed{CHC^{-1} \overset{!}{=} H}$$

(244) $$H = \sum_{\vec{k}} \left(\underline{\tilde{c}}^{+}(\vec{k})\right)^{T} \underline{\underline{h}}(\vec{k})\,\underline{\tilde{c}}(\vec{k})$$

$\underline{\underline{h}}(\vec{k})$ matrix in internal space (spin and Bloch bands)

(A) spinless particles

$$C \sum_{\vec{k}} \left(\underline{\tilde{c}}^{+}(k)\right)^{T} \underline{\underline{h}}(\vec{k})\,\underline{\tilde{c}}(\vec{k})\,C^{-1} =$$

$$= \sum_{\vec{k}} \left(C\,\tilde{c}^{+}(k)\,C^{-1}\right)^{T} C\,\underline{\underline{h}}(\vec{k})\,C^{-1}\,C\,\underline{\tilde{c}}(\vec{k})\,C^{-1}$$

$$= \sum_{\vec{k}} \left(\underline{\tilde{c}}(-k)\right)^{T} C\,\underline{\underline{h}}(\vec{k})\,C^{-1}\,\underline{\tilde{c}}^{+}(-k)$$

$$= -\sum_{\vec{k}} \left(\underline{\tilde{c}}^{+}(-k)\right)^{T} \underline{\underline{h}}^{T}(\vec{k})\,\underline{\tilde{c}}(-k)$$

anti commutator

$$= -\sum_{\vec{k}} \left(\tilde{\underline{C}}^{+}(k)\right)^{T} \underline{\underline{h}}^{T}(-\vec{k})\, \tilde{\underline{C}}(k)$$

$$\overset{!}{=} \sum_{\vec{k}} \tilde{\underline{C}}^{+}(k)^{T}\, \underline{\underline{h}}(\vec{k})\, \tilde{\underline{C}}(k)$$

$$\Rightarrow (245) \quad \boxed{\underline{\underline{h}}(\vec{k}) = -\underline{\underline{h}}^{T}(-\vec{k})}$$

(B) Spin full particles

$$(246) \quad \boxed{\underline{\underline{h}}(\vec{k}) = -\underline{\underline{\sigma}}_{y}\, \underline{\underline{h}}^{T}(-\vec{k})\, \underline{\underline{\sigma}}_{y}}$$

4.5 Combination of time-reversal and charge conjugation: Chiral transformation

Lets consider the combined trafo

$$(247) \quad \boxed{S \equiv T \circ C}$$

S is anti-unitary

$$(248) \quad S\, i\, S^{-1} = -i$$

$$(249) \quad \boxed{S^{2} = (TC)^{2} = +1} \quad \text{always } +1$$

consider spin-full fermions

$$S \underline{\hat{c}}_j S^{-1} = TC\hat{c}_j C^{-1} T^{-1}$$

$$= T(-i\sigma_y \underline{\hat{c}}^{+}_{j}) T^{-1} = -i\sigma_y T \underline{\hat{c}}^{+}_{j} T^{-1}$$

$$= -i\sigma_y i\sigma_y \underline{\hat{c}}^{+}_{j} = \underline{\hat{c}}^{+}_{j}$$

$$\boxed{\begin{aligned} S\underline{\hat{c}}_j S^{-1} &= \underline{\hat{c}}^{+}_{j} \\ S\underline{\hat{c}}(\underline{k})S^{-1} &= \underline{\hat{c}}^{+}(-\underline{k}) \end{aligned}} \tag{250}$$

Invariance of Hamiltonian under chiral transformation

$$\boxed{\underline{\underline{h}}(k) = -\underline{\underline{\sigma}}_y \underline{\underline{h}}(k) \underline{\underline{\sigma}}_y} \tag{251}$$

4.6 Summary of Fock-space transformations

(252)

unitary	U	$U\hat{c}_j U^{-1} = u_{je}\hat{c}_e$	$UiU^{-1} = i$
time reversal	T	$T\hat{c}_j T^{-1} = u^{t}_{je}\hat{c}_e$	$TiT^{-1} = -i$
charge conjugation	C	$C\hat{c}_j C^{-1} = u^{c}_{je}\hat{c}^{+}_{e}$	$CiC^{-1} = i$
chiral	$S = T\circ C$	$S\hat{c}_j S^{-1} = u^{s}_{je}\hat{c}^{+}_{e}$	$SiS^{-1} = -i$

- This is an exhaustive list.

- The unitary matrices contain in the case of spin 1/2 particle

(253) $$-i\underline{\underline{\sigma_y}}$$ in spin space

- In momentum space

$$T\,\tilde{c}(k)\,T^{-1} \qquad C\,\hat{c}(k)\,C^{-1} \qquad k \to -k$$

- There are 10 different classes under Fock-space transformations

not invariant "0"

invariant and $G^2 = +1$ "+1"

invariant and $G^2 = -1$ "-1"

(254)

T	0	+1	-1
C	0	+1	-1
S	0	+1	

$\}$ $3 \times 3 = 9$ classes

only non-trivial if $T=0$ and $C=0$ $\Rightarrow$ 1 new class

Ten-fold way

single particle Hamiltonians in momentum space are invariant if

(255)

$$\begin{array}{ll} T & \underline{\underline{h}}(k) = U_t\, \underline{\underline{h}}^*(-k)\, U_t^{-1} \\ C & \underline{\underline{h}}(k) = -U_c\, \underline{\underline{h}}^T(-k)\, U_c^{-1} \\ S & \underline{\underline{h}}(k) = -U_s\, \underline{\underline{h}}(k)\, U_s \end{array}$$

$$U_{t,c} = -i\sigma_y\, u_{t,c} \quad \text{for spin } 1/2$$

periodic table of topological insulators

(256)

Class	T	C	S	1	2	3	4
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	+	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	+	0	0	0	0	0	$\mathbb{Z}$
BDI	+	+	+	$\mathbb{Z}$	0	0	0
D	0	+	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
DIII	−	+	+	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
AII	−	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CII	−	−	+	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
C	0	−	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
CI	+	−	+	0	0	$\mathbb{Z}$	0

← intrinsic topology

symmetry protected topology

We will see that all classes with $T \neq 0$ have vanishing Chern number and must be characterized with a different topological invariant

example

(I) SSH model

$$(257) \quad \underline{\underline{h}}(k) = \begin{bmatrix} 0 & -t_1 - t_2 e^{-ik} \\ -t_1 - t_2 e^{ik} & 0 \end{bmatrix} = -(t_1 + t_2 \cos k)\sigma_x - t_2 \sin k \, \sigma_y$$

- time reversal $\quad \sigma_y^* = -\sigma_y \quad \sigma_y^T = -\sigma_y$

$$\underline{\underline{h}}^*(-k) = -(t_1 + t_2 \cos k)\sigma_x - t_2 \sin k \, \sigma_y = \underline{\underline{h}}(k)$$

$T = +1$

- charge conjugation

$$\underline{\underline{h}}^T(-k) = -(t_1 + t_2 \cos k)\sigma_x - t_2 \sin k \, \sigma_y = -\sigma_z \underline{\underline{h}}(k) \sigma_z$$

$C = +1$

- chiral transformation

$$\underline{\underline{h}}(k) = -(t_1 + t_2 \cos k)\sigma_x - t_2 \sin k \, \sigma_y = -\sigma_z \underline{\underline{h}}(k) \sigma_z$$

$S = 1$

$\boxed{\text{BDI}}$

(II) Rice-Mele model (RMM)

$$\underline{h}(k) = \begin{bmatrix} \Delta & -t_1-t_2e^{-ik} \\ -t_1-t_2e^{ik} & -\Delta \end{bmatrix} = -(t_1+t_2\cos k)\sigma_x - t_2\sin k\,\sigma_y + \Delta\sigma_z \quad (258)$$

- time reversal

$$\underline{\underline{h}}^*(-k) = \underline{\underline{h}}(k) \qquad T=+1$$

- charge conjugation

$$\underline{\underline{h}}^T(-k) \neq -\sigma_z\,\underline{\underline{h}}(k)\,\sigma_z \qquad C=0$$

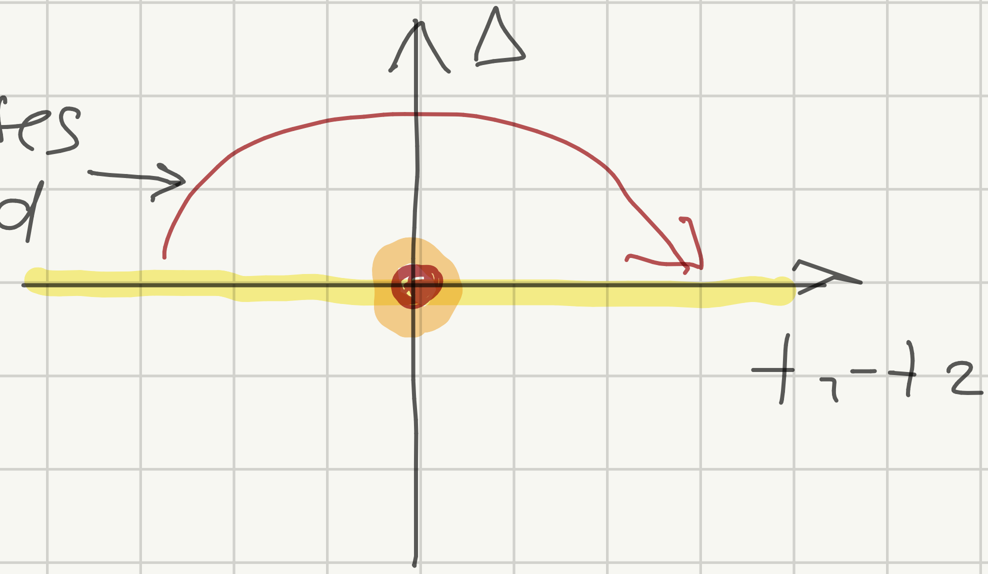
- chiral transformation

$$\underline{\underline{h}}(k) \neq -\sigma_z\,\underline{\underline{h}}(k)\,\sigma_z \qquad C=0$$

$\boxed{\text{AI}}$

not topological in 1D

topological phases can be connected without closing the gap



(III) reinstalling topology in the RMM

$$\Delta \to \Delta(t) \qquad t_1-t_2 \to (t_1-t_2)(t)$$

- time reversal

  $T = 0$

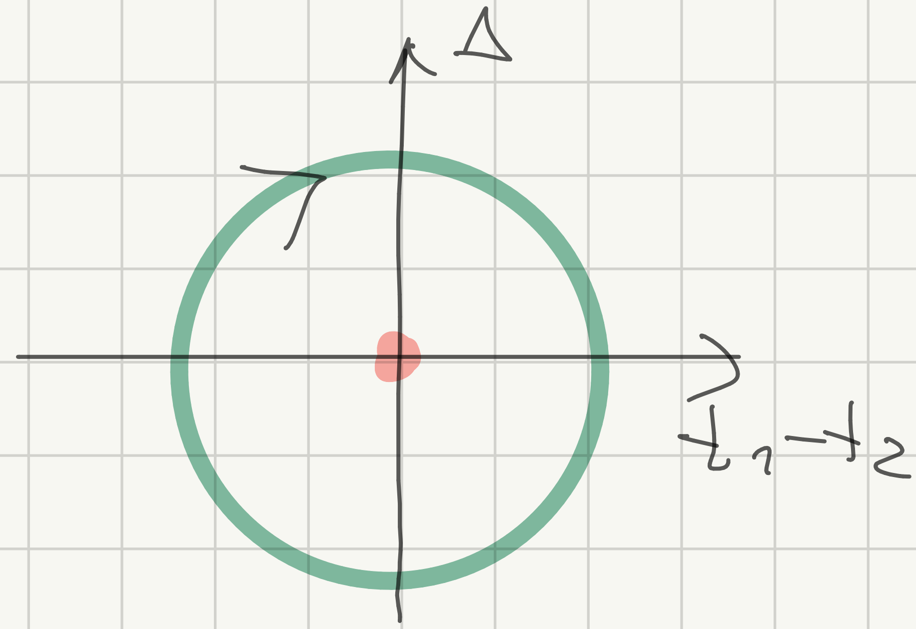
- charge conjugation

  $C = 0$

- chiral symmetry

  $S = 0$

$\Big\} \boxed{A}$



topological in 2D

$\hat{=} \ (k_x, t)$

(IV) Qi-Wu-Zhang model

(259) $$\underline{h}(\vec{k}) = \sin k_x \sigma_x + \sin k_y \sigma_y + [u + \cos k_x + \cos k_y]\sigma_z$$

- time reversal

$$\underline{h}(-\vec{k}) = -\sin k_x \sigma_x - \sin k_y \sigma_y + [u + \cos k_x + \cos k_y]\sigma_z$$

$$\underline{h}^*(\vec{k}) = +\sin k_x \sigma_x - \sin k_y \sigma_y + [u + \cos k_x + \cos k_y]\sigma_z$$

$T = 0$

- charge conjugation

$$\underline{h}^T(-\vec{k}) = -\sin k_x \sigma_x + \sin k_y \sigma_y + [u + \cos k_x + \cos k_y]\sigma_z$$

$C = 0$

- chiral symmetry

$$\sigma_x \underline{h}(\vec{k})\sigma_x = \sin k_x \sigma_x - \sin k_y \sigma_y - [u + \cos k_x + \cos k_y]\sigma_z$$

$$\sigma_y \underline{h}(\vec{k})\sigma_y = -\sin k_x \sigma_x + \sin k_y \sigma_y - [u + \cos k_x + \cos k_y]\sigma_z$$

$$\sigma_z \underline{h}(\vec{k})\sigma_z = -\sin k_x \sigma_x - \sin k_y \sigma_y + [u + \cos k_x + \cos k_y]\sigma_z$$

$S = 0$ $\boxed{A}$ topological in 2D