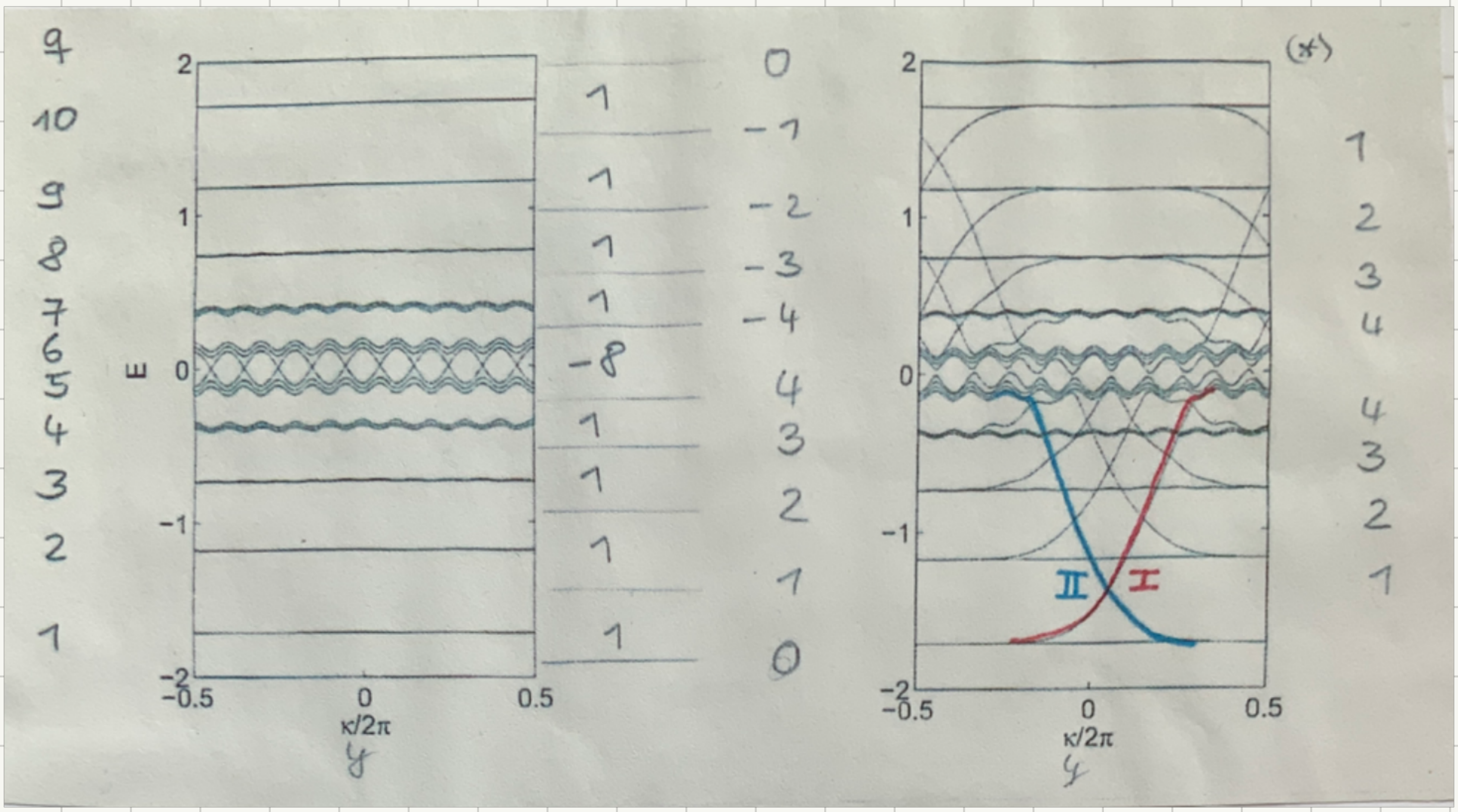
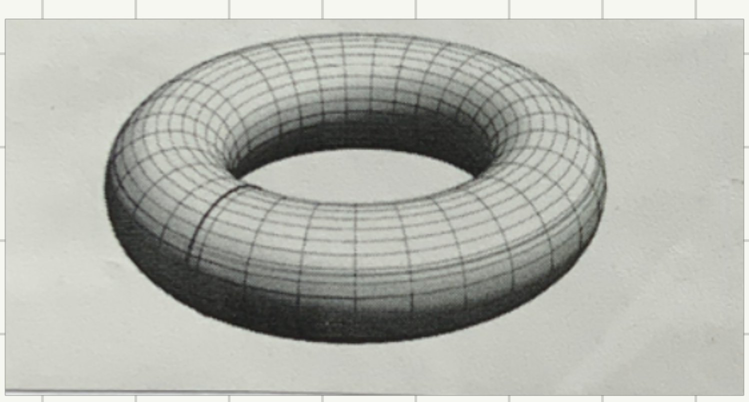
3.3 Edge modes and bulk-boundary correspondence

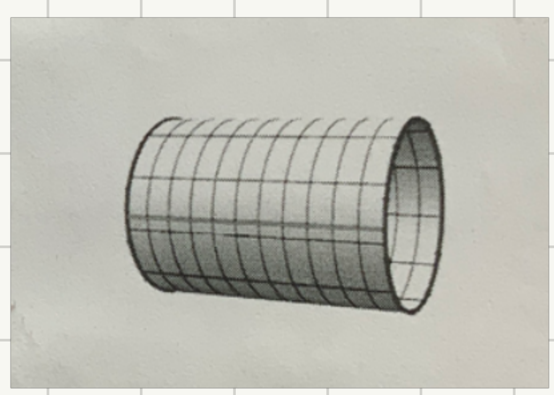
We have established the connection between the quantized Hall conductivity σ_{xy} and the Chern number in a QH system or Chern insulator. But what is the origin of the (almost) vanishing longitudinal resistivity? The system is an insulator after all!

(A) Some numerical findings

periodic BC   open BC







- isolated modes in the case of open boundary conditions. These are edge modes similar to those we have encountered in the Qi-Wu-Zhang model.

- There are two types of modes

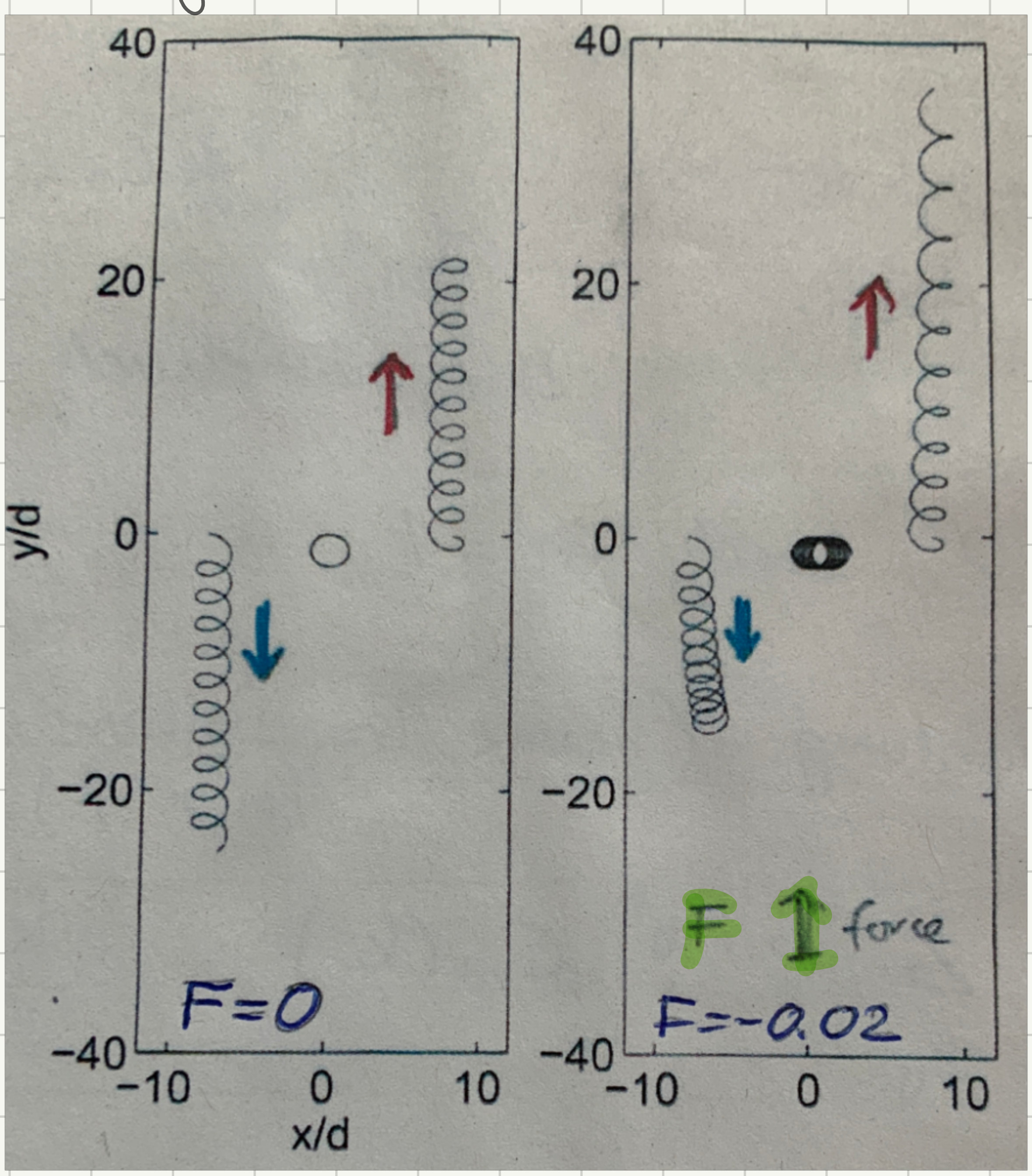
type I $\quad \frac{d\omega}{dk} > 0 \quad$ transport in "+" direction

type II $\quad \frac{d\omega}{dk} < 0 \quad$ transport in "−" direction

Chiral edge modes

- Level crossings are true level crossings!
  In finite systems they are avoided but with exponentially small gap

single particle trajectories under a force $\vec{F}$ parallel to the strip



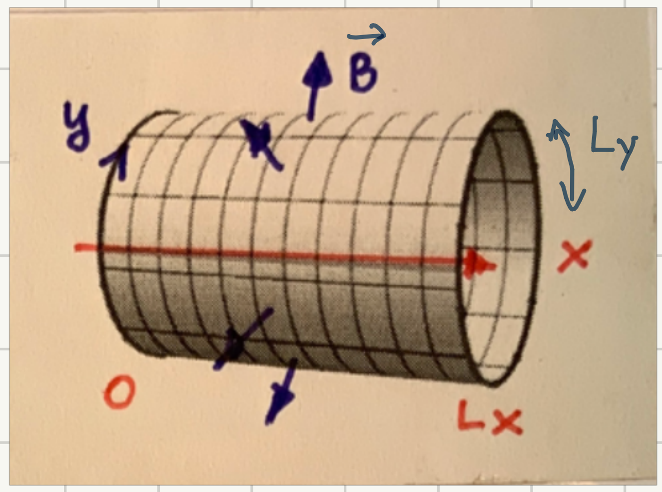
Without a proof:

$$\boxed{\text{The total number of edge modes (per direction) is equal to the absolute value of the sum of Chern numbers of all bands below the considered gap}}$$

bulk-boundary correspondence

(B) How to calculate the spectrum with open boundaries: transfer matrix method

Let us consider a cylinder geometry, i.e. periodic in y but with open boundaries in x direction in a magnetic field with flux per plaquette α



(178) $$H = -t_x \sum_{m,n} c^\dagger_{m+1,n} c_{mn} - t_y \sum_{m,n} c^\dagger_{mn+1} c_{mn} e^{2\pi i \alpha m} + h.c.$$

edges are at $m = 0$ and $m = L_x$

F-trafo with respect to y (periodicity)

$n_y \in \mathbb{Z}$,

(179) $$c_{mn} = \frac{1}{L_y} \sum_{k_y} e^{ik_y n} \tilde{c}_m(k_y) \qquad k_y = \frac{2\pi}{L_y} n_y$$

$\Rightarrow$

(180) $$H = -t_x \sum_{m,k_y} \tilde{c}^\dagger_{m+1}(k_y)\tilde{c}_m(k_y) - t_y \sum_{m,k_y} \tilde{c}^\dagger_m(k_y)\tilde{c}_m(k_y) e^{-ik_y + 2\pi i \alpha m}$$

Now we want to find the solution of the stationary Schrödinger eq.

$$H|\psi\rangle = E|\psi\rangle$$

for a single-particle state

$$(181) \qquad |\psi\rangle = \sum_{m,k_y} \psi_m(k_y)\, \tilde{c}_m^\dagger(k_y)\, |0\rangle$$

inserting into the SE yields for each k_y

$$(182) \qquad -t_x(\psi_{m+1} + \psi_{m-1}) - 2t_y \cos(k_y - 2\pi\alpha m)\psi_m = E\,\psi_m$$

This can be written in the form of a matrix recursion

$$(183) \qquad \boxed{\begin{pmatrix} \psi_{m+1}(k_y,\varepsilon) \\ \psi_m(k_y,\varepsilon) \end{pmatrix} = \underline{\underline{M}}_m(k_y,\varepsilon) \begin{pmatrix} \psi_m(k_y,\varepsilon) \\ \psi_{m-1}(k_y,\varepsilon) \end{pmatrix}}$$

where $\varepsilon = E/t_x$ and

$$(184) \qquad \boxed{\underline{\underline{M}}_m = \begin{pmatrix} -\varepsilon - \frac{2t_y}{t_x}\cos(k_y - 2\pi\alpha m) & -1 \\ -1 & 0 \end{pmatrix}}$$

m'th transfer matrix

- assume rational $\alpha = P/q$ and L_x commensurate with magnetic unit cell, i.e.

$$L_x = q\,\ell \qquad \ell \in \mathbb{N}$$

$\Rightarrow$ transfer matrices periodic in m with period q

$\Rightarrow$ total transfer matrix over q sites

(185) $$\underline{\underline{M}}(k_y, \varepsilon) = \underline{\underline{M}}_q(\varepsilon)\, \underline{\underline{M}}_{q-1}(\varepsilon) \cdots \underline{\underline{M}}_1(\varepsilon)$$

(we dropped k_y for simplicity) $\underline{\underline{M}}(\varepsilon)$ is 2×2

$$\underline{\underline{M}}(\varepsilon) = \begin{pmatrix} M_{11}(\varepsilon) & M_{12}(\varepsilon) \\ M_{21}(\varepsilon) & M_{22}(\varepsilon) \end{pmatrix}$$

- From eq. (184) we see:
  - $M_{11}(\varepsilon)$ polynom in ε of degree q
  - $M_{12}(\varepsilon), M_{21}(\varepsilon)$ polynom in ε of degree $q-1$
  - $M_{22}(\varepsilon)$ polynom in ε of degree $q-2$

since $L_x = q\ell \Rightarrow$ from (183)

(186) $$\begin{pmatrix} \psi_{L_x+1} \\ \psi_{L_x} \end{pmatrix} = \left(\underline{\underline{M}}(\varepsilon)\right)^{\ell} \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix}$$

bulk modes

open boundary conditions imply (ψ_{L_x+1} is irrelevant)

(187) $$\psi_{L_x} = \psi_0 \overset{!}{=} 0 \qquad \begin{pmatrix} \psi_{L_x+1} \\ 0 \end{pmatrix} \overset{!}{=} \underline{\underline{M}}^{\ell} \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix}$$

$\Rightarrow$

(188) $$\boxed{\left[\left(\underline{\underline{M}}(\varepsilon)\right)^{\ell}\right]_{21} \overset{!}{=} 0}$$ for every k_y

$\left(\underline{\underline{M}}^{\ell}\right)_{21}$ is a polynom of degree $q\ell-1 = L_x - 1$

$\Rightarrow$ $L_x - 1$ real roots $\mathcal{E}_n = \mathcal{E}_n(k_y)$ which are the eigenvalues of the problem with open BC

(C) Edge modes

Let us construct the edge modes. Eq. (188) can be fulfilled if we choose

(189) $$\underline{\underline{M}}_{21}(\mathcal{E}) = 0$$ (note $\underline{\underline{M}}$ not $\underline{\underline{M}}^{\ell}$)

Since a product of triangular matrices is again a triangular matrix. $\underline{\underline{M}}_{21}(\mathcal{E})$ is a polynomial in $\mathcal{E}$ of degree $q-1$ i.e. (189) has $q-1$ solutions

(190) $\mathcal{E}_j \quad j = 1, 2, \dots, q-1 \qquad$ set $\quad \mathcal{E}_i < \mathcal{E}_j \quad$ for $i < j$

One can show that (189) does not have degenerate solutions i.e. all $\mathcal{E}_j$ are different for a fixed value of k_y.

- Classification of the $q-1$ solutions

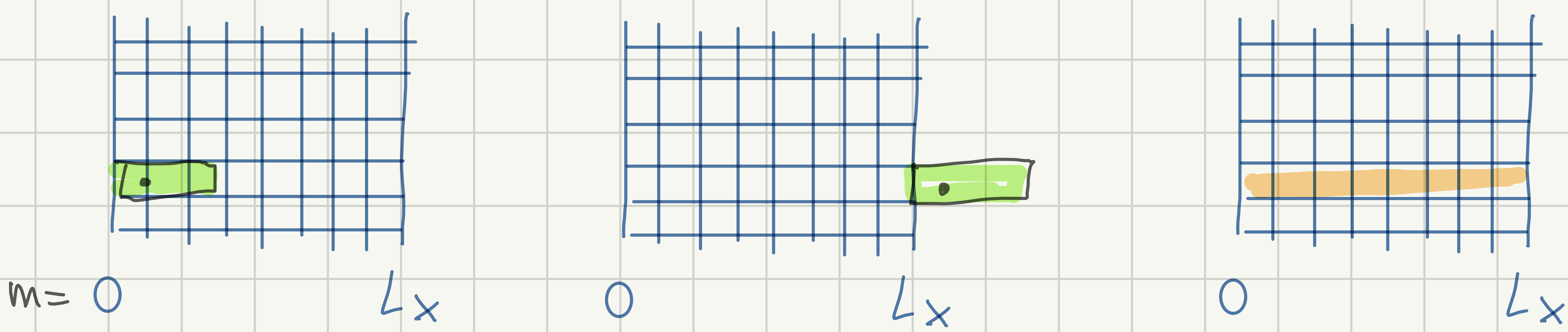
from (186) $(L_x = q\ell)$ $\quad \ell' < \ell$

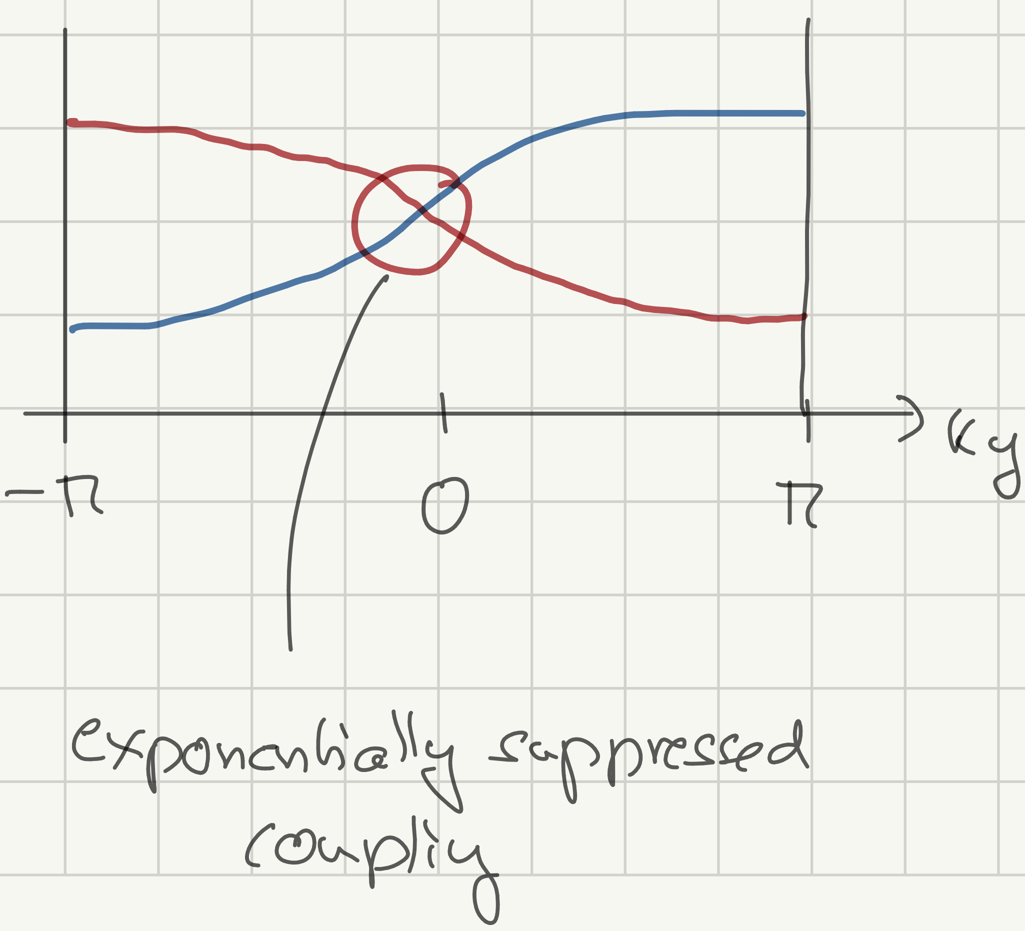
$$\begin{pmatrix} \psi_{q\ell'+1}(\mathcal{E}_j) \\ \psi_{q\ell'}(\mathcal{E}_j) \end{pmatrix} = \begin{pmatrix} M_{11}(\mathcal{E}_j)^{\ell'} & B \\ 0 & C \end{pmatrix} \begin{pmatrix} \psi_1(\mathcal{E}_j) \\ 0 \end{pmatrix}$$

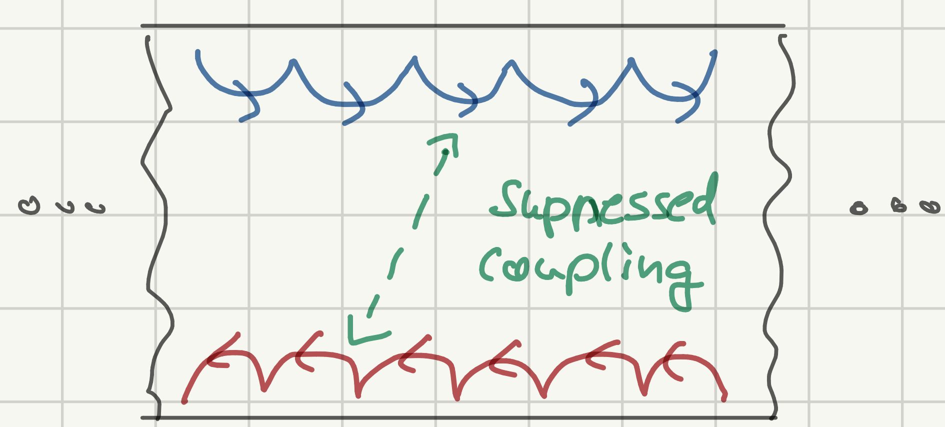
$\Rightarrow$

$$\frac{\psi_{ql'+1}}{\psi_1} = M_{11}(\varepsilon_j)^{l'}$$

$\boxed{\lvert M_{11}(\varepsilon_j)\rvert < 1}$	$\boxed{\lvert M_{11}(\varepsilon_j)\rvert > 1}$	$\boxed{\lvert M_{11}(\varepsilon_j)\rvert = 1}$
mode localized at magnetic unit cell containing $m=1$	mode localized at magnetic unit cell containing $m = ql+1 = L_x+1$	delocalized mode
edge mode on the left	edge mode on the right	bulk mode







$$\Rightarrow \boxed{\sigma_{xx} = 0} \qquad (191)$$