3. Chern insulators

3.1 The Harper Hofstadter model

The Harper-Hofstadter model describes electrons on a rectangular lattice in a constant transverse magnetic field. So we first have to discuss lattice models in the presence of a magnetic field

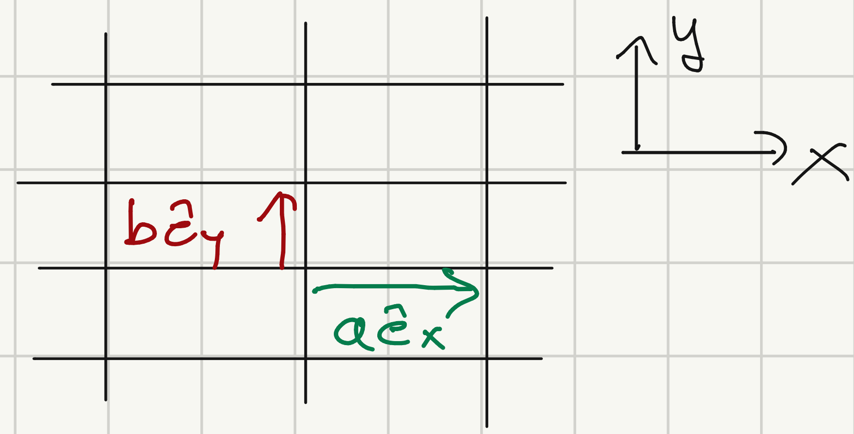
(A) Bloch states in a magnetic field

- without magnetic field

(135) $$V(\vec{r}) = V(\vec{r}+\vec{R})$$

(136) $$\vec{R} = n a \hat{e}_x + m b \hat{e}_y$$

$$n, m \in \mathbb{Z}$$



lattice constants a, b

the Hamiltonian is invariant under discrete lattice translations

(137) $$T_{\vec{R}} = e^{\frac{i}{\hbar}\vec{R}\vec{\hat{P}}} \qquad [T_{\vec{R}}, H] = 0$$

$$T_{\vec{R}_1} T_{\vec{R}_2} = T_{\vec{R}_2} T_{\vec{R}_1} = T_{\vec{R}_1+\vec{R}_2}$$

the eigenstates of H are Bloch functions

(138) $$\psi_{\vec{b}}(\vec{r}) = e^{i\vec{b}\cdot\vec{r}} u_{\vec{b}}(\vec{r}) \qquad u_{\vec{b}}(\vec{r}+\vec{R}) = u_{\vec{b}}(\vec{r})$$

lattice momenta in 1. BZ

$$-\frac{\pi}{a} \leq k_x \leq \frac{\pi}{a} \qquad -\frac{\pi}{b} \leq k_y \leq \frac{\pi}{b} \tag{139}$$

- now with magnetic field

$\boxed{\vec{A}(\vec{r}) \text{ is not translational invariant !}}$

e.g. $\vec{A}(\vec{r}) = -By\vec{e}_x$

but

$$T_{\vec{R}}\vec{A}(\vec{r}) = \vec{A}(\vec{r}+\vec{R}) = \vec{A}(\vec{r}) + \delta\vec{A}(\vec{r}) \tag{140}$$

$$\delta\vec{A}(\vec{r}) = \vec{D}\,\xi_{\vec{R}}(\vec{r})$$ → consider as gauge trafo !

Hamiltonian may be invariant under combined translation and gauge transformation

$$\boxed{\bar{T}_{\vec{R}} = e^{i\frac{e}{\hbar c}\xi_{\vec{R}}(\vec{r})}\,T_{\vec{R}}} \tag{141}$$ magnetic translation

but magnetic translations are in general not Abelian

$$\text{in general} \qquad [\bar{T}_{\vec{R}_1}, \bar{T}_{\vec{R}_2}] \neq 0 \tag{142}$$ ☹

(B) Hofstadter Butterfly ; magnetic unit cell

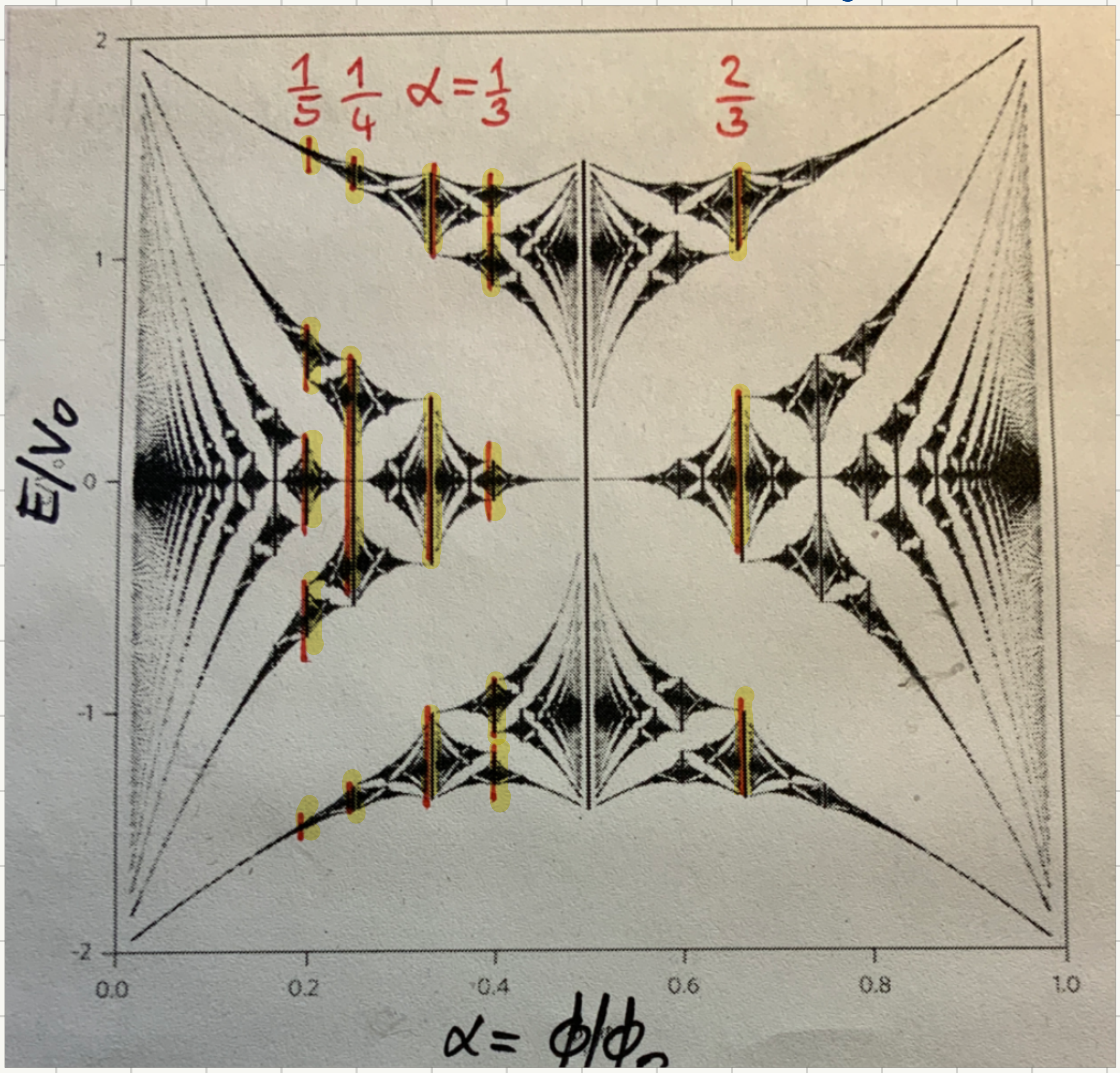
Due to (142) there is in general no Bloch theorem and the spectrum of the Harper-Hofstadter model is in general **fractal**

Hofstadter butterfly

Φ: flux per plaquette

(143) $$\alpha \equiv \frac{\Phi}{\Phi_0}$$

D. R. Hofstadter
Phys. Rev. B 14
2239 (1976)



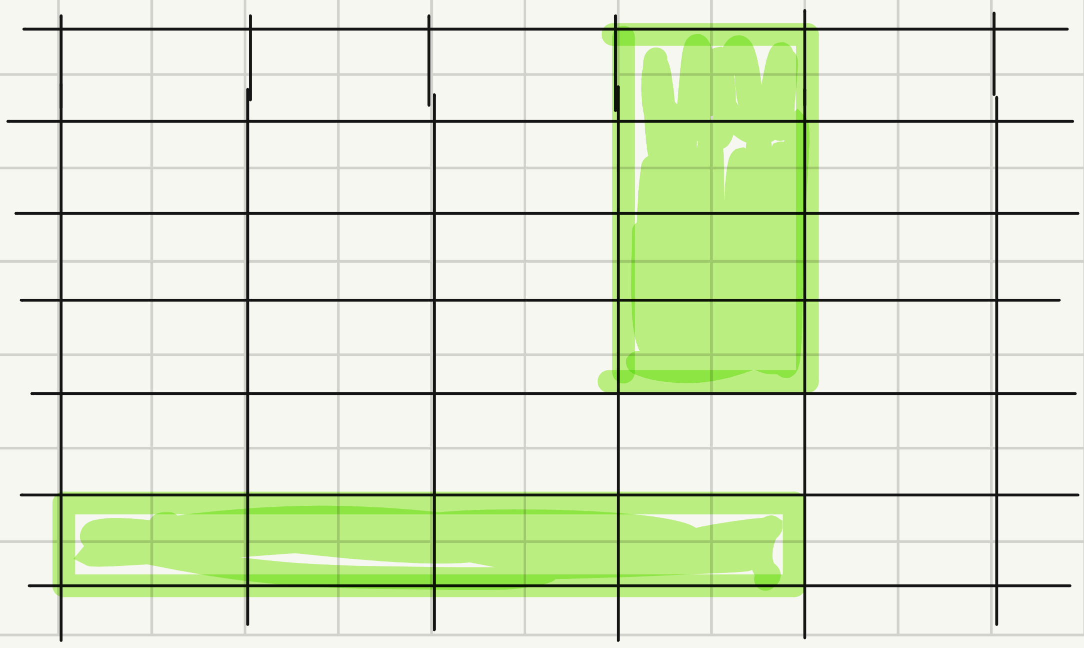
However at rational values of α the spectrum seems to have **bands**

$$\Rightarrow \quad \boxed{\alpha = \frac{\Phi}{\Phi_0} = \frac{p}{q}} \qquad (144)$$

p, q relative prime

introduce **magnetic unit cell** consisting of q normal unit cells

$q=4$



magnetic unit cell has flux

$$\boxed{\Phi_{tot} = p\,\Phi_0}$$

integer multiple of Φ_0

So let's define a magnetic lattice vector

(145) $$\boxed{\vec{R}_m = n q a \vec{e}_x + m b \vec{e}_y} \qquad (\alpha = p/q)$$

The corresponding magnetic translation operators now form an Abelian group! ☺

Choose Landau gauge $\vec{A} = -By\vec{e}_x$

(146) $$\overline{T}_{\vec{R}} = e^{-\frac{i}{\ell_0^2} Yx}\, T_{\vec{R}} = \exp\left(-\frac{i}{\ell_0^2} Yx\right) \exp\left(\frac{i}{\hbar} \vec{R}_m \cdot \vec{p}\right)$$

using Baker Hausdorff one easily shows

(147) $$\boxed{\overline{T}_{\vec{R}_1}\, \overline{T}_{\vec{R}_2} = \overline{T}_{\vec{R}_1 + \vec{R}_2} = \overline{T}_{\vec{R}_2}\, \overline{T}_{\vec{R}_1}}$$

where $\vec{R}_1, \vec{R}_2$ are magnetic lattice vectors!

- Since the unit cell is now q times enlarged (here in x direction) the Brillouin zone (in k_x) is reduced

$\Longrightarrow$ q magnetic subbands

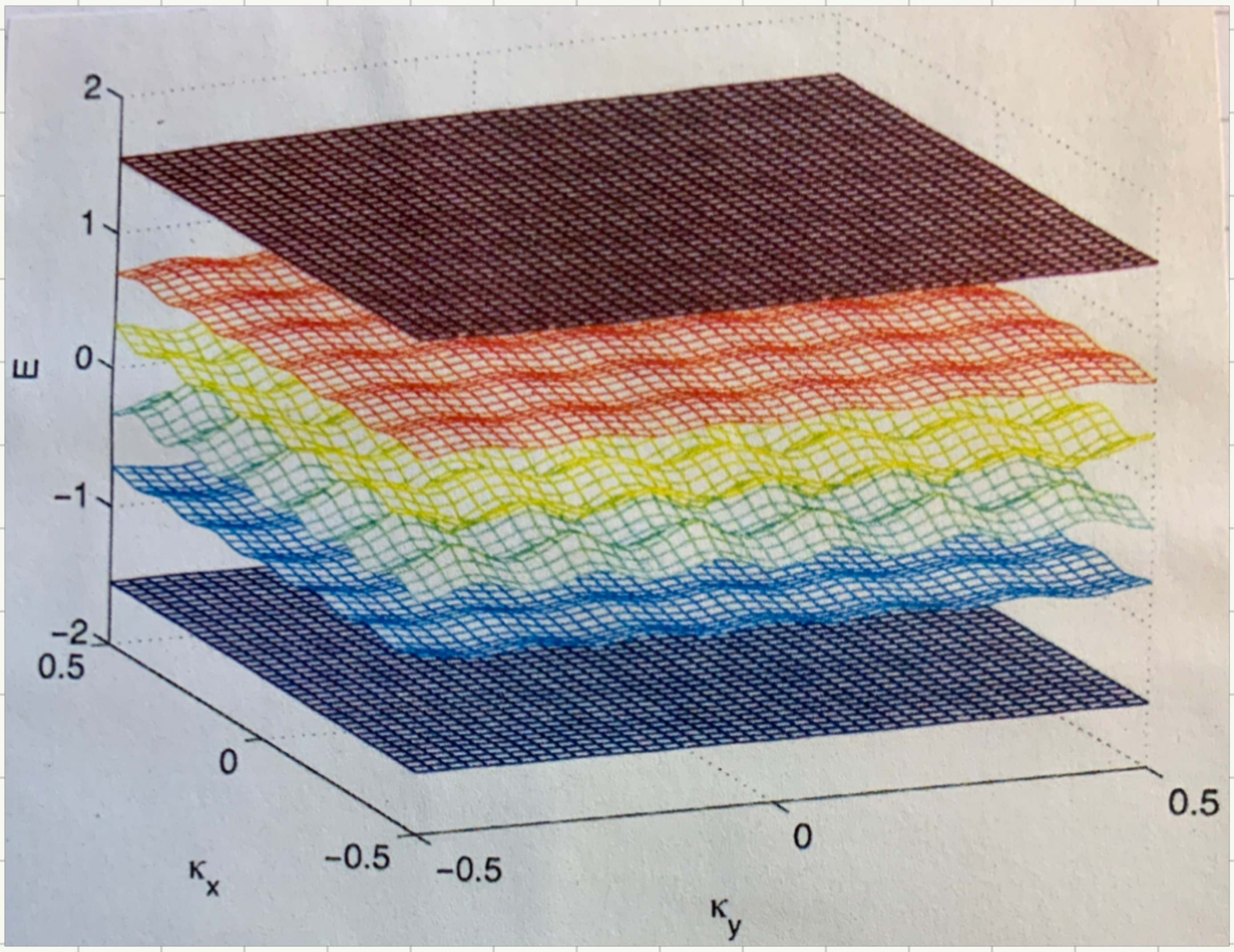
(148) $$\boxed{-\frac{1}{q}\frac{\pi}{a} \leq k_x \leq \frac{1}{q}\frac{\pi}{a}} \qquad \boxed{-\frac{\pi}{b} \leq k_y \leq \frac{\pi}{b}}$$

Rem.: One could have chosen the y direction instead of x

E.g. if the lattice without a magnetic field has a unit cell of one site and we add a magnetic field with $q=6$ we get 6 magnetic subbands

6 subbands

note the 6-fold periodicity in both k_x and k_y



(C) Harper equation and calculation of the spectrum of the H-H model

We want to diagonalize the Harper-Hofstadter Hamiltonian with rational flux per plaquette

$$H = -t \sum_{m,n} \left(c^{+}_{m+1\,n} c_{mn} + c^{+}_{m\,n+1} c_{mn} \underset{\uparrow}{e^{2\pi i \frac{p}{q} m}} + h.a. \right)$$

(149) Peierls phase

Fourier transformation with respect to original unit cell does not give diagonal hamiltonian. We have to use the magnetic unit cell as proper unit cell. (now $a=b=1$)

$$(150)\quad C_{m,n} = \frac{1}{(2\pi)^2} \int_{-\pi/q}^{\pi/q} dk_x \int_{-\pi}^{\pi} dk_y \; e^{ik_x m} e^{ik_y n} \, \tilde{c}_{k_x k_y}$$

$$\Rightarrow$$

$$(151)\quad H = \frac{1}{(2\pi)} \int_{-\pi/q}^{\pi/q} dk_x \int_{-\pi}^{\pi} dk_y \underbrace{\sum_{\nu=0}^{q-1}}_{\text{magnetic sub-bands}} H^{\nu}_{k_x k_y}$$

where

$$(152)\quad H^{\nu}_{k_x k_y} = -t \Big[2\cos\left(k_x + 2\pi \tfrac{p}{q}\nu\right) \tilde{c}^{+}_{k_x + 2\pi\frac{p}{q}\nu, k_y} \tilde{c}_{k_x + 2\pi\frac{p}{q}\nu, k_y} + e^{-ik_y} \tilde{c}^{+}_{k_x + 2\pi\frac{p}{q}(\nu+1), k_y} \tilde{c}_{k_x + 2\pi\frac{p}{q}\nu, k_y} + e^{ik_y} \tilde{c}^{+}_{k_x + 2\pi\frac{p}{q}(\nu-1), k_y} \tilde{c}_{k_x + 2\pi\frac{p}{q}\nu, k_y} \Big]$$

Obviously $\sum_{\nu=0}^{q-1} H^{\nu}_{k_x k_y}$ is still not diagonal in k_x, but it corresponds to a 1D lattice Hamiltonian along "k_x" wit q lattice sites and "hopping"

$$k_x + 2\pi \frac{p}{q}\nu \quad \underset{te^{ik_y}}{\overset{te^{-ik_y}}{\rightleftarrows}} \quad k_x + 2\pi \frac{p}{q}(\nu+1)$$

and periodic boundary conditions. This can be solved like any 1D tight binding Hamiltonian,

$$(153)\quad H_{k_x k_y} |\psi\rangle = \sum_{\nu=0}^{q-1} H^{\nu}_{k_x k_y} |\psi\rangle = E_{k_x k_y} |\psi\rangle$$

Ansatz:

$$|\psi\rangle = \sum_{\nu=0}^{q-1} \alpha_\nu \, \tilde{c}^{+}_{k_x + 2\pi \frac{p}{q}\nu,\, k_y} |0\rangle$$

$\Rightarrow \dots \Rightarrow$ Harper equations

(154)
$$\boxed{-2\cos\left(k_x + 2\pi \frac{p}{q}\nu\right)\alpha_\nu - e^{-ik_y}\alpha_{\nu-1} - e^{ik_y}\alpha_{\nu+1} = \frac{1}{t} E_{k_x k_y} \alpha_\nu}$$

with boundary condition

(155)
$$\alpha_{\nu+q} = \alpha_\nu$$

The exponential terms $e^{\pm ik_y}$ can be removed by $\alpha_j = \beta_j e^{-ik_y j}$. This gives

(156)
$$\boxed{-\beta_{\nu-1} - \beta_{\nu+1} - 2\cos\left(k_x + 2\pi \frac{p}{q}\nu\right)\beta_\nu = \frac{1}{t} E_{k_x k_y} \beta_\nu}$$

with $\beta_{\nu+q} = \beta_\nu e^{ik_y q}$. Note k_y disappeared and is hidden in the boundary conditions $\Rightarrow$

(157)
$$\boxed{E(k_x, k_y) = E\left(k_x, k_y + \frac{2\pi}{q} n\right)} \qquad n = 1, \dots, q-1$$

$\Rightarrow$ q-fold periodicity of spectrum within the normal Brillouin zone with respect to k_y

Some general properties of solutions of Harper equations :

- for even q there are solutions with $E=0$

▶ $q=4n$ $E=0$ at $k_y=\frac{2m}{q}\pi$ $k_x=0$

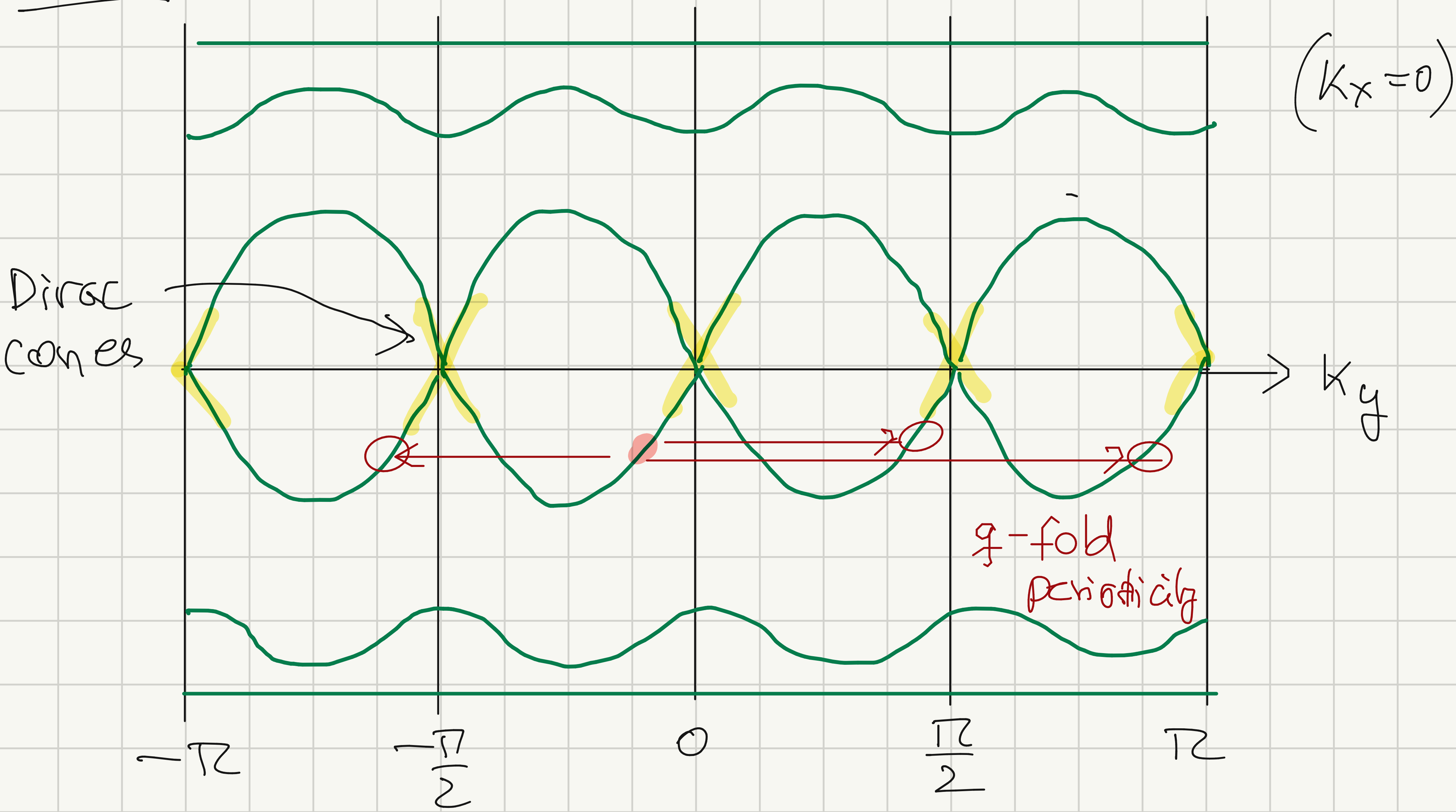
▶ $q=4n+2$ —"— $k_y=\frac{(2m+1)}{q}\pi$ $k_x=\pm\frac{\pi}{q}$

Close to these points the dispersion reads

$$\boxed{E \simeq \pm \gamma_q t^{q/2}|\vec{k}| = \pm \gamma_q t^{q/2}\sqrt{k_x^2+k_y^2}} \tag{158}$$

i.e. Dirac cones

$q=4$



In the chosen gauge ($\hat{=}$ choice of magnetic unit cell)

$$-\frac{\pi}{q} \leq k_x \leq \frac{\pi}{q} \qquad -\pi \leq k_y \leq \pi$$

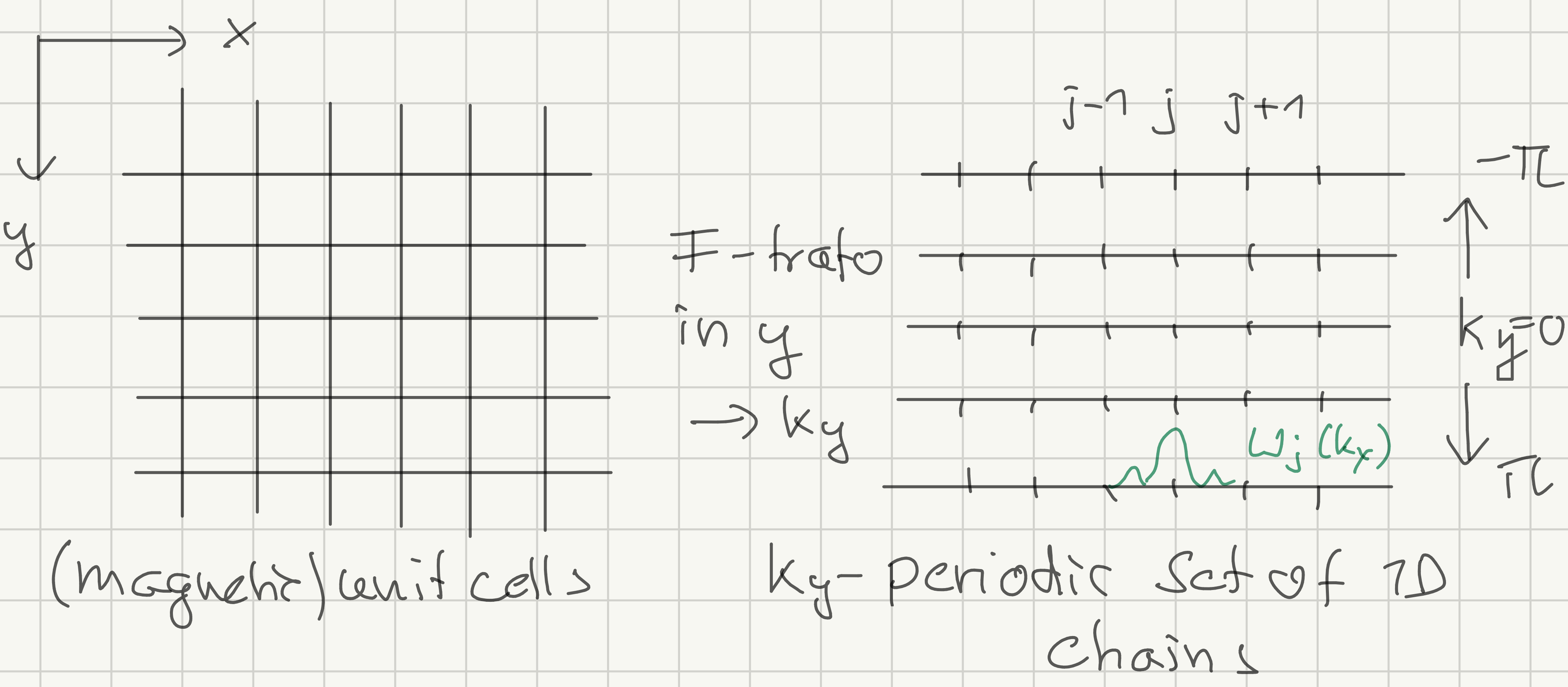
i.e. also periodicity in k_x !

3.2 Hall conductivity and Chern number

Now we want to argue that electrons hopping on a 2D lattice subject to an external magnetic field with commensurate flux per plaquette can have a quantized Hall conductivity determined by a topological invariant, the Chern number

(A) Hall conductivity

Harper-Hofstadter model $\alpha = p/q$



separate Hamiltonian $h(k_y)$

Center of mass of Wannier center $\omega_j^\nu(k_y)$ in x direction for every value of k_y (polarization)

$$(159) \quad \boxed{\begin{aligned} P_\nu(k_y) &= \langle \omega_j^\nu(k_y) | \hat{x} | \omega_j^\nu(k_y) \rangle \\ &= \frac{1}{2\pi} \phi_\nu^{2Ak}(k_y) \end{aligned}}$$

Consider a filled magnetic subband (insulator) and apply a small electric field in the y-direction

$$(160) \quad \frac{dk_y}{dt} = E_y \qquad \vec{E} = E_y \vec{e}_y$$

If $\vec{E}$ is small enough not to close insulating many-body gap $\rightarrow$ adiabatic following

$$|\omega_j^\nu(k_y)\rangle \longrightarrow |\omega_j^\nu(k_y(t))\rangle$$

Hall conductivity

$$(161) \quad \sigma_{xy} = \frac{j_x}{E_y} \qquad (\text{charge } e \stackrel{!}{=} 1)$$

$$(162) \quad j_x = \int_{BZ} dk_y \frac{d P_\nu(k_y(t))}{dt}$$

contribution from all electrons occupying the magnetic subband

$$\sigma_{xy} = \int_{BZ} dk_y \frac{dP_\nu}{dt} \cdot \frac{dt}{dk_y} = \int_{BZ} dk_y \frac{\partial P_\nu(k_y)}{\partial k_y}$$

(163)
$$= \frac{1}{2\pi} \oint_{BZ} dk_y \frac{\partial \phi_\nu^{ZAK}(k_y)}{\partial k_y}$$

$$= \frac{1}{2\pi} \oiint dk_x dk_y \, \underbrace{i\left[\langle \partial_{k_y} u^\nu | \partial_{k_x} u^\nu \rangle - \langle \partial_{k_x} u^\nu | \partial_{k_y} u^\nu \rangle\right]}_{\text{Berry curvature}}$$

$$= C \qquad \text{Chern number}$$

$\Rightarrow$ quantization of Hall conductivity in insulating bulk state

(B) Streda formula

To find the Chern number of a particular magnetic sub-band of the Harper Hofstadter model one could just numerically integrate the Berry curvature. It turns out however that there is an alternative way to determine C_ν via the Streda formula that connects the Hall conductivity with the change of the electron density with magnetic field while keeping the Fermi energy constant

- Current density in 2D electron gas

(164) $$j_k = -\sigma_{xy}\, \varepsilon_{kl} E_l$$ ε_{kl} 2dim Levi Civita

continuity equation

$$\frac{\partial}{\partial t}\rho = -\vec{\nabla}\cdot\vec{j} = -(\partial_x j_x + \partial_y j_y)$$

$$= +\sigma_{xy}(\partial_x E_y - \partial_y E_x) = \sigma_{xy}(\vec{\nabla}\times\vec{E})_z$$

with Maxwell eqs

$$\vec{\nabla}\times\vec{E} = \frac{1}{c}\frac{\partial\vec{B}}{\partial t} \Rightarrow \frac{\partial\rho}{\partial t} = \sigma_{xy}\frac{1}{c}\frac{\partial B}{\partial t}$$

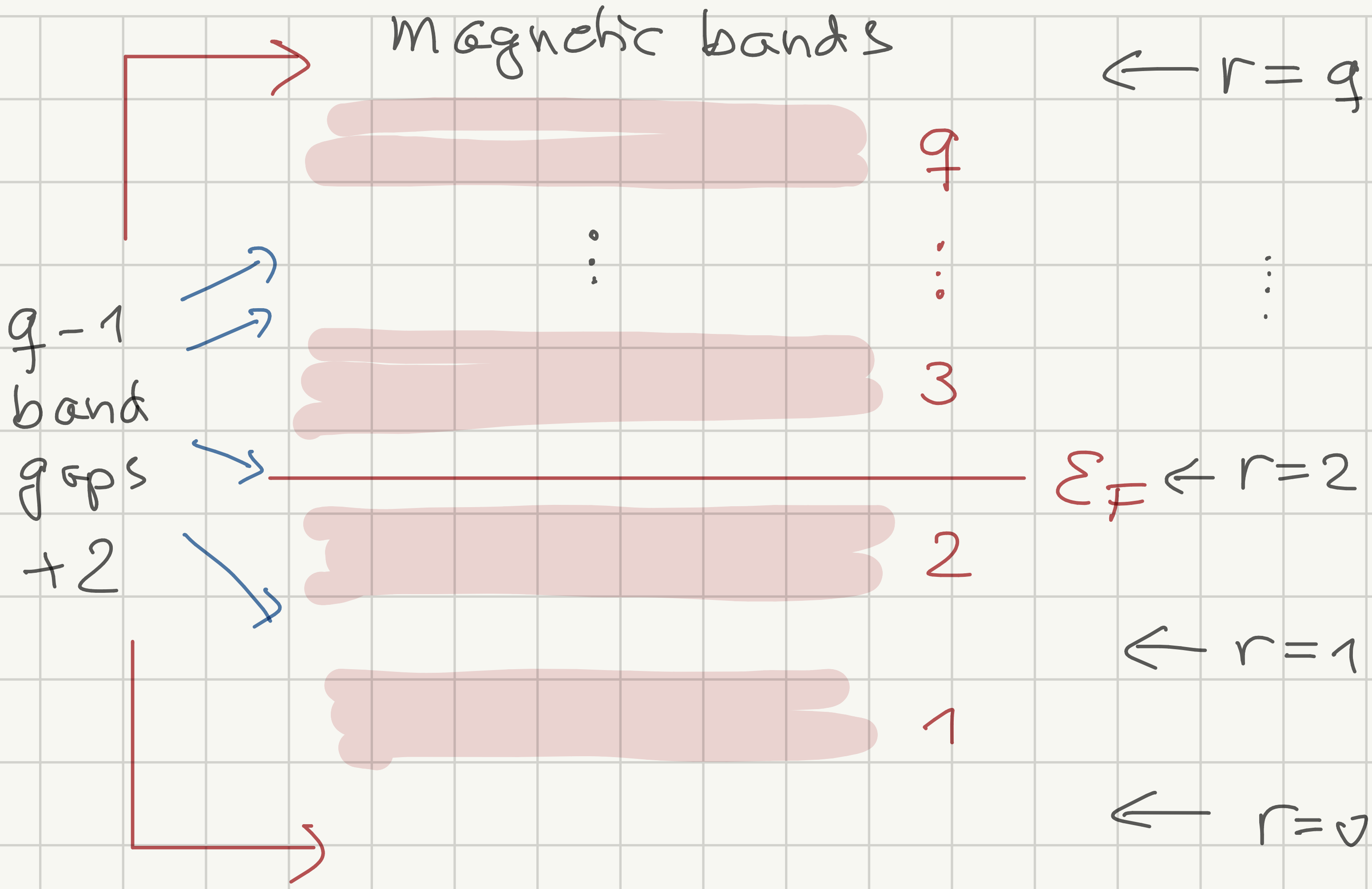
(165) $$\boxed{\sigma_{xy} = c\,\frac{\partial\rho}{\partial B}}$$ Streda formula (c: speed of light)

Changing B changes the number of available states below the Fermi energy. So if we keep the Fermi energy constant additional electrons must flow into the sample

- Now assume flux per plaquette

(166) $$\Phi = \frac{p}{q}\Phi_0 \Rightarrow q \text{ subbands}$$

ε_F in the r'th gap between subbands



$r, p, q \in \mathbb{N}$

$0 \leq r \leq q$

One can easily see that there are integer numbers s_r and t_r such that

(167) $$r = q s_r + p t_r \qquad |t_r| \leq \frac{q}{2}$$

Now consider a finite lattice with M lattice sites. Then the number of occupied lattice sites is $N = r/q\, M$ (each magn. subband has the same number of states)

(168) $$\rho = e\frac{N}{A} = e\frac{r}{q}\frac{M}{A} = \frac{er}{qA_0}$$

where A_0 is area of unit cell. With the Streda formula

$$\sigma_{xy} = c\frac{\partial \rho}{\partial B} = ec\frac{1}{A_0}\frac{\partial}{\partial B}\left(s_r + \frac{p}{q}t_r\right)$$

Since $\frac{p}{q} = \frac{\Phi}{\Phi_0} = \frac{BA_0}{\Phi_0}$ only $\frac{\partial}{\partial B}\frac{p}{q} = \frac{A_0}{\Phi_0} \neq 0$

This finally gives the quantization of σ_{xy}

(169) $$\boxed{\sigma_{xy} = \frac{ec}{\Phi_0} t_r = \frac{e^2}{h} t_r}$$ t_r integer $|t_r| \leq \frac{q}{2}$

Darboux Theorem

(i) $\boxed{q \text{ even}}$ lets assume $p=1$ for simplicity i.e.

$$0 \leq r = q s_r + t_r \leq q$$

- $r < \frac{q}{2} \Rightarrow s_r = 0 \quad t_r = r$

contribution of each band to total Hall conducting (in units of e^2/h)

(170) $$\boxed{\Delta t_r = t_{r+1} - t_r = 1}$$ increase by 1

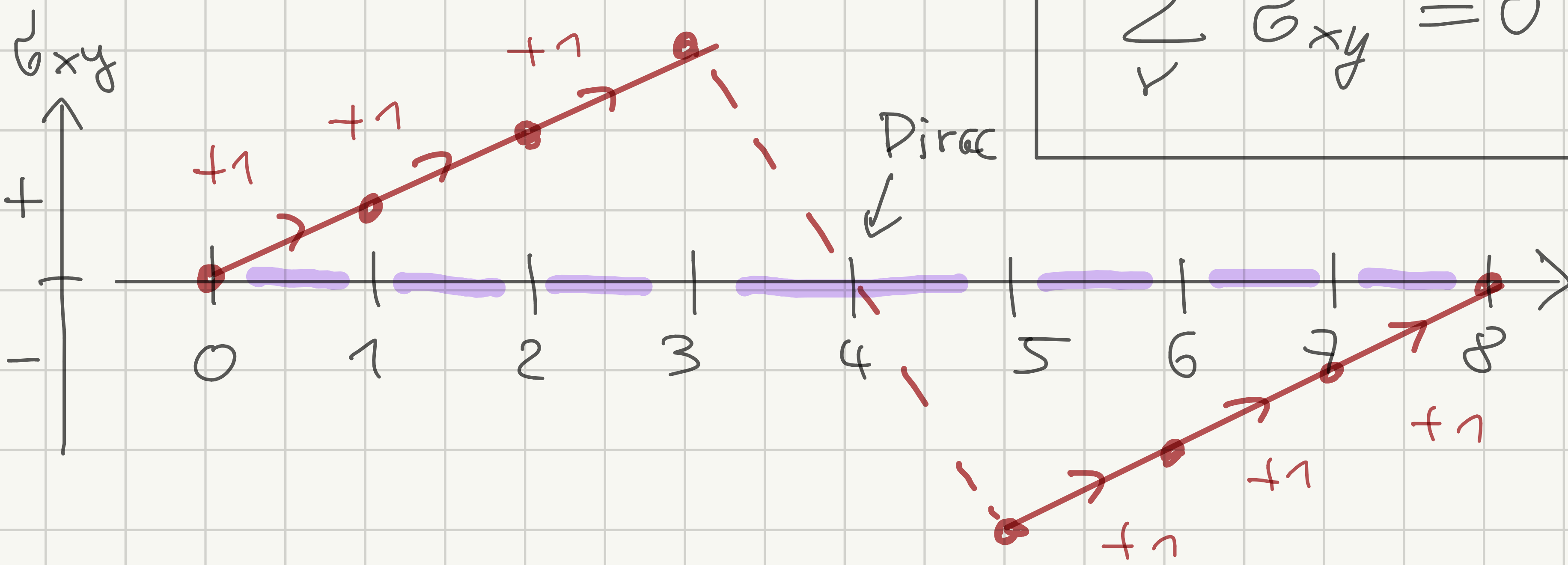
- $r > \frac{q}{2} \Rightarrow s_r = 1 \quad t_r = r - q$

(171) $$\boxed{\Delta t_r = t_{r+1} - t_r = 1}$$ increase by 1

- $r = \frac{q}{2}$ not a true gap (Dirac point)

$$\Delta t_{q/2} = t_{\frac{q}{2}+1} - t_{\frac{q}{2}-1} = -\frac{q}{2} + 1 - \frac{q}{2} + 1$$

(172) $$\boxed{\Delta t_{q/2} = 2 - q}$$ jump!



$q = 8$

$$\boxed{\sum_r \sigma_{xy}^r = 0} \tag{173}$$

(ii) $\boxed{q \text{ odd}}$

- $r \leq \frac{q-1}{2} \Rightarrow S_r = 0 \quad t_r = r$

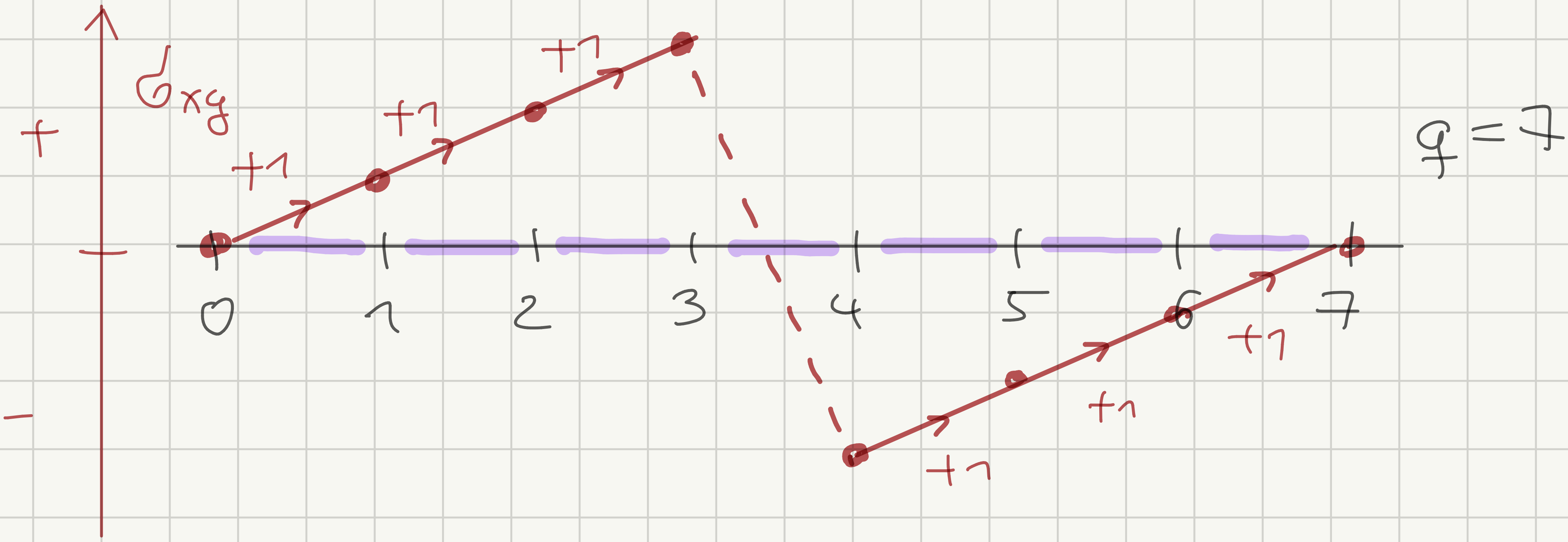
$$\boxed{\Delta t_r = t_{r+1} - t_r = 1} \tag{174}$$ increase by 1

- $r \geq \frac{q+1}{2} \Rightarrow S_r = 1 \quad t_r = r - q$

$$\boxed{\Delta t_r = t_{r+1} - t_r = 1} \tag{175}$$ increase by 1

- central energy band

$$\boxed{\Delta t_{q/2} = t_{\frac{q+1}{2}} - t_{\frac{q-1}{2}} = 1 - q} \tag{176}$$ jump!



(C) Continuum limit: Landau levels

The Harper-Hofstadter model smoothly connects to the continuum model of a 2D electron gas in a magnetic field by considering the limit

$$\alpha = \frac{p}{q} \longrightarrow 0 \qquad \text{i.e. } q \rightarrow \infty \tag{177}$$

and having only a finite number of magnetic sub-bands occupied.

In the limit (177) all bands become flat (Landau levels)