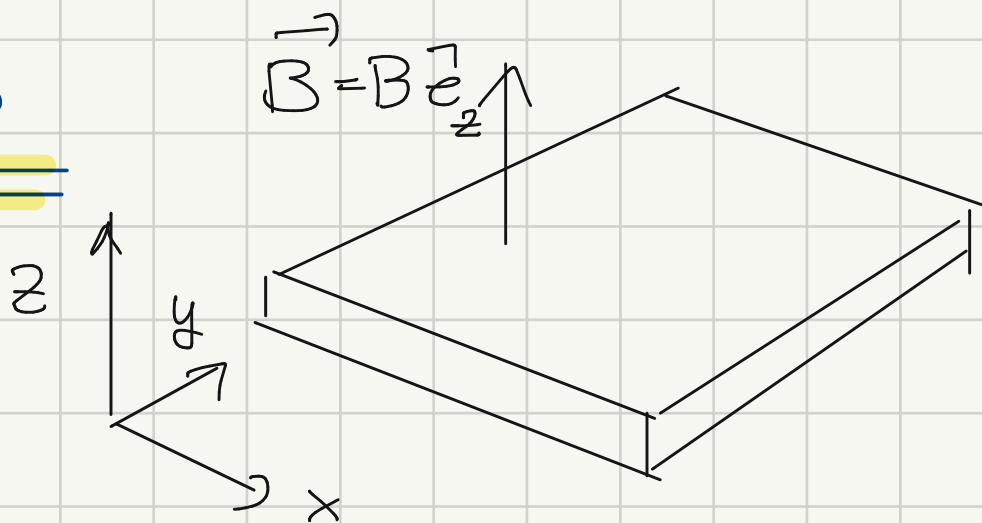


2. Integer quantum Hall system

2.1 Landau levels

free electrons in magnetic field
+ 2D confinement



$$(100) \quad H = \frac{1}{2m} \left(\vec{p} + \frac{e}{c} \vec{A} \right)^2$$

$$\vec{B} = \nabla \times \vec{A}$$

(A) Landau gauge

$$(101) \quad \vec{A} = B(-y, 0, 0)$$

gauge breaks rotational symmetry

$$(102) \quad H = \frac{1}{2m} \left[\left(\hat{p}_x - \frac{eB}{c} \hat{y} \right)^2 + \hat{p}_y^2 \right]$$

$$[H, \hat{p}_x] = 0 \quad \Rightarrow \quad \hat{p}_x = p_x = \hbar k_x$$

conserved

characteristic length

$$(104) \quad \boxed{l_0 = \sqrt{\frac{\hbar c}{eB}}}$$

magnetic length

introduce dimensionless variables

$$y \rightarrow y' = \frac{y}{l_0} - l_0 k_x \quad p_y \rightarrow p_y' = \frac{l_0}{\hbar} p_y$$

- now let's fix k_x (constant of motion)

$$(105) \quad H[k_x] = \hbar \omega_c \left(\frac{1}{2} y'^2 + \frac{1}{2} p_{y'}^2 \right) + \text{const.}$$

this is an harmonic oscillator

(106)

$$\omega_c = \frac{eB}{m\hbar}$$

cyclotron frequency

energies
(107)

$$E_n = \left(n + \frac{1}{2}\right) \hbar \omega_c$$

Landau levels

- does not depend on $k_x \Rightarrow$ degeneracy
- y' -coordinate of oscillators depends on k_x

(B) symmetric gauge

$$(108) \quad \vec{A} = \frac{1}{2} (\vec{B} \times \vec{r}) = \frac{B}{2} (-y, x, 0)$$

gauge respects
rotational
symmetry

in dimensionless units: (l_0, ω_c)

$$(109) \quad H = \frac{1}{2} \left[\left(-i \frac{\partial}{\partial x} - \frac{y}{2} \right)^2 + \left(-i \frac{\partial}{\partial y} + \frac{x}{2} \right)^2 \right]$$

$\Rightarrow$ introduce complex coordinates in (x, y) plane

$$(110) \quad \boxed{z = x - iy \quad z^* = x + iy = r e^{i\theta}}$$

$$(111) \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial z^*} \quad \frac{\partial}{\partial y} = -i \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*} \right)$$

where z and z^* are treated as independent variables

$\Rightarrow$ define two types of ladder operators

$$(112) \quad \boxed{\begin{aligned} \hat{a} &= \frac{1}{\sqrt{2}} \left(\frac{z}{2} + 2 \frac{\partial}{\partial z^*} \right) \\ \hat{a}^\dagger &= \frac{1}{\sqrt{2}} \left(\frac{z^*}{2} - 2 \frac{\partial}{\partial z} \right) \end{aligned}}$$

$$\frac{\partial}{\partial z^*} = \frac{1}{2} (\partial_x - i \partial_y)$$

$$\frac{\partial}{\partial z} = \frac{1}{2} (\partial_x + i \partial_y)$$

$$(113) \quad \boxed{\begin{aligned} \hat{b} &= \frac{1}{\sqrt{2}} \left(\frac{z^*}{2} + 2 \frac{\partial}{\partial z} \right) \\ \hat{b}^\dagger &= \frac{1}{\sqrt{2}} \left(\frac{z}{2} - 2 \frac{\partial}{\partial z^*} \right) \end{aligned}}$$

$$(114) \quad [\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = 1 \quad [\hat{a}, \hat{b}^{(\dagger)}] = 0$$

one finds that the Hamiltonian depends only on $\hat{a}, \hat{a}^\dagger$

$$(115) \quad \boxed{\hat{H} = \hbar \omega_c \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right)}$$

again
Landau levels

$\Rightarrow$ degeneracy of Landau levels with respect
to $\hat{b}^\dagger \hat{b}$

Angular momentum in z direction

$$[H, \hat{L}_z] = 0 \quad \text{by symmetry}$$

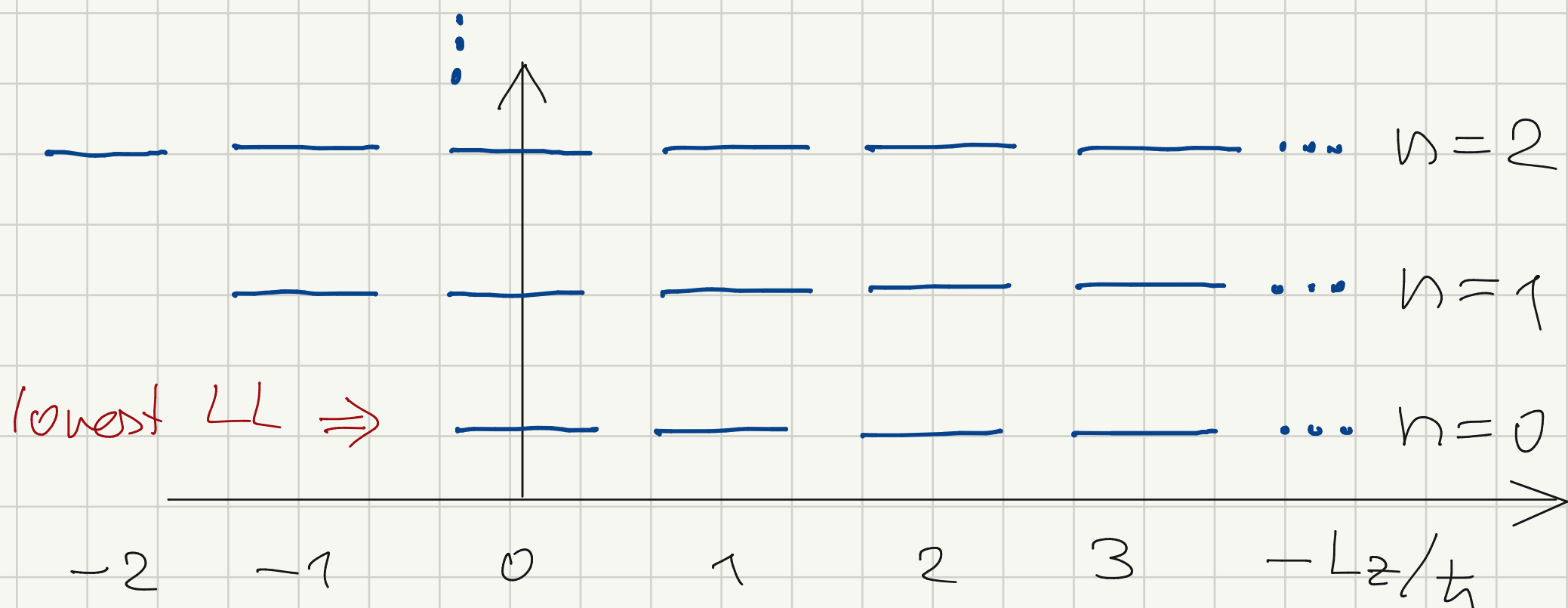
$$L_z = -i\hbar \frac{\partial}{\partial \theta} = -\hbar \left(z \frac{\partial}{\partial z} - z^* \frac{\partial}{\partial z^*} \right)$$

$$(116) \quad \boxed{\hat{L}_z = \hbar (a^\dagger a - b^\dagger b)}$$

eigenvalues of $\hat{L}_z$: $\hbar m$

$$m = n, n-1, n-2, \dots$$

spectrum of Landau levels (LL)



wavefunction

$$(117) \quad |n, m\rangle = \frac{b^{\dagger n+m} a^{\dagger m} |0\rangle}{\sqrt{(n+m)! m!}}$$

$$L_z = -\hbar m$$
$$E = \hbar \omega_c n$$

$$(118) \quad \psi_{nm}(z) = N \left(\frac{z^*}{2} - 2 \frac{\partial}{\partial z} \right)^n \left(\frac{z}{2} - 2 \frac{\partial}{\partial z^*} \right)^{m+n} e^{-|z|^2/4}$$

(in normalized units)

2.2 Lowest Landau level (LLL)

For the physics at low temperature the lowest Landau level is of particular importance

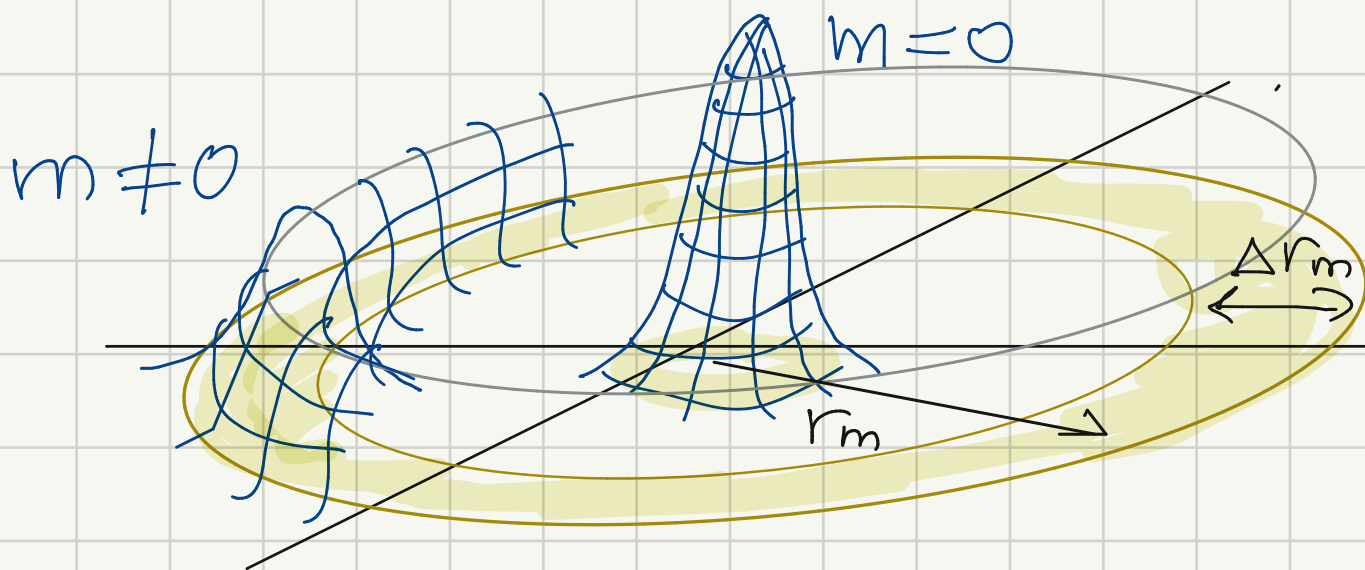
$$n=0$$

$$(119) \quad \psi_{0m}(z) = N \left(\frac{z}{2} - 2 \frac{\partial}{\partial z^*} \right)^m e^{-|z|^2/4}$$

So
(120)

$$\psi_{0m}(z) = N z^m e^{-|z|^2/4}$$

analytic wavefunction (apart from Gaussian)



$$\Delta r_m = \frac{\ell_0}{\sqrt{2m}}$$

$$r_m = \ell_0 \sqrt{2m}$$

(121)

area:

$$A_m = 2\pi r_m \Delta r_m = 2\pi \ell_0^2$$

Number of states in an area A ($\hat{=}$ degeneracy)

$$(122) \quad N = \frac{A}{2\pi l_0^2} = \frac{AB}{\Phi_0} = \frac{\Phi}{\Phi_0}$$

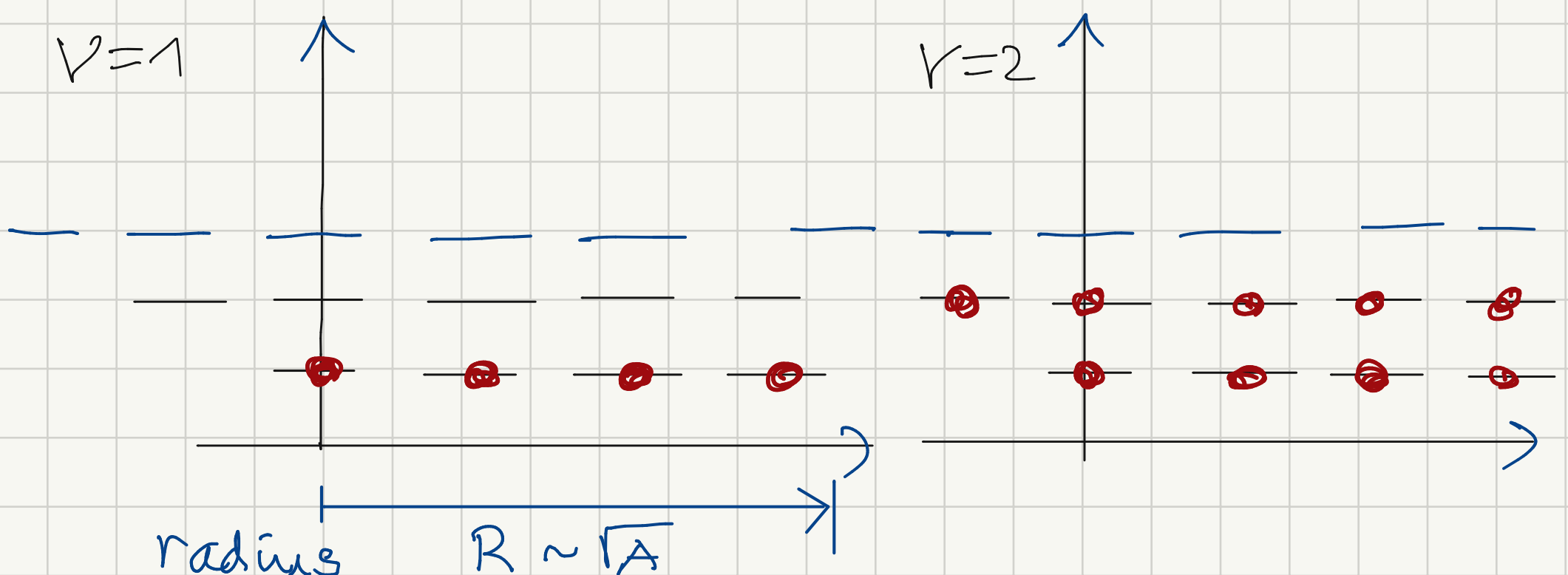
where $\Phi_0 = \frac{hc}{e}$

Dirac flux quantum

Since we consider (spin-polarized) electrons, every state can be occupied by one electron (Pauli principle) there can be at most one particle in $2\pi l_0^2$ if we stay in the LL

in general: filling fraction

$$(123) \quad \nu = \frac{N_e}{N} = \frac{N_e 2\pi l_0^2}{A} = 2\pi \rho_e l_0^2 = \frac{\rho_e}{B/\Phi_0}$$



• many-body wavefunction of filled LLL ($\nu=1$)

Slater determinant

(124)

$$\psi_{\nu=1}(z_1, \dots, z_N) \sim \begin{vmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_N \\ z_1^2 & z_2^2 & \dots & z_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ z_1^N & z_2^N & \dots & z_N^N \end{vmatrix} e^{-\frac{1}{4} \sum_{j=1}^N |z_j|^2}$$

Every electron can have all angular momenta, i.e. all wavefunctions $z_j^m e^{-\frac{1}{4} |z_j|^2}$ with $m=0, 1, \dots, N$ occur in many-body state

$$\psi_{\nu=1}(z_1, \dots, z_N) \sim \prod_{j < k} (z_j - z_k) \exp \left\{ -\frac{1}{4} \sum_j |z_j|^2 \right\}$$

(125)

Pauli exclusion
+ antisymmetry

normalization
for $|z_j| \rightarrow \infty$

2.3 Discretization and lattice model

We will see now that the integer quantum Hall effect can be considered as limiting case of a Dirac Hamiltonian with topological properties

discretization in real space (square lattice)

$$(126) \quad \begin{aligned} x &\rightarrow x_j = j \Delta x \\ y &\rightarrow y_\ell = \ell \Delta x \\ \psi(\vec{r}) = \psi(x, y) &\rightarrow \psi_{j\ell} \end{aligned}$$

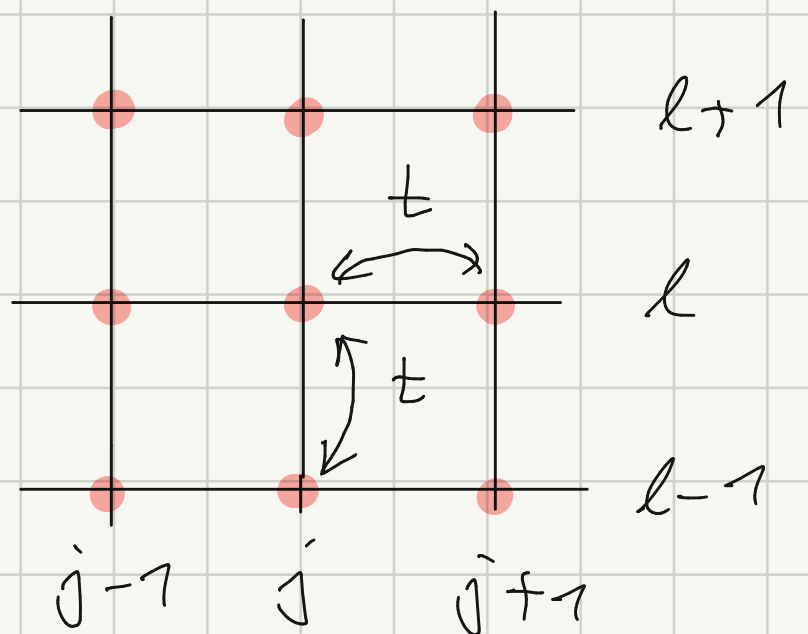
(127) kinetic energy

$$\Delta \psi = \frac{1}{\Delta x^2} \left(\psi_{j+1, \ell} + \psi_{j-1, \ell} + \psi_{j, \ell+1} + \psi_{j, \ell-1} - 4\psi_{j\ell} \right)$$

$$i\hbar \frac{\partial}{\partial t} \psi = H \psi = -\frac{\hbar^2}{2m} \Delta \psi$$

$$(128) \quad H \psi_{em} = -t \left(\psi_{j+1, \ell} + \psi_{j-1, \ell} + \psi_{j, \ell+1} + \psi_{j, \ell-1} - 4\psi_{j\ell} \right)$$

$$t = \frac{\hbar^2}{2m(\Delta x)^2}$$



in second quantization

$$(129) \quad H = -t \sum_{\vec{j}, e} \left(C_{\vec{j}+1, e}^\dagger C_{\vec{j}, e} + C_{\vec{j}, e+1}^\dagger C_{\vec{j}, e} + h.a. \right)$$

Now let's add the interaction with the magnetic field in Landau gauge.

$$\vec{A} = -By \vec{e}_x$$

$$(130) \quad H\psi = -\frac{\hbar^2}{2m} \left[\left(\frac{\partial}{\partial x} - \frac{i}{\hbar} \frac{eBy}{c} \right)^2 + \left(\frac{\partial}{\partial y} \right)^2 \right] \psi$$

$$\left(\frac{\partial}{\partial x} - \frac{ieBy}{\hbar c} \right) \psi(x, y) = \frac{1}{\Delta x} \left(\psi_{\vec{j}+1, e} - \psi_{\vec{j}, e} - \frac{ieB\Delta x^2}{\hbar c} e \psi_{\vec{j}, e} \right)$$

$$= \frac{1}{\Delta x} \left(\psi_{\vec{j}+1, e} - \psi_{\vec{j}, e} - \frac{ieB\Delta x^2}{\hbar c} e \psi_{\vec{j}+1, e} \right)$$

$$(131) \quad = \frac{1}{\Delta x} \left(\underbrace{\psi_{\vec{j}+1, e} \left(1 - \frac{ieB\Delta x^2}{\hbar c} e \right)}_{\approx \exp\left(-\frac{ieB\Delta x^2}{\hbar c} e\right)} - \psi_{\vec{j}, e} \right)$$

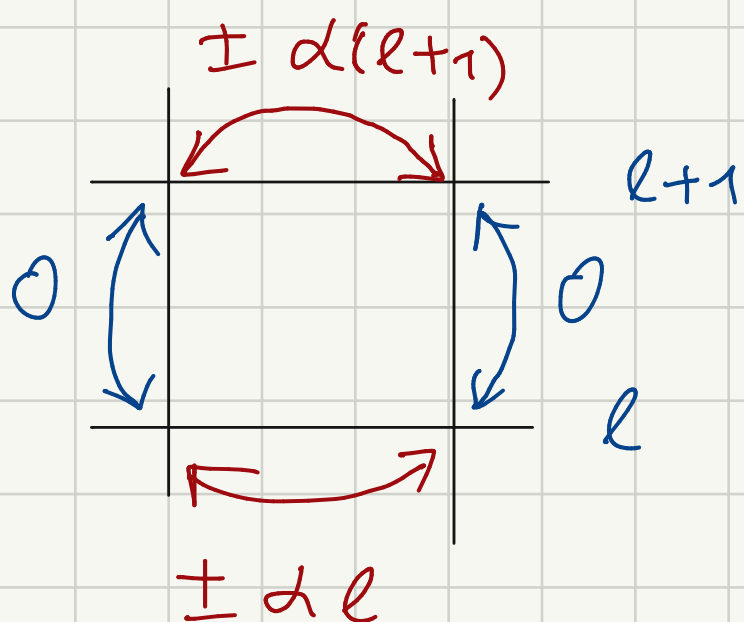
$$\approx \exp\left(-\frac{ieB\Delta x^2}{\hbar c} e\right) \quad \text{Peierls phase}$$

$$H\psi_{\vec{j}, e} = -t \left(e^{-i\alpha e} \psi_{\vec{j}+1, e} + e^{i\alpha e} \psi_{\vec{j}-1, e} + \psi_{\vec{j}, e+1} + \psi_{\vec{j}, e-1} - 4\psi_{\vec{j}, e} \right)$$

$$(132) \quad \alpha = \frac{eB}{\hbar c} (\Delta x)^2 = \left(\frac{\Delta x}{l_0} \right)^2 \quad \text{Peierls phase}$$

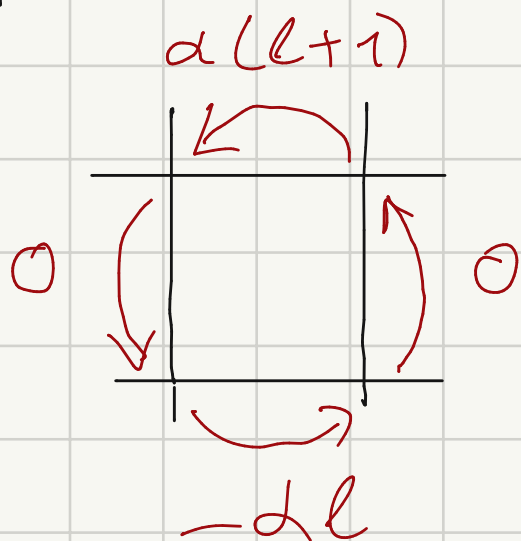
so in second quantization we obtain complex hopping amplitudes

$$(133) \quad H = -t \sum_{j\ell} \left(e^{-i\alpha\ell} c_{j\ell+1}^\dagger c_{j\ell} + \text{h.c.} \right) - t \sum_{j\ell} \left(c_{j\ell+1}^\dagger c_{j\ell} + \text{h.c.} \right) + 4t \sum_{j\ell} c_{j\ell}^\dagger c_{j\ell}$$



in chosen gauge:
phases when hopping
in x-direction

The phase picked up upon a closed loop in any plaquette



$$\Delta\phi = \alpha = \left(\frac{\Delta x}{l_0} \right)^2 = \frac{B \Delta x^2}{\Phi_0} = \frac{\text{magn. flux per plaquette}}{\text{Dirac flux quantum}}$$