12. Floquet topological insulators

We have seen in the discussion of TR invariant topological insulators that a proper bandstructure and sufficiently strong spin-orbit coupling is needed to generate a non-trivial topology. In particular in solid-state systems it is very difficult to find appropriate systems and the manipulation of Hamiltonian parameter is hard. Here periodic driving, e.g. with an external laser field can offer a new twist.

12.1 The principle of Floquet engineering: Effective Hamiltonians

Consider e.g. a 2band Blochhamiltonian in 2D

$$\underline{\underline{h}}_0(\vec{k}) = \varepsilon_0 \mathbb{1} + \vec{d}(\vec{k}) \cdot \vec{\underline{\underline{\sigma}}} \tag{523}$$

and add a time-periodic potential

$$\underline{\underline{h}}(\vec{k},t) = \underline{\underline{h}}_0(\vec{k}) + \underline{\underline{V}}(\vec{k},t) \qquad \underline{\underline{V}}(\vec{k},t+T) = \underline{\underline{V}}(\vec{k},t) \tag{524}$$

whose periodicity T is short compared to all other time scales of the system, then at stroboscopic times the system is described by the Floquet Hamiltonian H_F defined as

$$\boxed{e^{i\underline{\underline{h}}_F(\vec{k})T} = U_{\vec{k}}(t_0+T, t_0)} \tag{525}$$

where

$$U_{\vec{k}}(t_0+T, t_0) = T\exp\left\{-i\int_{t_0}^{t_0+T} d\tau\, \underline{\underline{h}}(\vec{k},\tau)\right\} \tag{526}$$

$\underline{\underline{h}}_F(\vec{k})$ can have different symmetries and topological properties than the original static Hamiltonian.

In the above case $\underline{\underline{h}}_F(\vec{k})$ is again a 2band Bloch Hamiltonian

$$\boxed{\underline{\underline{h}}_F(\vec{k}) = \varepsilon_F(\vec{k})\mathbb{1} + \vec{n}(\vec{k})\cdot\underline{\underline{\vec{\sigma}}}} \tag{527}$$

and one can define a Chern number

$$\boxed{C_F = \frac{1}{4\pi}\int_{BZ} d^2k \left(\frac{\partial\hat{n}(\vec{k})}{\partial k_x} \times \frac{\partial\hat{n}(\vec{k})}{\partial k_y}\right)\cdot\hat{n}(\vec{k})} \tag{528}$$

where $\hat{n}(\vec{k}) = \vec{n}(\vec{k})/|\vec{n}(\vec{k})|$ which may be completely different from that of the original static Hamiltonian

$$C = \frac{1}{4\pi}\int_{BZ} d^2k \left(\frac{\partial\hat{d}(\vec{k})}{\partial k_x} \times \frac{\partial\hat{d}(\vec{k})}{\partial k_y}\right)\cdot\hat{d}(\vec{k}) \tag{529}$$

where $\hat{d} = \vec{d}/|\vec{d}|$. Note that (529) is equivalent to a description in terms of Bloch states.

We will see later that the Floquet Hamiltonian does not capture all topological properties of the periodically driven system.

(A) Mathematical basis of Floquet theory

Consider a time-periodic Hamiltonian

(530) $$H(t+T) = H(t)$$

Then the time-evolution operator can be written as

(531) $$U(t+T,0) = U(t,0)\,U(T,0)$$

So it is sufficient to know $U(t,0)$ for $t \in (0,T)$ and $U(T,0)$, which is called the one-cycle evolution operator, which we write as $(\hbar = 1)$

(532) $$\boxed{U(T,0) = \exp(-i H_{eff} T)} \quad H_{eff}^{\dagger} = H_{eff}$$

With this we define

(533) $$P(t) := U(t,0)\, e^{+iH_{eff} t}$$

$$\Rightarrow \quad P(t+T) = U(t+T,0)\, e^{iH_{eff}(t+T)}$$

$$= U(t,0)\, \cancel{U(T,0)\, e^{iH_{eff}T}}\, e^{iH_{eff}t} = P(t)$$

Thus

(534) $$\boxed{U(t,0) = P(t)\, e^{-iH_{eff}t}}$$

Micromotion (T periodic)

effective evolution.

where the unitary operator P is T-periodic

(535) $$\boxed{P(t+T) = P(t)}$$

Floquet states and quasienergies

The eigenstates of the time independent hermitian operator H_{eff} are and form a complete set

(536) $$\boxed{H_{eff} \, |n\rangle = \mathcal{E}_n |n\rangle}$$

(assume finite Hilbert space for simplicity)

$\mathcal{E}_n$: quasi energies

An initial state

(537) $$|\psi(0)\rangle = \sum_n a_n |n\rangle$$

then evolves in time according to

(538) $$|\psi(t)\rangle = \mathcal{U}(t,0)|\psi(0)\rangle = \sum_n a_n P(t) e^{-iH_{eff}t} |n\rangle$$
$$= \sum_n a_n P(t)|n\rangle e^{-i\mathcal{E}_n t}$$

with time-independent a_n and $\mathcal{E}_n$

(539) $$\boxed{|\mathcal{E}_n(t)\rangle \equiv P(t)|n\rangle}$$

Floquet states

They are periodic in time

(540) $$|\mathcal{E}_n(t+T)\rangle = |\mathcal{E}_n(t)\rangle$$

from time-periodic to time-independent problem

Let us consider a unitary transformation

(5.41) $$\boxed{|\psi(t)\rangle = P(t)\,|\tilde{\psi}(t)\rangle}$$

Then

$$i\frac{d}{dt}|\tilde{\psi}(t)\rangle = i\dot{P}(t)^{-1}|\psi(t)\rangle + iP(t)^{-1}\frac{d}{dt}|\psi(t)\rangle$$

$$= i\frac{d}{dt}\left(e^{-iH_{\rm eff}t}U^{-1}(t,0)\right)|\psi(t)\rangle + P^{-1}(t)H(t)\underbrace{P(t)|\tilde{\psi}(t)\rangle}_{|\psi(t)\rangle}$$

$$= H_{\rm eff}\underbrace{e^{-iH_{\rm eff}t}U^{-1}(t,0)}_{P^{-1}}|\psi(t)\rangle \cancel{- \underbrace{e^{-iH_{\rm eff}t}U^{-1}}_{P^{-1}}H(t)}P(t)|\tilde{\psi}(t)\rangle$$

$$\cancel{+ P^{-1}(t)H(t)}P(t)|\tilde{\psi}(t)\rangle$$

$$= H_{\rm eff}|\tilde{\psi}(t)\rangle$$

I.e.

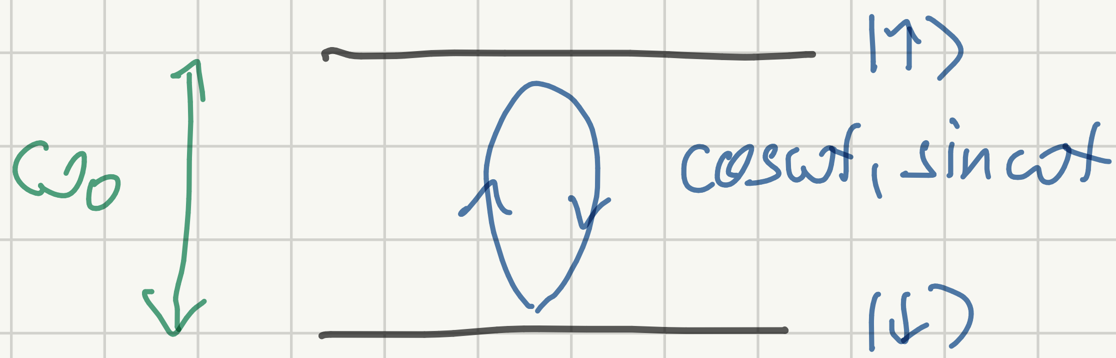
(5.42) $$\boxed{i\frac{d}{dt}|\tilde{\psi}(t)\rangle = H_{\rm eff}\,|\tilde{\psi}(t)\rangle}$$

time-independent problem in rotating frame

This allows the solution of certain integrable time-periodic problems.

example (i) two-level system in time-periodic circular polarized light
(without rotating wave approximation!)

(543) $$H(t) = \frac{\omega_0}{2}\sigma_z + \frac{\mu E}{2}\left(\sigma_x \cos\omega t + \sigma_y \sin\omega t\right)$$



Choose

(544) $$P(t) = e^{-i\frac{\omega}{2}\sigma_z t}\, e^{i\frac{\omega}{2}t} = \exp\left(i\frac{\omega}{2}t(1-\sigma_z)\right)$$

with

$$i\frac{d}{dt}|\psi(t)\rangle = i\dot{P}(t)|\tilde{\psi}(t)\rangle + iP\frac{d}{dt}|\tilde{\psi}(t)\rangle$$

$\Rightarrow$

$$i\frac{d}{dt}|\tilde{\psi}(t)\rangle = \left(P^{-1}(t)H(t)P(t) - iP^{-1}(t)\dot{P}(t)\right)|\tilde{\psi}(t)\rangle$$

so

$$G = P^{-1}(t)\left[H(t) - i\frac{d}{dt}\right]P(t)$$

(545) $$= \frac{\omega}{2}\mathbb{1} + \frac{1}{2}(\omega_0 - \omega)\sigma_z + \frac{\mu E}{2}\sigma_x$$

<u>quasi energies:</u>

(546) $$\underline{\underline{\varepsilon_\pm = \frac{1}{2}(\omega \pm \Omega)}} \qquad \underline{\underline{\Omega = \sqrt{(\omega_0-\omega)^2 + (\mu E)^2}}}$$

This sounds all too simple! So where is the catch?

(i) the transformation $P(t)$ can in general only be found approximately
$\Rightarrow$ high frequency expansion

(ii) the quasi energy spectrum has a Brillouin zone structure and is not unique

lets look at the limit of $\varepsilon_\pm$ if $E \to 0$

$$\omega < \omega_0 \qquad \varepsilon_+ \to +\frac{\omega_0}{2}$$

$$\varepsilon_- \to -\frac{\omega_0}{2} + \omega$$

?

$$\omega > \omega_0 \qquad \varepsilon_+ \to +\frac{\omega_0}{2} + \omega$$

$$\varepsilon_- \to -\frac{\omega_0}{2}$$

In fact we realize from (535) that a transformation

(547) $$P(t) \to P(t)\, e^{\pm i n \omega t} \qquad \omega = \frac{2\pi}{T}$$

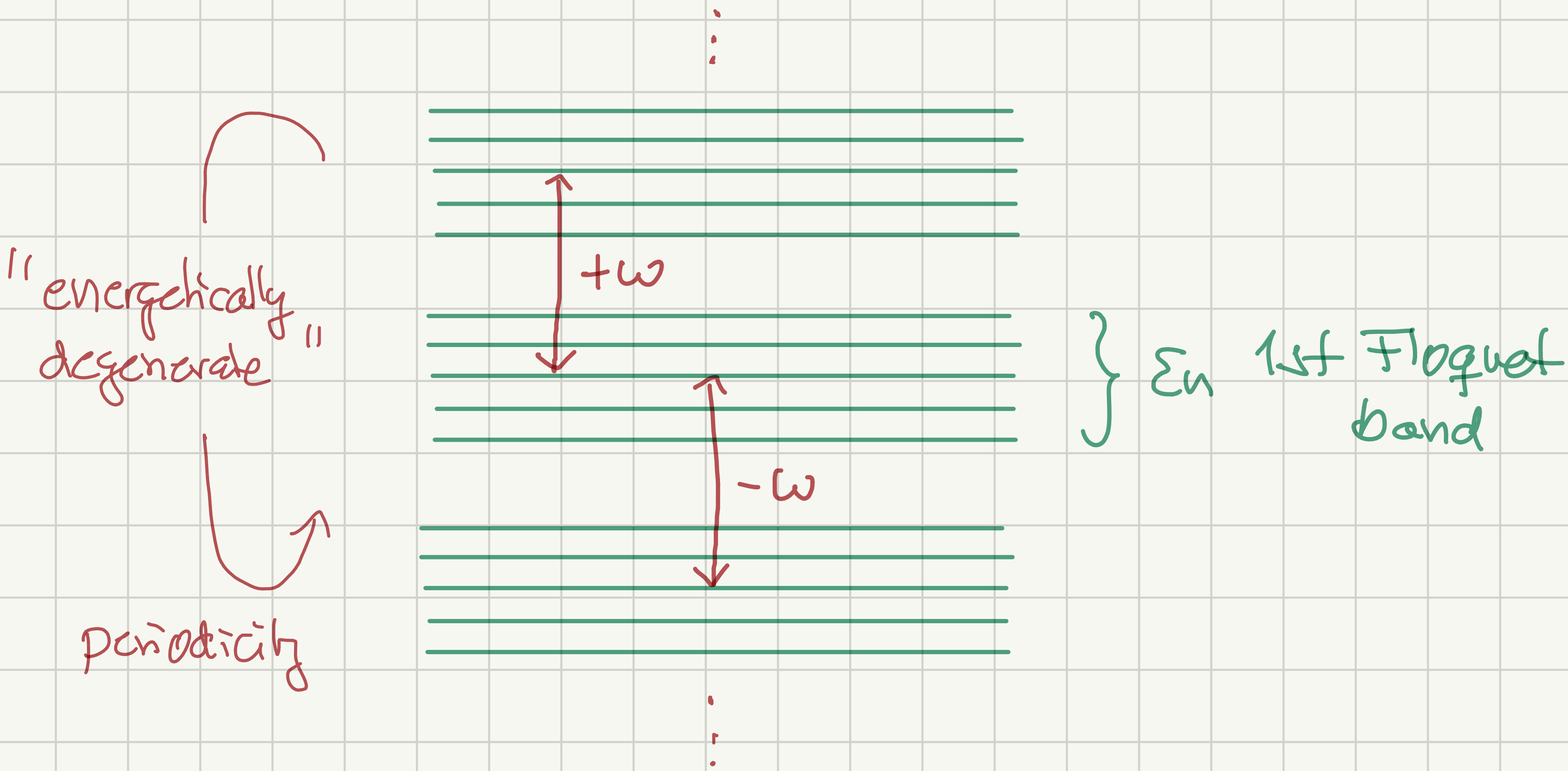
does not change the results. So the quasi energies are only defined modulo ω

(548) $$\boxed{\varepsilon_n \to \varepsilon_{n,m} = \varepsilon_n + m\omega} \qquad m = 0, \pm 1, \pm 2$$

$\Rightarrow$ Brillouin zone structure of quasi-energy spectrum

1. Floquet band

(549) $$\boxed{-\frac{\omega}{2} < \varepsilon \leq \frac{\omega}{2}}$$



(B) Approximate solutions: high-frequency expansion

Let's return to the two-level example in Sec. (A) but now with linear polarized light

(550) $$H(t) = \frac{\omega_0}{2}\sigma_z + \mu E \sigma_x \cos\omega t$$

Here we do not find a simple analytic function $P(t)$. However if we use $P(t)$ from eq. (544) we would get

(551) $$P^{-1}_{(t)}\left[H(t) - i\frac{d}{dt}\right]P(t) =$$

$$= H_{eff} + \frac{\mu E}{2}\left(\sigma_x \cancel{\cos 2\omega t}_{\approx 0} - \sigma_y \cancel{\sin 2\omega t}_{\approx 0}\right)$$

with H_{eff} from (545), i.e. the circular polarized case.
We realize that the remaining time dependent terms are fast oscillating and one can neglect than hoping that their effect cancels out when time averaging over times $\Delta t \gg T$.
This is the famous rotating wave approximation.

small parameter $\boxed{\frac{\varepsilon_{typical}}{\omega}}$

General goal:

- find a unitary $\boxed{P(t) = e^{-iK}}$ such that

$$\boxed{P^{-1}(t)\left[H(t) - i\frac{d}{dt}\right]P(t) \simeq H_{eff} = \text{const}_t} \tag{552}$$

procedure: expansion in ω^{-1}

$$H_{eff} = \sum_{n=0}^{\infty} \frac{1}{\omega^n} H_{eff}^{(n)} \tag{553}$$

$$K(t) = \sum_{n=0}^{\infty} \frac{1}{\omega^n} K^{(n)} \tag{554}$$

$$P^{-1} H P = e^{iK} H e^{-iK} \qquad \text{Baker Hausdorff}$$

$$= H + i \underbrace{[K, H]}_{\omega^{-1}} - \frac{1}{2!} [K, \underbrace{[K, H]}_{\omega^{-2}}] + \cdots \qquad \text{etc.}$$

$$-i P^{-1} \frac{d}{dt} P = i\left(\frac{d}{dt} P^{-1}\right) P$$

$$= \underbrace{-\frac{\partial K}{\partial t}}_{\omega^{-1}} - \frac{i}{2} \underbrace{\left[K, \frac{\partial K}{\partial t}\right]}_{\omega^{-2}} + \cdots \qquad \text{etc.}$$

- We now apply this expansion to a Hamiltonian

$$\boxed{\begin{aligned} &H(t) = H_0 + \hat{V}(t) && (555) \\ &\text{with} \\ &\hat{V}(t) = \sum_{n=1}^{\infty} \left(\hat{V}^{(n)} e^{in\omega t} + \hat{V}^{(-n)} e^{-in\omega t} \right) && (556) \end{aligned}}$$

- up to zeroth order in $1/\omega$ one finds

$$H_{eff}^{(0)} = H_0 + \hat{V}(t) - \frac{\partial K^{(1)}(t)}{\partial t} \qquad (556)$$

choosing

(557) $$\boxed{K^{(1)}(t) = \int_0^t d\tau\, V(\tau)}$$

makes $H_{eff}^{(0)}$ time independent.

- up to first order in $1/\omega$ one finds for the effective Hamiltonian

(558) $$\boxed{H_{eff} = H_0 + \frac{1}{\omega} \sum_{n=1}^{\infty} \frac{1}{n} \left[\hat{V}^{(n)}, \hat{V}^{(-n)}\right] + \mathcal{O}\left(\frac{1}{\omega^2}\right)}$$

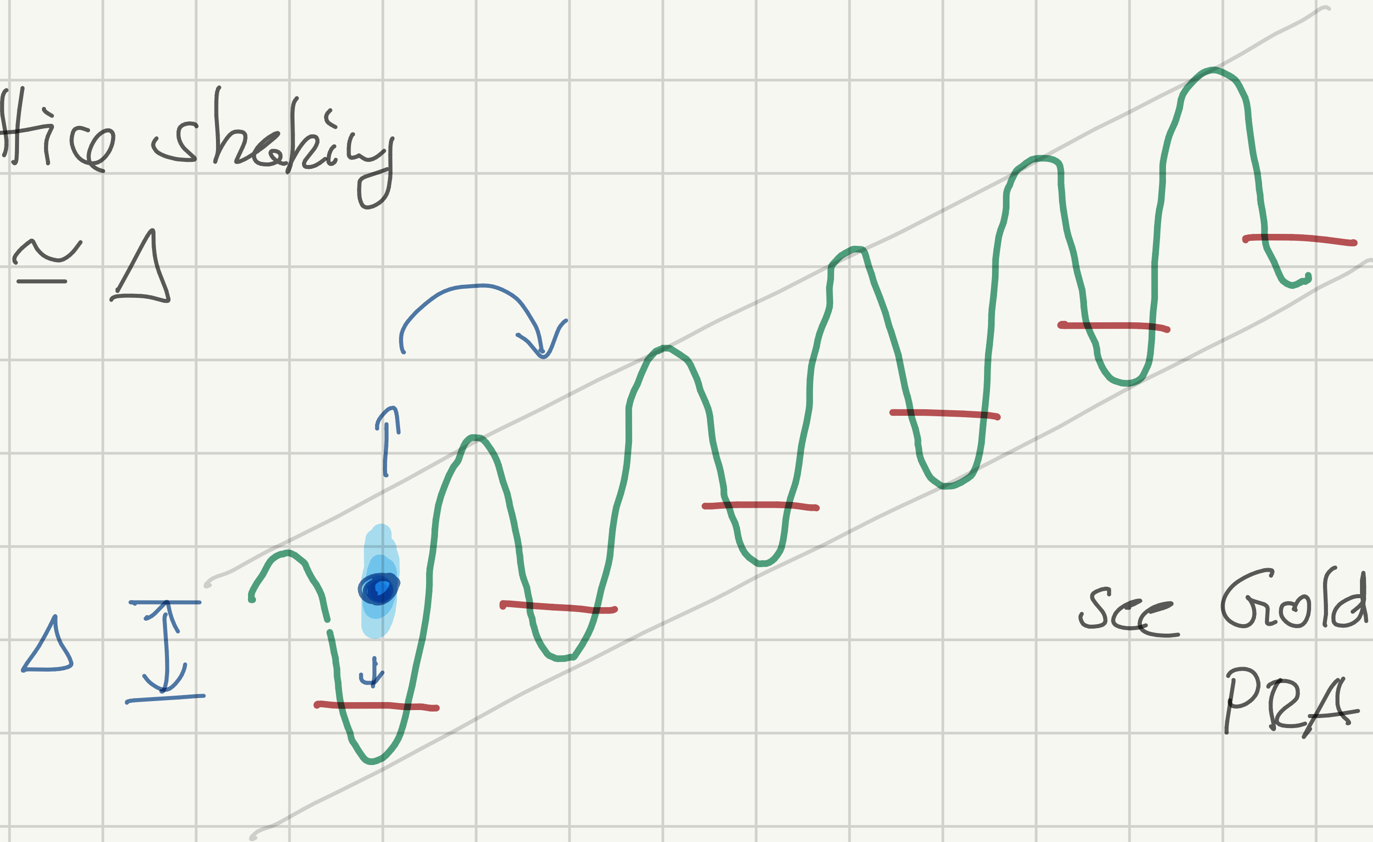
and for the kick operator determining the micromotion

(559) $$\boxed{K = \frac{1}{i\omega} \sum_{n=1}^{\infty} \frac{1}{n} \left(\hat{V}^{(n)} e^{in\omega t} - \hat{V}^{(-n)} e^{-in\omega t}\right) + \mathcal{O}\left(\frac{1}{\omega^2}\right)}$$

(C) Generalization to systems with resonant modulation *

Floquet techniques are used e.g. for optical lattice schemes where a linear potential is added forming a so-called Wannier-Stark ladder lattice shaking

$\omega \simeq \Delta$



see Goldmann et al. PRA <u>91</u>, 033632 (2015)

For these systems $H_0 \sim \Delta = \omega$ and the above derivation of the Floquet Hamiltonian must be slightly revised. We assume that the static Hamiltonian H_0 has the form

(560) $$H_0 = \sum_{\alpha,\beta} \hat{\Pi}^\alpha H^0_{\alpha\beta} \hat{\Pi}^\beta + \omega \sum_\alpha \alpha \hat{\Pi}^\alpha$$

where $\hat{\Pi}^\alpha$ are orthogonal projectors in some sectors labelled by α. Then

(561) $$\hat{V}^{(\pm n)} = \sum_{\alpha\beta} \hat{\Pi}^\alpha \hat{V}^{(\pm n)}_{\alpha\beta} \hat{\Pi}^\beta \qquad n = 1, 2, 3, \ldots$$

Now we add a <u>unitary trafo</u>

(562) $$\boxed{R(t) = \exp\left\{ i \sum_\alpha \alpha \omega t \hat{\Pi}^\alpha \right\} = R(t+T)}$$

i.e.

(563) $$|\psi\rangle \longrightarrow |\psi'\rangle = R(t)\,|\psi\rangle$$

and

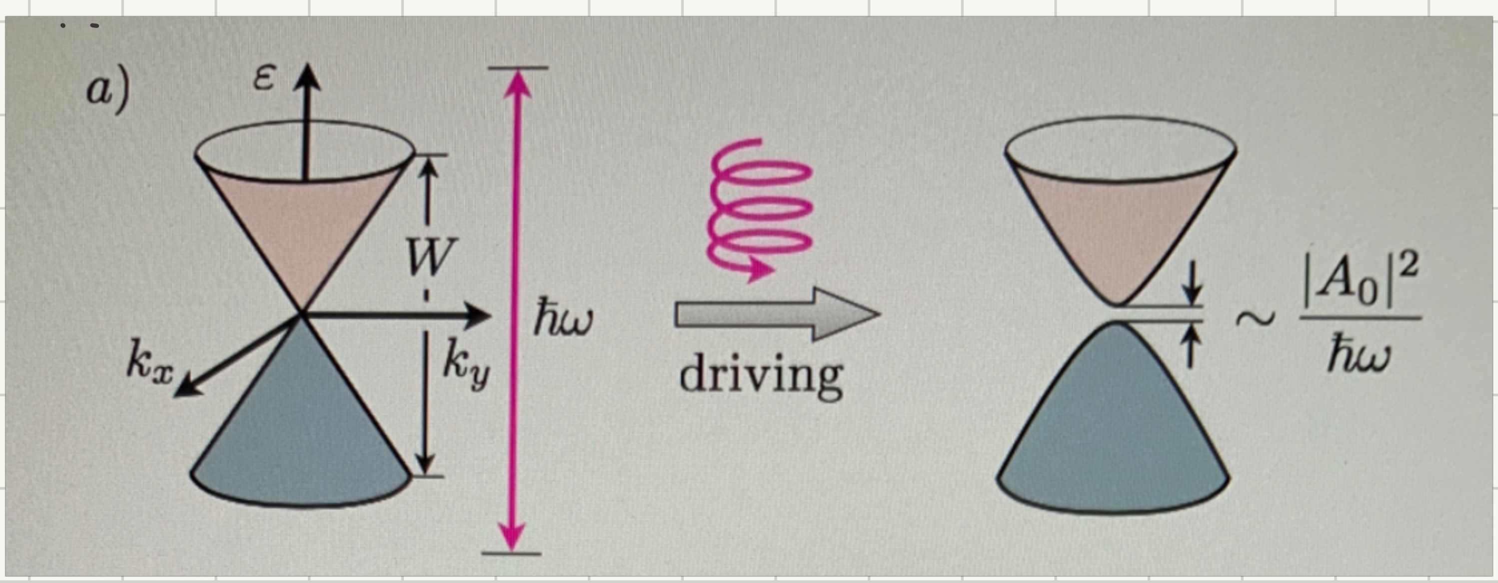
(564) $$\boxed{H(t) \rightarrow \tilde{H}(t) = RHR^{+} - iR\frac{d}{dt}R^{+}}$$

In $\tilde{H}$ we now got rid of terms $\sim \omega$ so we can apply the technique from Sec. (B).

12.2 Topology of Floquet bands

(A) Creating topological phases by high-frequency modulation

- Graphene irradiated by circular polarized light



low-energy Hamiltonian of Graphene

(565) $$\underline{h_0}(\vec{k},t) = v_F\left(\sigma_x \tau_x q_x + \sigma_y q_y\right)$$