11. Some comments on the general classification of topological systems with interactions

Topological system with interactions are not fully understood.
In the following we will scetch a classification scheme by

Xia-Gang Wen et al. PRB **82**, 155138 (2010)

phase transitions

Spontaneous symmetry-breaking	topological

11.1 Symmetry-breaking phase transitions and entanglement

<u>example</u> 1D Ising model in transverse field

(510) $$H = -B \sum_j \sigma_j^x - J \sum_j \sigma^z_{j+1} \sigma^z_j$$

has a global $\mathbb{Z}_2$ symmetry

$$|\uparrow\rangle \longleftrightarrow |\downarrow\rangle \quad \text{i.e.} \quad \sigma^z \longrightarrow -\sigma^z$$

ground state

$\boxed{B \gg J}$

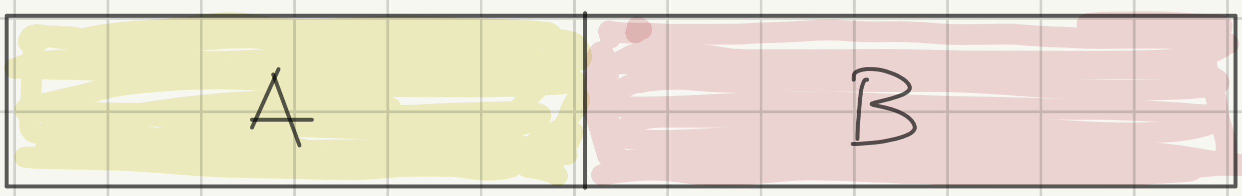
(5.11) $$|\phi^P_{gs}\rangle = \prod_j \left(\frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}\right)_j = \prod_j |+\rangle_j$$

$\boxed{B \ll J}$

(5.12) $$|\phi^F_{gs}\rangle = \begin{cases} |\uparrow\rangle|\uparrow\rangle|\uparrow\rangle \cdots \\ \text{or} \\ |\downarrow\rangle|\downarrow\rangle|\downarrow\rangle \cdots \end{cases}$$ break $\mathbb{Z}_2$ symmetry!

In both limits $B \gg J$ and $B \ll J$ ground state is a product state!

Consider bi-partition



Schmidt theorem

(5.13) $$|\psi\rangle = \sum_{j=1}^{n} \sqrt{\lambda_j}\, |j\rangle_A |j\rangle_B$$

$$n \le \min\left(\dim(\mathcal{H}_A), \dim(\mathcal{H}_B)\right)$$

note that $|j\rangle_A$ and $|j\rangle_B$ can have completely different representations in terms of local states

$|j\rangle_A$ and $|j\rangle_B$ are ONB

von-Neumann entropy of half system as measure of entanglement between A and B

$$\rho_A = \mathrm{Tr}_B\{|\psi\rangle\langle\psi|\} \qquad \rho_B = \mathrm{Tr}_A\{|\psi\rangle\langle\psi|\}$$

in Schmidt representation

$$\rho_A = \sum_j \lambda_j |j\rangle_A {}_A\langle j| \qquad \rho_B = \sum_j \lambda_j |j\rangle_B {}_B\langle j|$$

$\Rightarrow$ von Neumann entropy

$$S_A = -\mathrm{Tr}_A\{\rho_A \ln \rho_A\} = S_B = -\mathrm{Tr}_B\{\rho_B \ln \rho_B\}$$

(5.4) $$= -\sum_{j=1}^{n} \lambda_j \ln \lambda_j$$

In general

(5.5) $$n \leq p^{L/2}$$

p: local dimension ; L : total # of lattice sites

Thus in general

$$S_A = -\sum_{j=1}^{n} \lambda_j \ln \lambda_j \leq -\sum_{j=1}^{n} \frac{1}{n} \ln \frac{1}{n} \leq \ln n \leq \ln p^{L/2}$$

(5.6) $$\boxed{S \leq L \ln \sqrt{p}}$$ "volume" law (here in 1D)

This is to be expected in general since the entropy is an extensive thermodynamic quantity

For gapped many-body states one finds however in general a area law

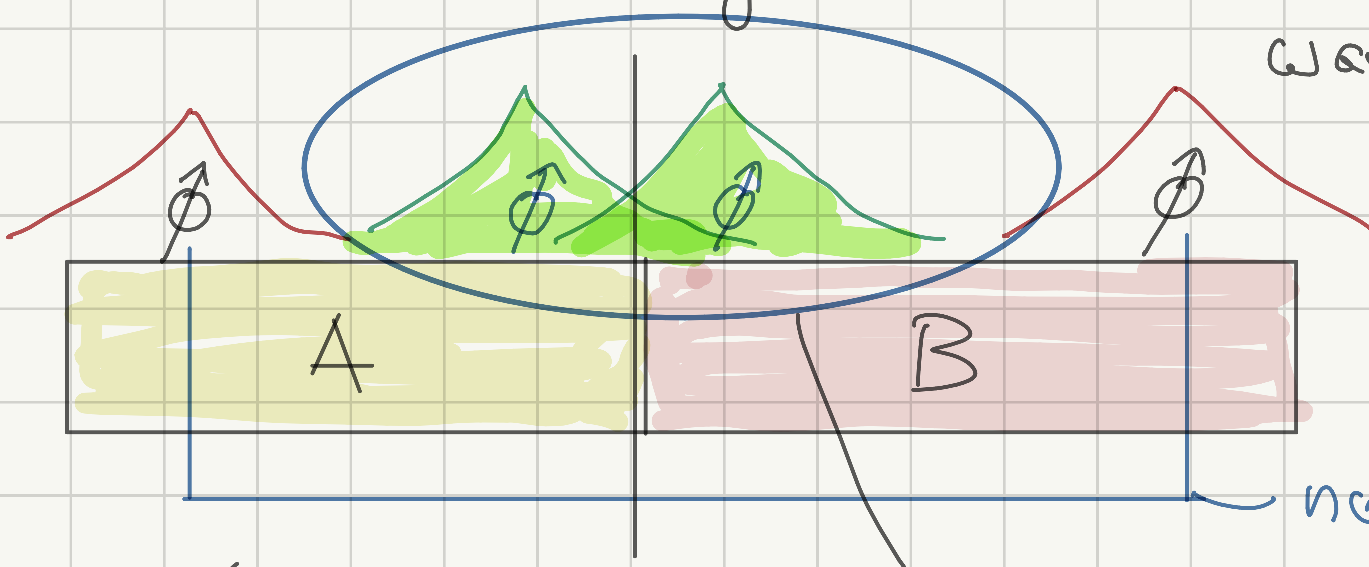
(517) $$\boxed{S \leq \eta\, \mathcal{O}(L) \sim \eta L^{d-1}} \quad \begin{cases} d=1 & \text{const!} \\ d=2 & S \sim L \end{cases}$$

At the critical point one gets an extra factor $\ln V$.

$\Rightarrow$ both phases of Ising model have much less entropy than maximally possible!

This is because there are only short-range correlations in the system

gapped state:



only wavefunction close to cut contribute to entanglement

short-range entanglement

Conjecture	gapped ground states that can be characterized by a spontaneously broken symmetry are short-range entangled

precise definition of "short range entanglement" will follow later

11.2 Quantum phase transitions, local unitary evolution and topological order

quantum phase transition

ground state:

(518) $$H = H(\lambda) \qquad 0 \leq \lambda \leq 1 \qquad |\phi(\lambda)\rangle$$

If a local observable exists which has a non-analytic behavior at some value λ_c in the thermodynamic limit $L \to \infty$

(519) $$\langle \hat{O} \rangle(\lambda) = \langle \phi(\lambda)| \hat{O} |\phi(\lambda)\rangle$$

then the system is said to have a quantum phase transition. The thdyn. limit is important since in finite systems there are no non-analyticities (except in pathological, non-generic cases)

The states $|\phi(\lambda=0)\rangle$ and $|\phi(\lambda=1)\rangle$ are said to be in the same phase (or same equivalence class) if we can find a smooth deformation of the local Hamiltonian $H \to H(\lambda)$ such that $|\phi(0)\rangle$ and $|\phi(1)\rangle$ are ground states of $H(\lambda)$ at $\lambda=0$ and $\lambda=1$ and there was no gap closing on the λ-path i.e.:

$$|\phi(1)\rangle \hat{=} |\phi(0)\rangle \quad \text{iff} \quad |\phi(1)\rangle = \hat{T} e^{-i\int_0^1 d\lambda H(\lambda)} |\phi(0)\rangle \tag{520}$$

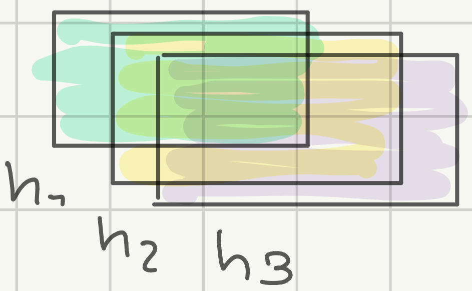
Here

$$\boxed{U_L \equiv \hat{T} e^{-i\int_0^1 d\lambda H(\lambda)}} \tag{521}$$

is called a Local unitary map LU

Rem: $H(\lambda)$ is called local if it can be written as a sum of terms that connect only local sites or decay exponentially with separation

$$H = \sum_j h_j$$



With this we can now formally define what short-range entangled means

Def: A many-body state is short-range entangled if and only if it can be transformed into a product state by a finite number of LU transformations

SRE

Set of all product states

Set of all SRE states

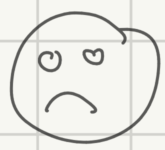
$$|\phi\rangle = \prod_j |\phi_j\rangle \rightarrow \boxed{LU} \rightarrow |\tilde{\phi}\rangle$$

Def: Quantum many-body states that cannot be connected to product states by a finite number of LU's are called **long-range entangled** — **LRE**

X.-G. Wen: Long-range entangled states have **intrinsic topological order**

One can show that ground states of local Hamiltonians in 1D are all SRE (see discussion on Ising chain)

$\Rightarrow$ **no intrinsic topological order in 1D**



but there is a second class of states with topological order if we require certain symmetries

10.3 Symmetry-protected topological order

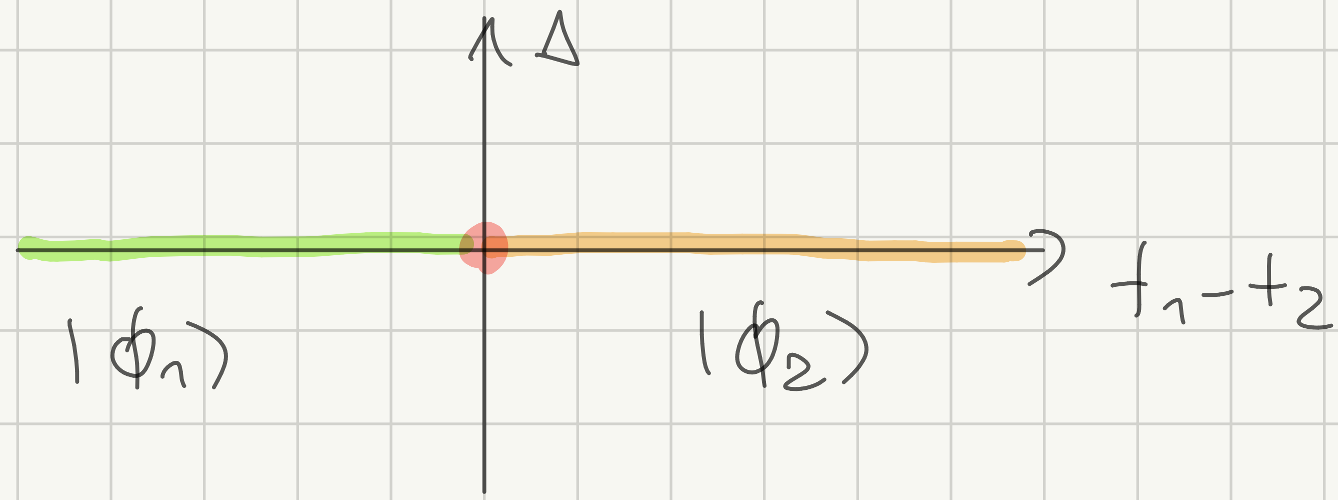
$$|\phi(1)\rangle \underset{S}{\hat{=}} |\phi(0)\rangle \quad \text{iff} \quad |\phi(1)\rangle = T e^{-i\int_0^1 d\lambda H(\lambda)} |\phi(0)\rangle$$

(5.22) $$\underline{\text{and}} \quad [H(\lambda), S] = 0$$

$\Rightarrow$ restriction to allowed transformations

The states $|\phi(\lambda=0)\rangle$ and $|\phi(\lambda=1)\rangle$ are said to be in the same symmetry protected phase (eq. class) if we can find a smooth deformation of $H \to H(\lambda)$ such that $[H(\lambda), S] = 0$ and that $|\phi(0)\rangle$ and $|\phi(1)\rangle$ are ground states of $H(\lambda)$ at $\lambda=0$ and $\lambda=1$ and there was no gap closing on the λ-path

example 1D Rice-Mele model vs. SSH



$|\phi_1\rangle, |\phi_2\rangle$ not in the same symmetry-protected phase but if we lift symmetry protection they are in the same phase