

(c) Symmetry protected topology of SLBHM

We have seen that for finite U there is no longer particle-hole symmetry (charge conjugation) in SLBHM.

So what protects the topology in this 1D system?

$$H = -t_1 \sum_{j \text{ even}} a_{j+1}^\dagger a_j - t_2 \sum_{j \text{ odd}} a_{j+1}^\dagger a_j + \text{h.c.} + \frac{U}{2} \sum_j a_j^\dagger a_j^2 \quad (\Delta=0)$$

consider twisted Hamiltonian $a_j \rightarrow a_j e^{-\beta j}$

$$(471) \quad H \rightarrow \tilde{H} = -t_1 e^{i\beta} \sum_{j \text{ even}} a_{j+1}^\dagger a_j - t_2 e^{i\beta} \sum_{j \text{ odd}} a_{j+1}^\dagger a_j + \frac{U}{2} \sum_j a_j^\dagger a_j^2$$

Symmetry

$$[\tilde{H}, \tilde{S}] = 0$$

(472)

$$\tilde{S} = I K$$

I in version about $j=0$, K - complex conjugation

$$(473) \quad |E_0(\beta)\rangle = e^{i\chi_\beta} \hat{S} |E_0(\beta)\rangle$$

$$\begin{aligned} A(\beta) &= i \langle E_0 | \partial_\beta E_0 \rangle = i \langle S E_0 | \partial_\beta S E_0 \rangle - \partial_\beta \chi_\beta \\ &= i \langle K E_0 | \partial_\beta K E_0 \rangle - \partial_\beta \chi_\beta = i \langle E_0^* | \partial_\beta E_0^* \rangle - \partial_\beta \chi_\beta \end{aligned}$$

(474)

$$= i \langle \partial_\beta E_0 | E_0 \rangle - \partial_\beta \chi_\beta$$

$$\begin{aligned} \phi_{MB} &= -\frac{1}{2i} \int_{-\pi}^{\pi} d\beta \left(\langle E_0 | \partial_{\beta} E_0 \rangle - \langle \partial_{\beta} E_0 | E_0 \rangle \right. \\ &\quad \left. \langle \partial_{\beta} E_0 | E_0 \rangle + i \partial_{\beta} \chi_{\beta} \right) \\ (475) \quad &= \frac{i}{2} \int_{-\pi}^{\pi} d\beta \partial_{\beta} \chi_{\beta} = \frac{i}{2} (\chi_{\pi} - \chi_{-\pi}) \end{aligned}$$

Since $\chi_\pi = \chi_{-\pi} \pmod{2\pi}$

(476) $\phi_{MB} = 0, \pi$

As in the fermionic SSF model the many-body Zak phase is either 0 or π

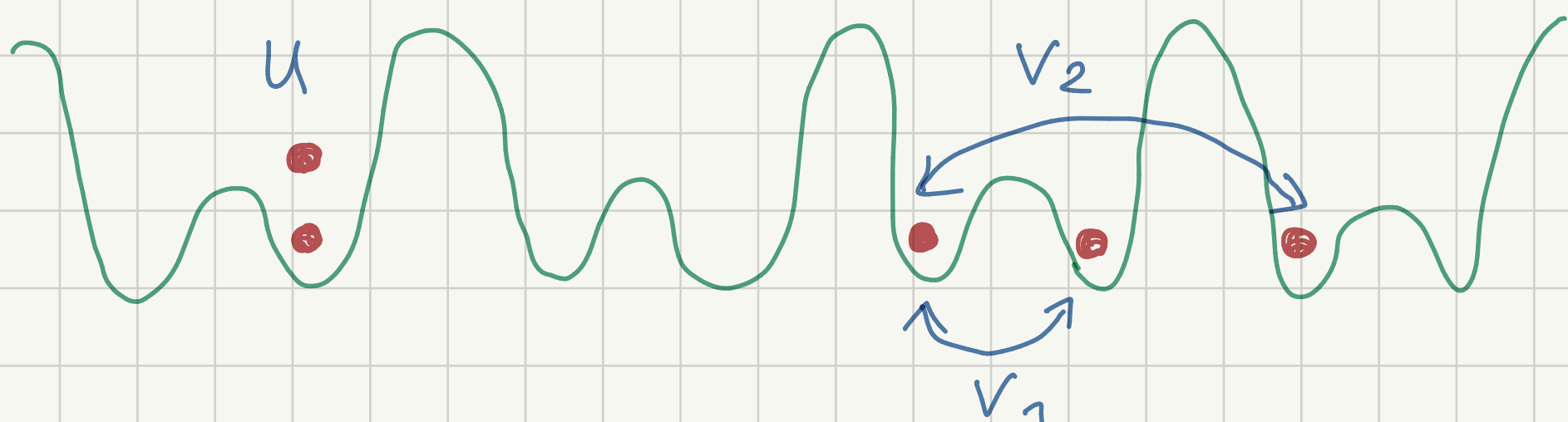
10.3 The extended SLBHM: fractional topological charges

As in the FQHE gapped phases with fractional topological charges can also appear in 1D lattice models:

For this we extend the SLBHM by non-local interactions

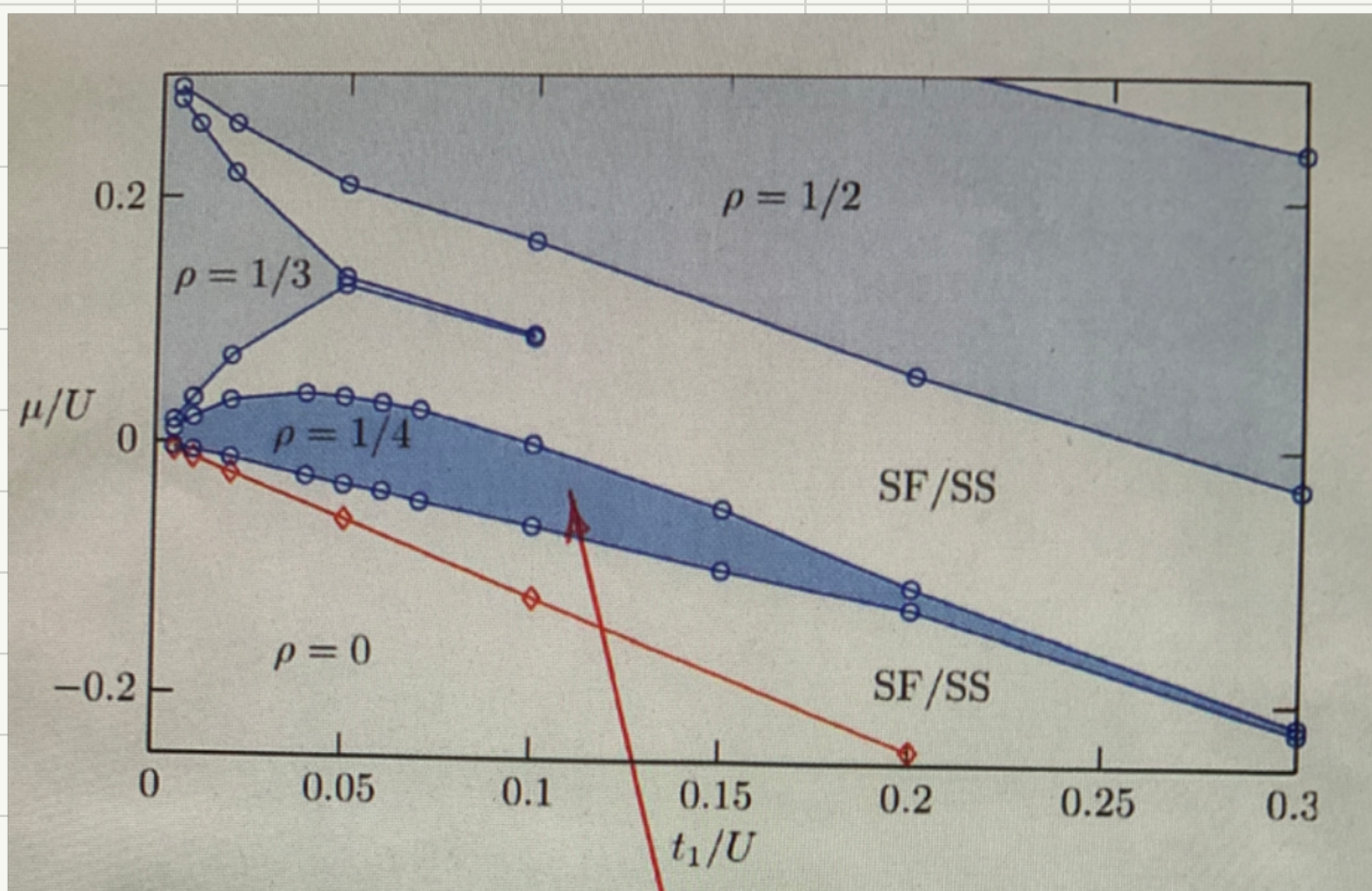
$$(477) \quad H = H_{\text{SLBHM}} + V_1 \sum_{\langle ij \rangle} \hat{n}_i \hat{n}_j + V_2 \sum_{\langle\langle ij \rangle\rangle} \hat{n}_i \hat{n}_j$$

NN interaction NNN interaction



(A) Fractional Mott insulator and fractional Thouless pump

phase diagram $\hat{H} - \mu \hat{N}$

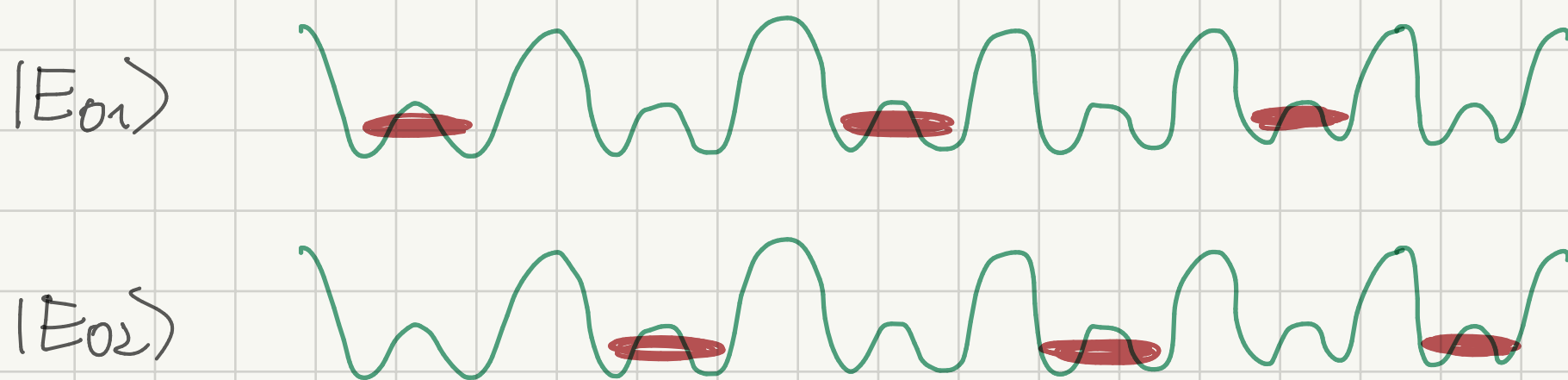


fractional MI

The MI phase at fractional filling is
degenerate

• atomic limit $t_2 \gg V_1, V_2 \gg t_1$

two density wave possible GS



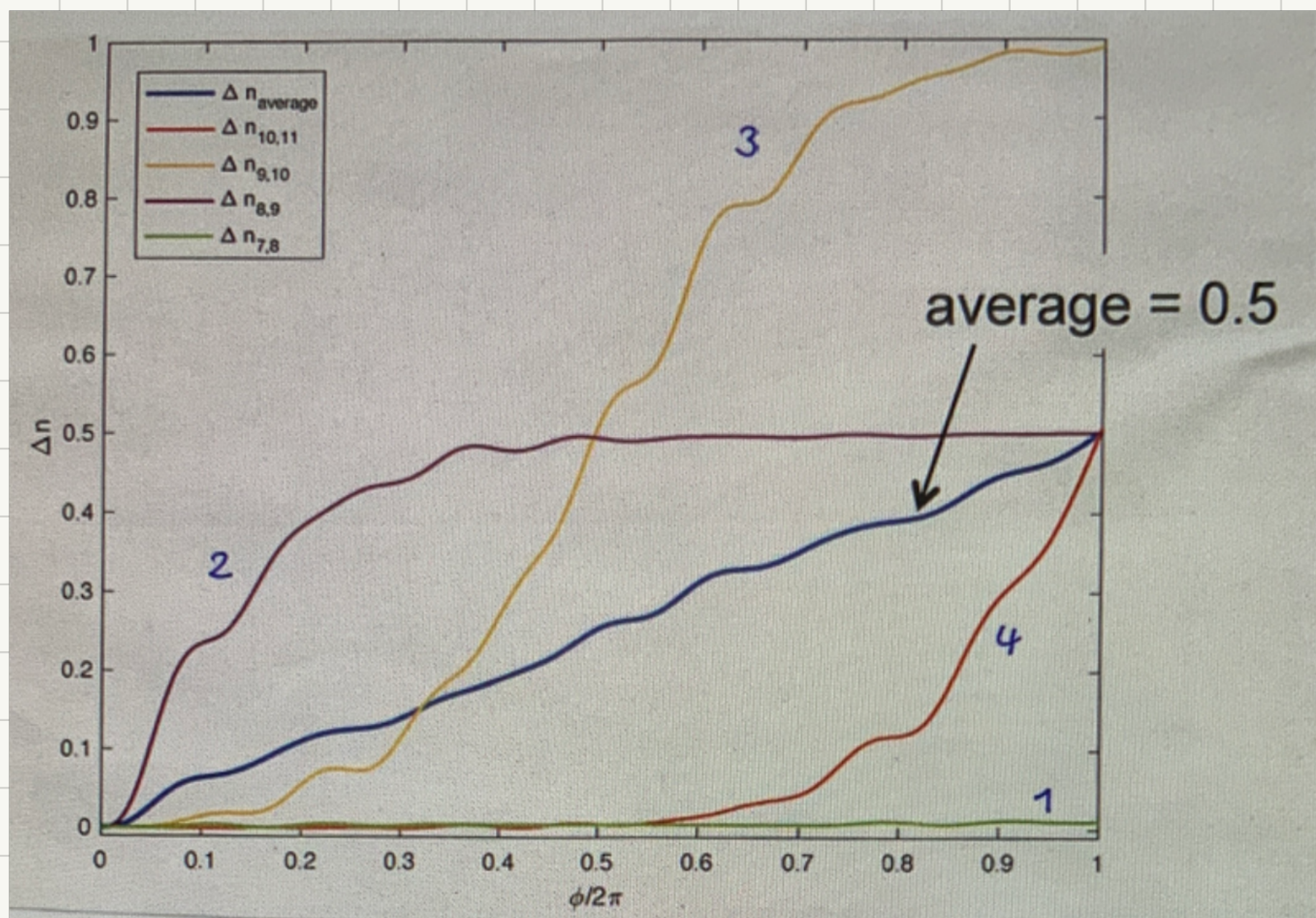
$$\langle E_{01} | E_{02} \rangle = 0$$

gapped many-body states with fractional filling
are degenerate and spontaneously break
a discrete symmetry (here lattice translation by
one unit cell) (Lieb-Schultz-Mattis)

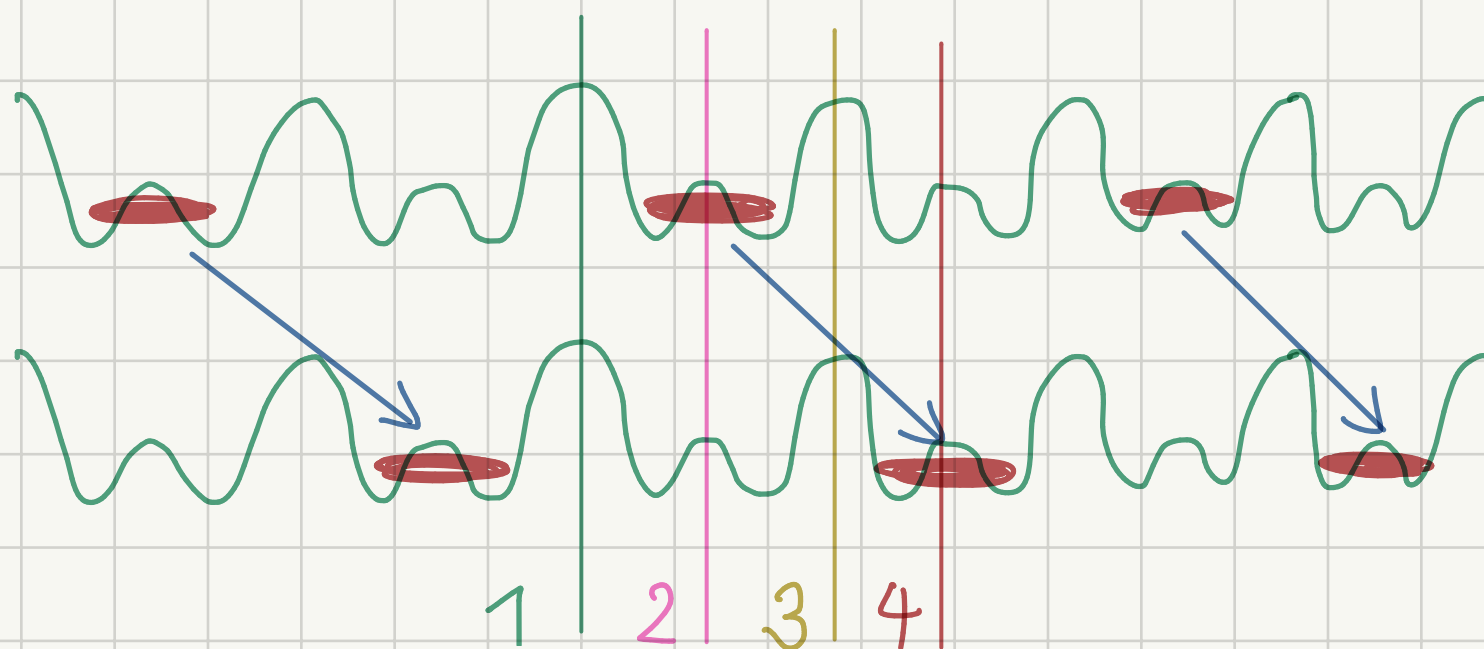
$\Rightarrow$ single cycle Thouless pump will in general not
return the bulk state into itself but

$$(478) \quad |E_{01}\rangle \longrightarrow e^{i\Delta\phi} |E_{02}\rangle \text{ possible}$$

p - degree of degeneracy: p Thouless cycles
needed to return to initial state



transport in atomic limit



Niu, Thouless, Wu PRB 1985

consider p -fold loop

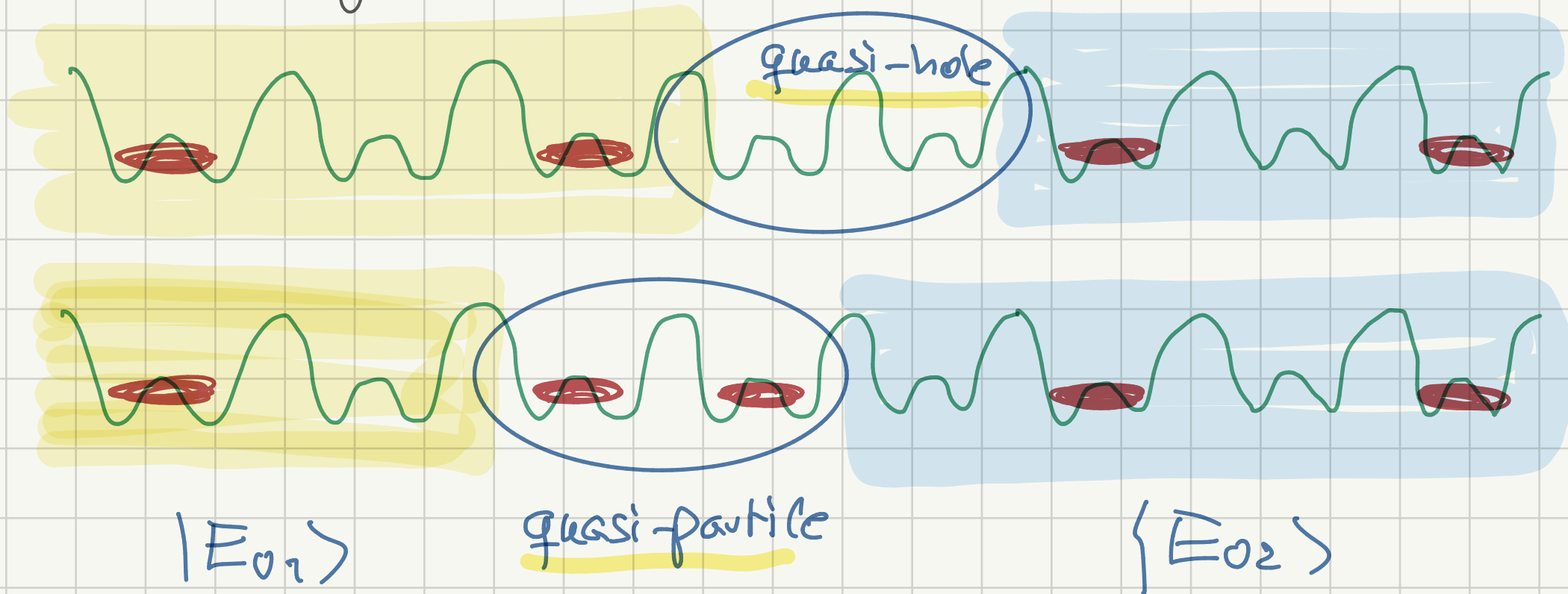
$$(479) \quad \nu = \frac{1}{2\pi} \int_0^{2\pi p} d\theta \operatorname{Im} \left\langle E_{0\mu} \left| \frac{\partial}{\partial \theta} E_{0\mu} \right. \right\rangle \quad \forall \mu=1,2,\dots,p$$

fractional transport (in one cycle)

$$(480) \quad \Delta n = \frac{1}{p} \nu$$

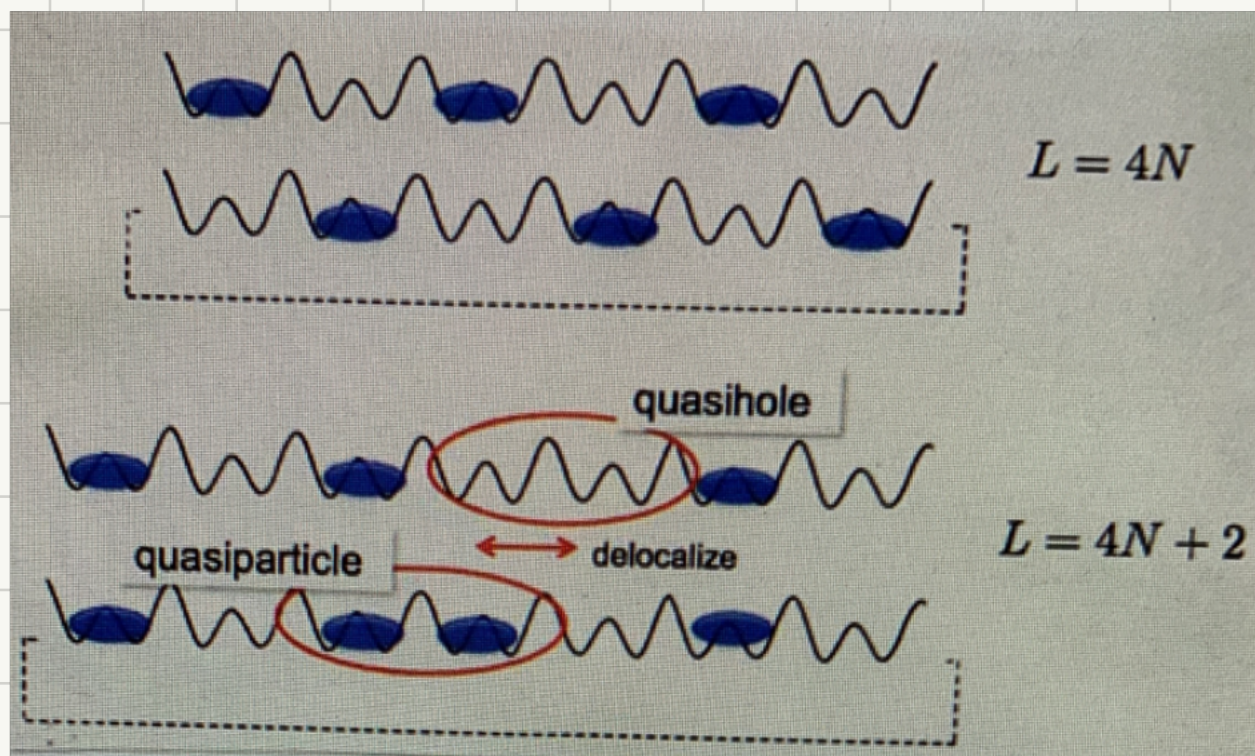
(B) topological excitations: domain walls

If there are degenerate ground states then low energy topological excitations exist at a spatial boundary between two different ground states

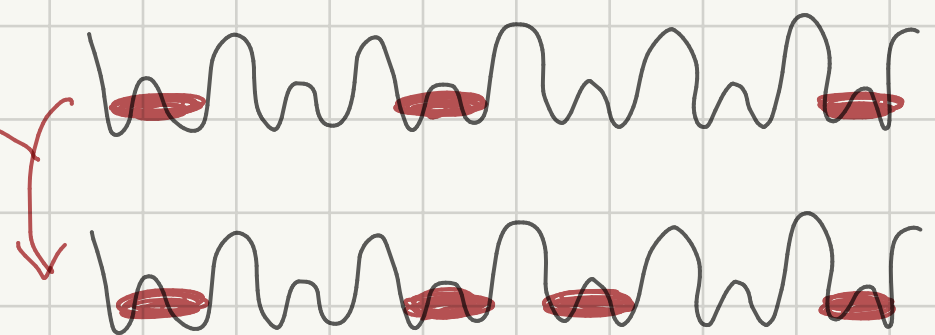
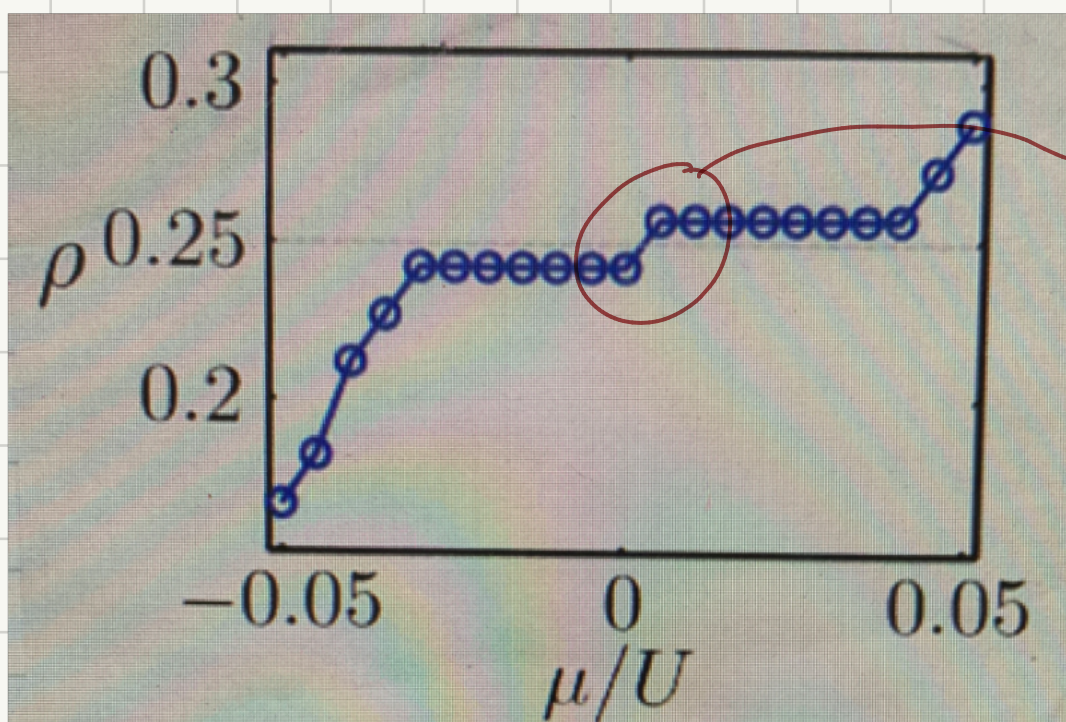


a quasi particle has an excitation energy $\sim \alpha V_1 + \beta V_2$

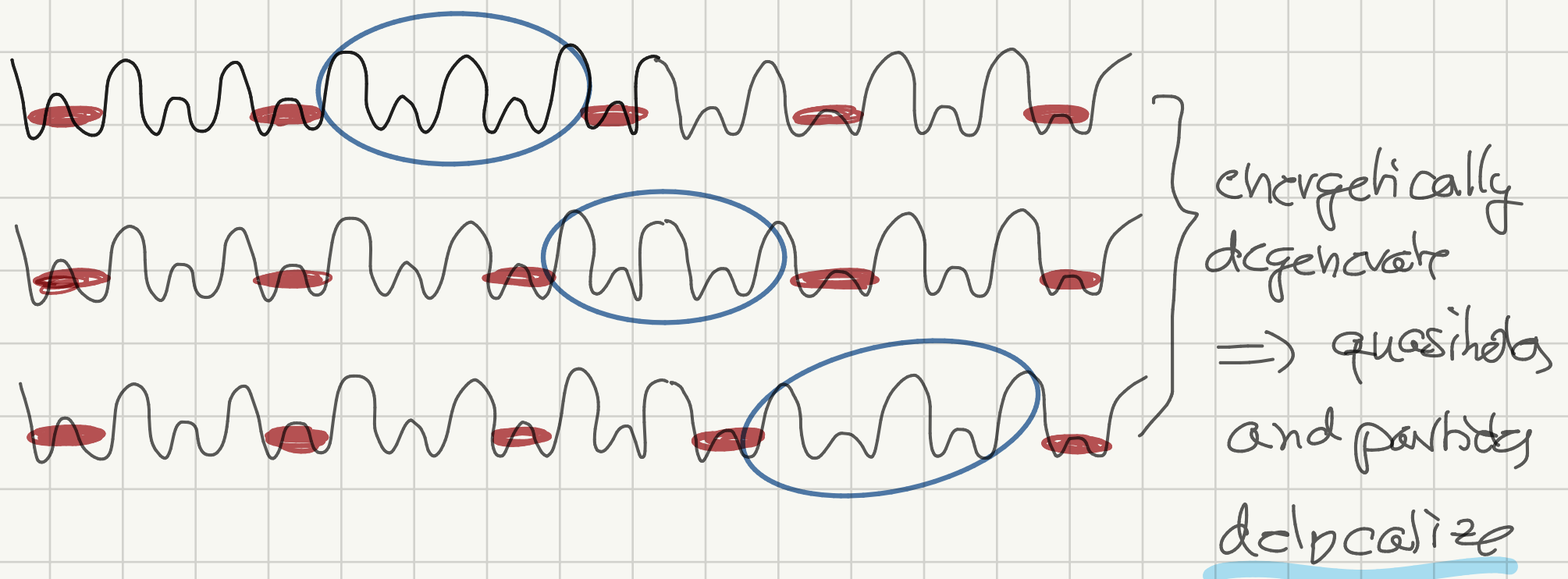
For a system with periodic boundary conditions the PBC may not be commensurate with the fractional filling and may enforce domain wall excitations



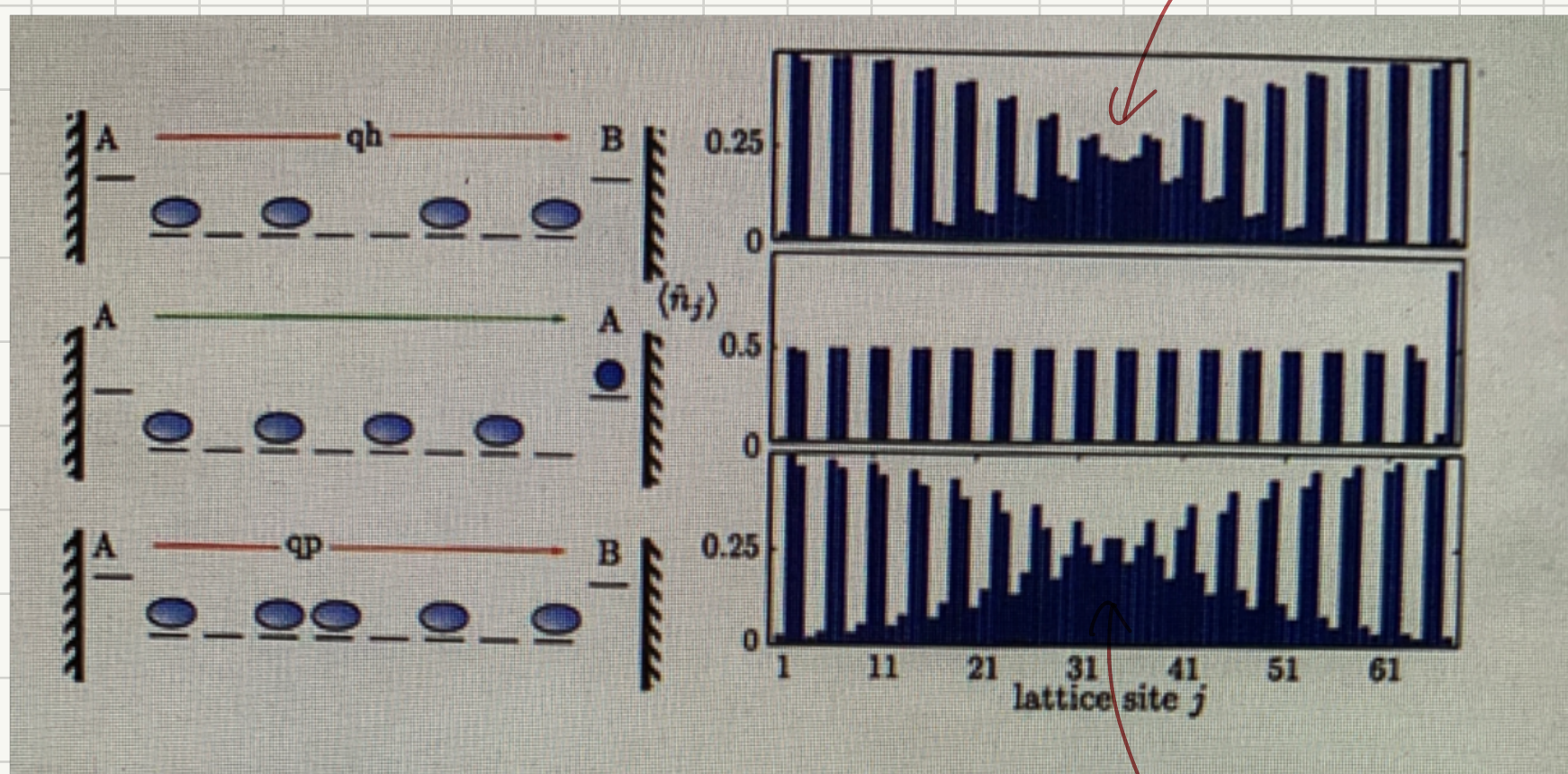
phase diagram in incommensurate case



domain walls are a 1D analogue of gh and qp in the FQHE



with open boundaries (DNRG)



10.4 Topological invariants for (quasi) degenerate systems

In the extended SCBHM we have seen that interaction-induced (topological) ground states

with fractional filling have degeneracies. This is a generic feature of systems with fractional topological charges. We thus need to discuss the generalization of topological invariants such as the single- or many-body Zak phase to systems with degeneracies, which also will give a justification for the NTW invariant, eq. 479.

(A) From Berry phase to Wilson loops

non degenerate state $|\phi(\lambda)\rangle$

Berry phase
(481)
$$\theta = \int d\lambda \langle \phi(\lambda) | \partial_\lambda \phi(\lambda) \rangle$$

This is connected with the $U(1)$ gauge freedom as discussed in Sect. 0.3. Apart from a dynamical phase an adiabatic change in parameter space yields

(482)
$$|\psi(t=0)\rangle = |\phi(\lambda=0)\rangle \rightarrow |\psi(t)\rangle = e^{i\theta(t)} |\phi(\lambda(t))\rangle$$

Wilczek and Lee (PRL 52, 2111 (1984)) have generalized this concept to the case of an n -fold degenerate (or partially degenerate) sub-space

which has a $U(n)$ gauge freedom since unitary transformations within the subspace are now possible, so

$$(483) \quad |\varphi(t=0)\rangle = |\phi_\alpha(\lambda=0)\rangle \rightarrow |\varphi(t)\rangle = \sum_{\beta} U_{\beta\alpha}(\lambda(t)) |\phi_\beta(\lambda(t))\rangle$$

where

$$(484) \quad \underline{U}(t) = T \exp \left\{ i \int_0^t d\tau \frac{d\lambda}{d\tau} \underline{A}(\lambda(\tau)) \right\}$$

$\underline{A}(\lambda)$ is now the (in general) non-Abelian
(i.e. matrix valued) Berry connection

$$(485) \quad A^{\alpha\beta} = i \left\langle \phi_\alpha(\lambda) \left| \frac{\partial}{\partial \lambda} \phi_\beta(\lambda) \right. \right\rangle \quad \alpha, \beta = 1, \dots, n$$

If we consider a closed loop in parameter space the exponential of the Berry phase

$$e^{i\Delta\theta} \quad \Delta\theta = \oint d\lambda \langle \phi(\lambda) | \partial_\lambda \phi(\lambda) \rangle$$

is replaced by the so-called Wilson loop

$$(486) \quad \underline{W} = P \exp \left\{ i \oint d\lambda \underline{A}(\lambda) \right\}$$

$n \times n$ matrix

$$(487) \quad \underline{\underline{W}}^\dagger \underline{\underline{W}} = \underline{\underline{1}}$$

The invariance of the (closed-loop) Berry phase under $U(1)$ gauge transformations (modulo 2π) translates into a $U(n)$ gauge invariance of the Wilson loop

$$(488) \quad \underline{\underline{W}} \rightarrow \underline{\underline{W}}' = \underline{\underline{U}}^\dagger \underline{\underline{W}} \underline{\underline{U}}$$

Note that $\underline{\underline{W}}' \neq \underline{\underline{W}}$ in general, but this is just the freedom of choice in the n -dimensional (degenerate) basis. Gauge invariant are

$$(489) \quad \boxed{\det \underline{\underline{W}}, \operatorname{tr} \underline{\underline{W}}, \text{eigenvalues}(\underline{\underline{W}})}$$

(B) Multi-band Chern number and Wilson loop

In the case of a non-degenerate ground state we found for the adiabatic transport

$$(490) \quad \Delta n = \frac{1}{2\pi} \int_0^T dt \frac{\partial}{\partial t} \int_{-\pi}^{\pi} d\beta \langle E_0 | \partial_\beta E_0 \rangle$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} dt \frac{\partial}{\partial t} \int_{-\pi}^{\pi} d\beta A(\beta, t)$$

where $A(\beta, t)$ is the 1+1D many-body Berry-connection.

If there are several (in some parameter points) degenerate ground states the derivation of $\Delta \bar{n}$ must be revised!

We have to choose a set of (instantaneous) eigenstates

$$|E'_{0\mu}\rangle$$

for which

$$(491) \quad \langle E'_{0\nu} | \frac{\partial \mathcal{H}}{\partial \beta} | E'_{0\mu} \rangle \sim \langle E'_{0\nu} | \partial_{\beta} E'_{0\mu} \rangle \sim \delta_{\mu\nu}$$

This is in principle possible, since the non-Abelian Berry connection is a hermitian matrix

$$(492) \quad A_{\mu\nu} = i \langle E'_{0\nu} | \partial_{\beta} E'_{0\mu} \rangle = A_{\mu\nu}^*$$

The transport in one of these eigenstates in a single loop of a Thouless pump is however

not necessarily quantized since after one loop the state $|E_{0\mu}\rangle$ may not return to itself!

However if we consider the total transport in all degenerate bands

$$(493) \quad \Delta \bar{h} = \frac{1}{2\pi} \int_0^T dt \frac{\partial}{\partial t} \sum_{\mu=1}^n \int_{-\pi}^{\pi} d\beta \, i \langle E_{0\mu} | \partial_{\beta} E_{0\mu} \rangle$$

the situation is different, as then

$$(494) \quad \Delta \bar{h} = \frac{1}{2\pi} \int_0^T dt \frac{\partial}{\partial t} \int_{-\pi}^{\pi} d\beta \, \text{Tr} \{ A_{\mu\nu} \}$$

and the trace (in n dimensional subspace) is invariant under all unitary trafo's within the degenerate subspace.

So we can introduce a non-Abelian generalization of the Zak-phase winding

(495)

$$C = \nu = \frac{1}{2\pi} \oint \frac{\partial}{\partial \tau} \int_{-\pi}^{\pi} d\beta \, \text{Tr} \{ \underline{A}(\beta) \}$$

total Chern number

We can express C in terms of the Wilson loop

$$\underline{W} = P \exp \left\{ i \int_{-\pi}^{\pi} d\beta \underline{A}(\beta) \right\} \quad P: \text{path ordering for } \beta$$

$$\text{Tr} \int_{-\pi}^{\pi} d\beta \underline{A}(\beta) = \text{Im} \text{Tr} \ln(\underline{W}) \quad \text{and using}$$

$$(496) \quad \boxed{\text{Tr} \ln(\underline{W}) = \ln \det(\underline{W})}$$

We find for (495):

$$(497) \quad \boxed{C = \frac{1}{2\pi} \oint d\lambda \frac{\partial}{\partial \lambda} \text{Im} \ln(\det \underline{W})} \quad \lambda = 2\pi i$$

for a 1D system with λ -periodic (time-periodic) Hamiltonian

Since $\det \underline{W}$ is gauge invariant under $U(1)$ gauge trans, so is C .

(C) Physical interpretation of Wilson loop for non-interacting systems

To get some physical insight into the Wilson loop let's go back to non-interacting systems:

- In the non-degenerate situation we have found the King-Smith Vanderbilt relation between Zak phase and center of mass of Wannier functions

$$(498) \quad \langle w_n | \hat{X} | w_n \rangle = \frac{1}{2\pi} \int_{BZ} dk A(k) = \frac{i}{2\pi} \int_{BZ} dk \langle u_n(k) | \partial_k u_n(k) \rangle$$

where $A(k)$ is the single-particle Berry connection.

- p -fold degeneracy (n = Bloch index)

$$(499) \quad |u_n(k)\rangle \longrightarrow |u_n^\alpha(k)\rangle \quad \alpha = 1, 2, \dots, p$$

$$A(k) \longrightarrow \tilde{A}_{\alpha\beta}(k) = i \langle u_n^\alpha(k) | \partial_k u_n^\beta(k) \rangle$$

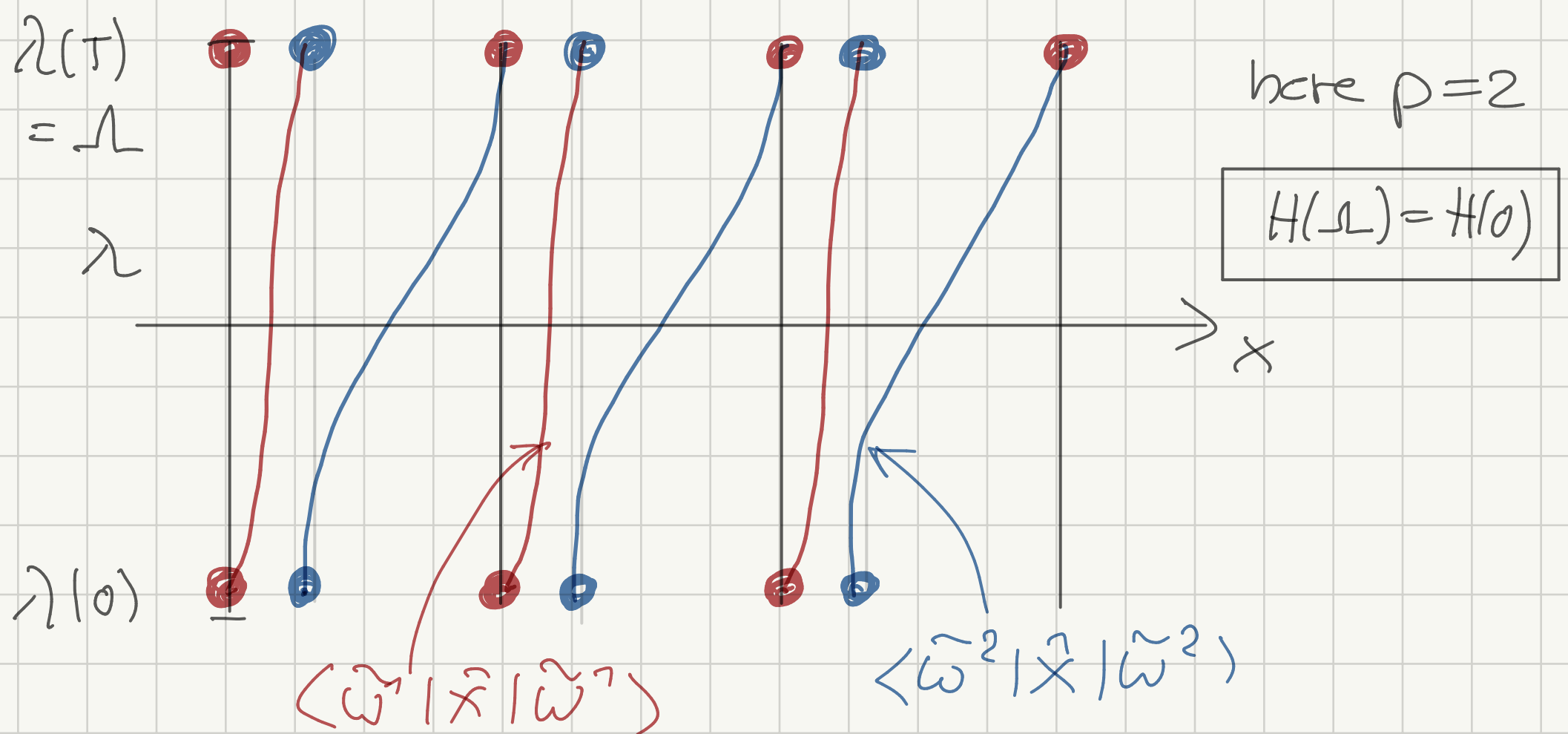
let us now use a basis

$$|\tilde{u}_n^\alpha(k)\rangle$$

in which $\tilde{A}_{\alpha\beta} \sim \delta_{\alpha\beta}$ is diagonal. Then

$$(500) \langle \tilde{\omega}_n^\alpha | \hat{X} | \tilde{\omega}_n^\alpha \rangle = \frac{i}{2\pi} \int d\mu \langle \tilde{u}_n^\alpha(\mu) | \partial_\mu \tilde{u}_n^\alpha(\mu) \rangle$$

As noted above, if we adiabatically change the parameter λ of a 1D Hamiltonian $H(\lambda)$ in a single period, the Wannier states may not return to themselves and Wannier centers may exchange places

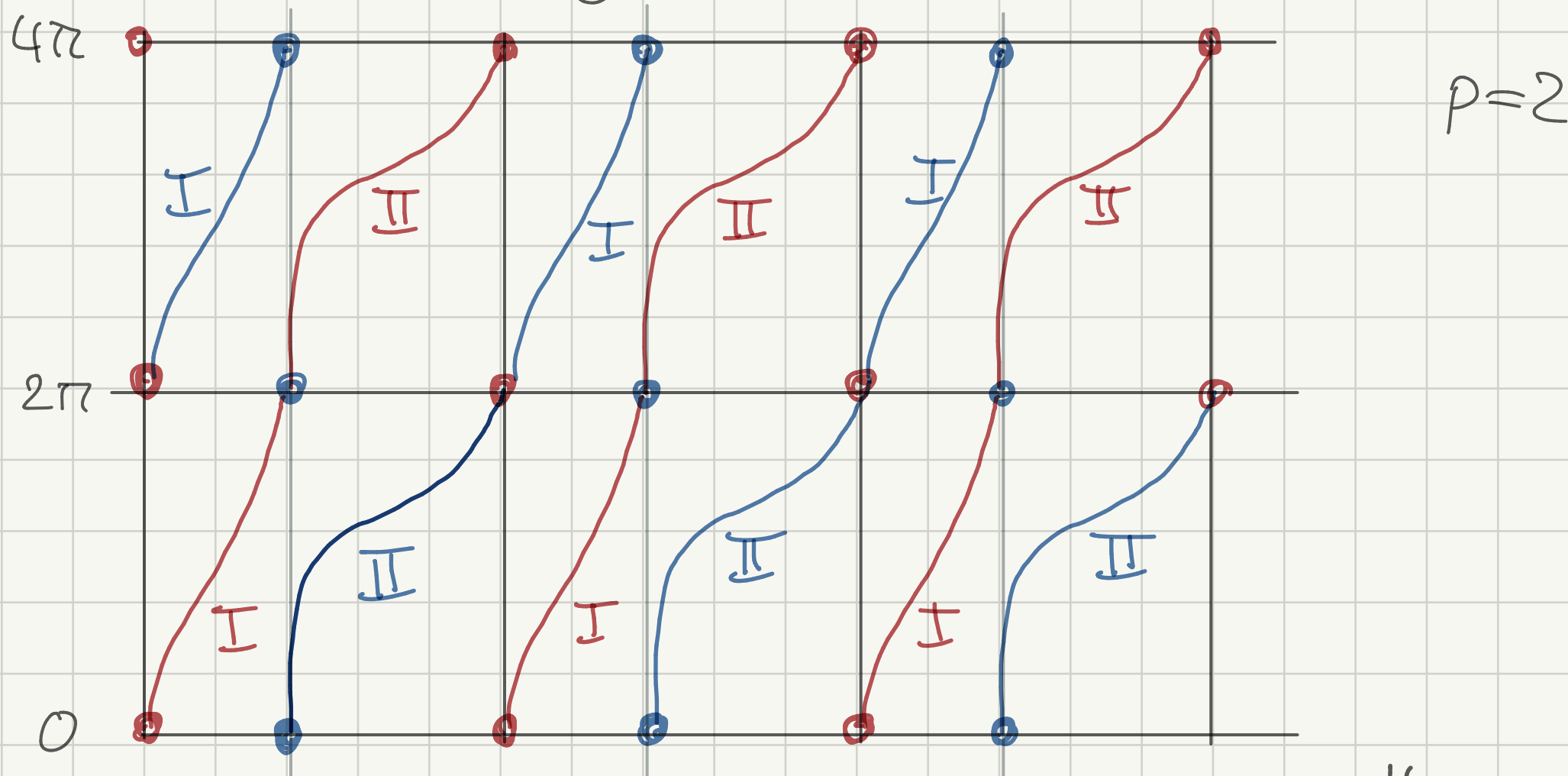


After one cycle of the pump the Wannier centers have to either return to their own positions modulo a lattice spacing or to exchange places.

After p cycles the Wannier centers must

come back to their initial places modulo integer number of lattice constants. Thus

$$(501) \quad \int_0^1 d\lambda \frac{\partial}{\partial \lambda} \langle \tilde{\omega}^\alpha | \hat{x} | \tilde{\omega}^\alpha \rangle = \int_0^1 d\lambda \frac{\partial}{\partial \lambda} \langle \tilde{\omega}^\beta | \hat{x} | \tilde{\omega}^\beta \rangle \quad \forall \alpha, \beta$$



exchange of Wannier particles after one cycle

Thus

$$(502) \quad C = \frac{1}{2\pi} \int_0^1 d\lambda \frac{\partial}{\partial \lambda} \int_{-\pi}^{\pi} d\beta \operatorname{Tr} \{ \underline{A}(\beta) \} = \frac{1}{2\pi} \int_0^1 d\lambda \frac{\partial}{\partial \lambda} \cdot \int_{-\pi}^{\pi} d\beta \sum_{\alpha=1}^P \tilde{A}^{\alpha\alpha}(\beta)$$

$$(503) \quad C = \frac{1}{2\pi} \int_0^{L_P} d\lambda \frac{\partial}{\partial \lambda} \int_{-\pi}^{\pi} d\beta A^{\alpha}(\beta) \quad \forall \alpha$$

total Chern number

(D) Wilson loop and many-body polarization

The relation between Wannier centers and many-body polarization must now be generalized to (partially) degenerate sub-bands. For $H(\lambda=L) = H(\lambda=0)$

$$(504) \quad C = \frac{1}{2\pi} \int_0^L d\lambda \frac{\partial}{\partial \lambda} \text{Im} \ln \det \underline{W}$$

As shown by Aligia and Ortiz PRL 82, 2560 (1999)

this can be written in terms of a generalized polarization.

$$(505) \quad C = \frac{1}{2\pi} \int_0^L d\lambda \frac{\partial}{\partial \lambda} \text{Im} \ln \langle \psi_0^{\alpha} | e^{\frac{2\pi i}{L} \hat{X}_P} | \psi_0^{\alpha} \rangle$$

$\forall \alpha = 1, 2, \dots, P$

Where the momentum shift operator is replaced

(506)

$$e^{\frac{2\pi i}{L} \hat{x}} \rightarrow e^{\frac{2\pi i}{L} \hat{x}_p}$$

- One can construct a complete set of states in the p -fold degenerate subband with

(507)

$$|\psi_0^\alpha\rangle = \hat{S}^\alpha |\psi_0^0\rangle \quad \alpha=1,2,\dots,p$$

where $\hat{S}$ is the lattice shift operator by one unit cell

$$(508) \quad \hat{S}^{-1} e^{\frac{2\pi i}{L} \hat{x}} \hat{S} = e^{\frac{2\pi i}{L} \sum_j (j+1) a_j^\dagger a_j} \\ = e^{\frac{2\pi i}{L} \hat{x}} e^{\frac{2\pi i}{L} N}$$

We see that

$$\langle \psi_0^\alpha | e^{\frac{2\pi i}{L} \hat{x}_p} | \psi_0^\alpha \rangle = \langle \psi_0^0 | \hat{S}^{-1} e^{\frac{2\pi i}{L} \hat{x}_p} \hat{S} | \psi_0^0 \rangle \\ = \langle \psi_0^0 | e^{\frac{2\pi i}{L} \hat{x}_p} e^{\frac{2\pi i}{L} N p} | \psi_0^0 \rangle$$

Since the ground state has filling $\frac{N}{L} = \frac{1}{p} \Rightarrow$

$$e^{\frac{2\pi i}{L} N p} = 1 \quad \text{and (505) applies to any of}$$

the degenerate ground states.

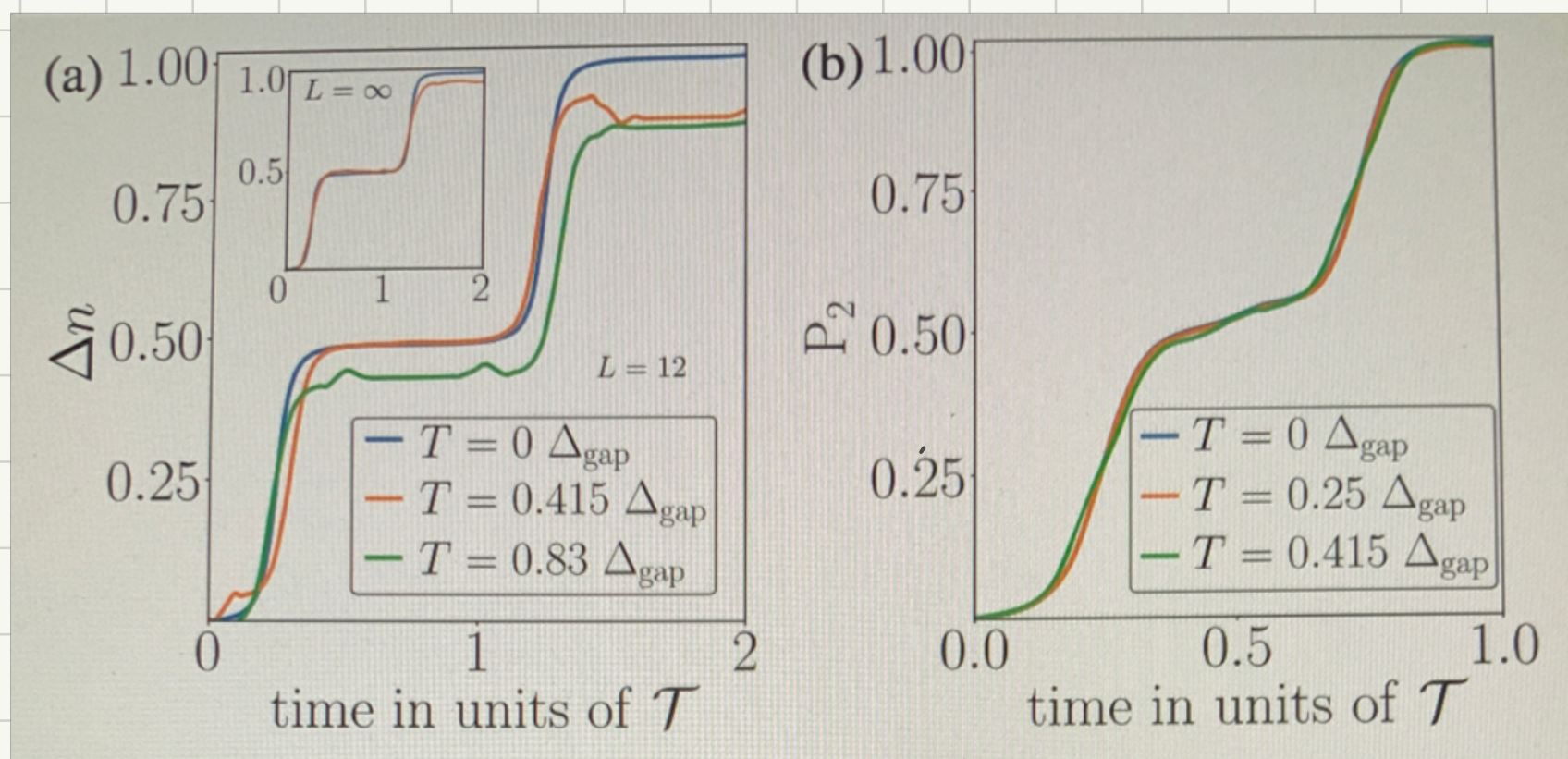
(E) Finite-temperature topological invariant for degenerate systems

Eq. (505) can immediately be extended to mixed state to define a generalized ensemble geometric phase for interacting systems

$$(509) \quad \phi_{EGP}^{(p)} = \text{Im} \ln \text{Tr} \left\{ \rho e^{\frac{2\pi i}{L} \hat{X}^p} \right\}$$

whose winding is a topological invariant.

Example: finite- T Ex SLBHM $v = \frac{1}{2}$ ($p=2$)



$P_2 = \frac{1}{2\pi} \phi_{EGP}^{(2)}$ from Unanyan et al. PRL 125 215701 (2020)

(a) particle transport

(b) EGP winding