10. Topological lattice models with interactions and fractional Chern insulators

10.1 Rice-Mele model with interactions

Superlattice Bose-Hubbard model

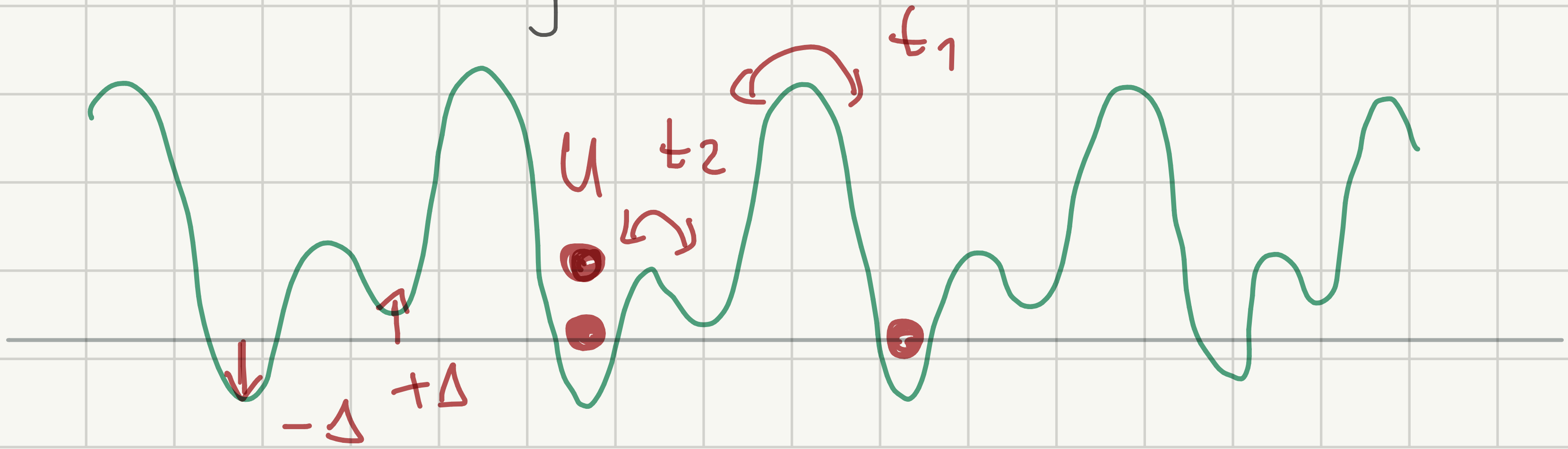
We first consider the simplest topological models in 1D, the SSH and Rice-Mele model with onsite interactions for bosonic particles

Superlattice Bose-Hubbard model (SLBHM)

RM
$$H = -t_1 \sum_{j\,\mathrm{even}} \left(a^\dagger_{j+1} a_j + h.a.\right) - t_2 \sum_{j\,\mathrm{odd}} \left(a^\dagger_{j+1} a_j + h.a.\right) + \Delta \sum_j (-1)^j a^\dagger_j a_j$$

(436)

repulsive interaction
$$+ \frac{U}{2} \sum_j a^{\dagger 2}_j a^2_j$$



The single particle Hamiltonian has chiral symmetry for $\Delta = 0$ and non-trivial winding of $\mathbb{Z}$-th phase when encircling the origin. The SSH model ($\Delta=0$) has edge states. What survives with interactions?

Let us consider the grand canonical setting

(437) $$K = H - \mu N \qquad N = \sum_j a_j^+ a_j$$

(A) Hardcore limit and mapping to fermionic RMM

hardcore limit $U \to \infty$ $\Rightarrow$ only $n=0$ and $n=1$ particles per lattice site

$$\frac{U}{2} a^{+2} a^2 = \frac{U}{2} a^+ a (a^+ a - 1)$$

hard core bosons = spin 1/2 particle

(438) $$\boxed{\begin{aligned} &[a_i, a_j^+] = \delta_{ij} \qquad a_j^{+2} = a_j^2 = 0 \\ &a_i = \sigma_i^- \qquad a_i^+ = \sigma_i^+ \end{aligned}}$$

Wigner-Jordan mapping to fermions

(439) $$\boxed{\sigma_\ell^- = \exp\left\{-i\pi \sum_{j<\ell} c_j^\dagger c_j\right\} c_\ell}$$

non-local transformation

(440) $$\{c_j, c_\ell^\dagger\} = \delta_{\ell j} \qquad \{c_j, c_\ell\} = 0$$

check: $\ell' < \ell$

$$\sigma_\ell^- \sigma_{\ell'}^- = e^{-i\pi\sum_{j<\ell} c_j^\dagger c_j} c_\ell \, e^{-i\pi\sum_{j<\ell'} c_j^\dagger c_j} c_{\ell'}$$

$$= e^{-i\pi\sum_{j=\ell'}^{\ell-1} c_j^\dagger c_j} c_\ell c_{\ell'} = -e^{-i\pi\sum_{j=\ell'}^{\ell-1} c_j^\dagger c_j} c_{\ell'} c_\ell$$

$$\sigma_{\ell'}^- \sigma_\ell^- = e^{-i\pi\sum_{j<\ell'} c_j^\dagger c_j} c_{\ell'} \, e^{-i\pi\sum_{j<\ell} c_j^\dagger c_j} c_\ell$$

$$= e^{-i\pi\sum_{j=\ell'+1}^{\ell} c_j^\dagger c_j} \underbrace{c_{\ell'} e^{-i\pi c_{\ell'}^\dagger c_{\ell'}}}_{= e^{-i\pi(c_{\ell'}^\dagger c_{\ell'}+1)} c_{\ell'}} c_\ell$$

$$= e^{-i\pi} e^{-i\pi\sum_{j=\ell'}^{\ell} c_j^\dagger c_j} c_{\ell'} c_\ell = \sigma_\ell^- \sigma_{\ell'}^-$$

$$\Rightarrow \quad [\sigma_\ell^-, \sigma_{\ell'}^-] = 0 \quad \square$$

- Wigner-Jordan trafo of Hamiltonian in hard-core limit

$$a_\ell^\dagger a_\ell = \sigma_\ell^+ \sigma_\ell^- = c_\ell^\dagger e^{+i\pi\sum_{j<\ell} c_j^\dagger c_j} e^{-i\pi\sum_{j<\ell} c_j^\dagger c_j} c_\ell$$

(440) $$= c_\ell^\dagger c_\ell$$

$$a^\dagger_{\ell+1} a_\ell = c^\dagger_{\ell+1}\, e^{\pi i \sum_{j<\ell+1} c_j^\dagger c_j}\, e^{-i\pi \sum_{j<\ell} c_j^\dagger c_j}\, c_\ell$$

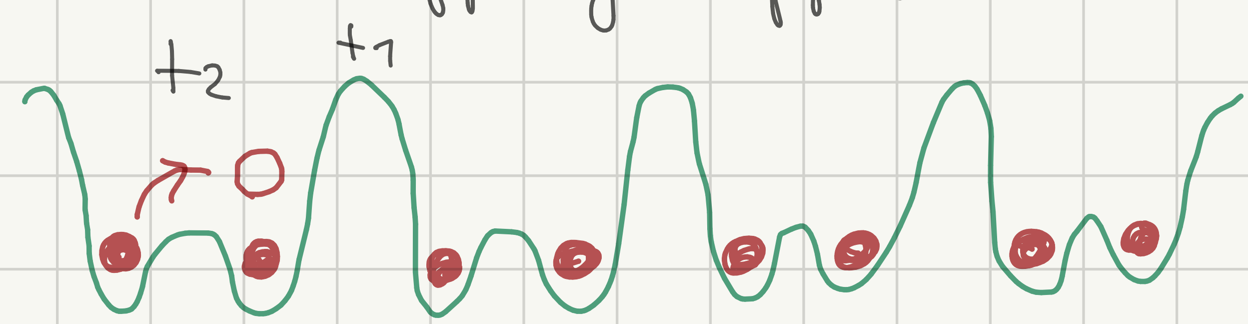
(441) $$= \underbrace{c^\dagger_{\ell+1}\, e^{i\pi c^\dagger_{\ell+1} c_{\ell+1}}}_{c^{\dagger\, n>1}_{\ell+1} = 0} c_\ell = c^\dagger_{\ell+1} c_\ell$$

SLBHM in hard core limit is isomorph to the fermionic RMM $\Rightarrow$ identical topological features

(B) Going away from the hard-core limit

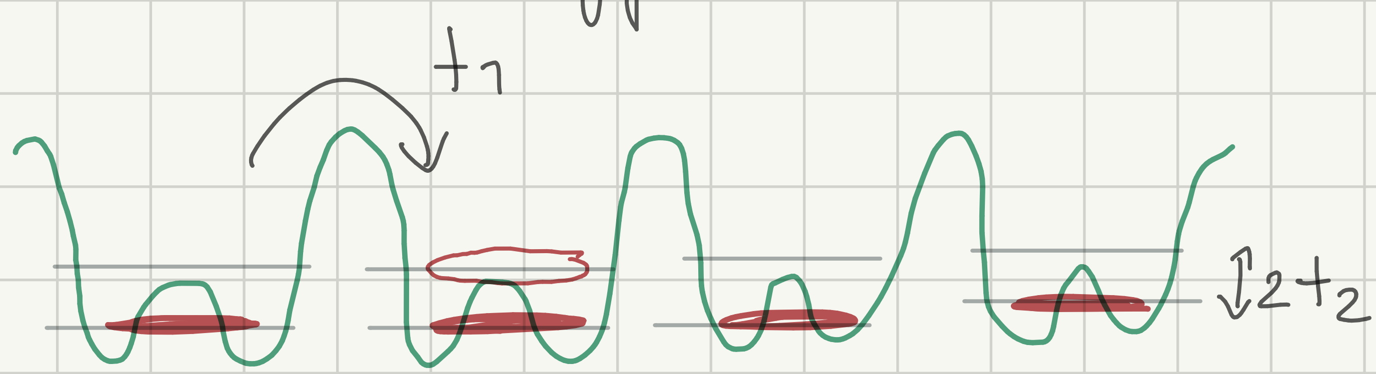
- small t/U Mott insulator (MI)
  hopping suppressed

$\nu = 1$



hopping costs interaction energy U
gains kinetic energy t_2

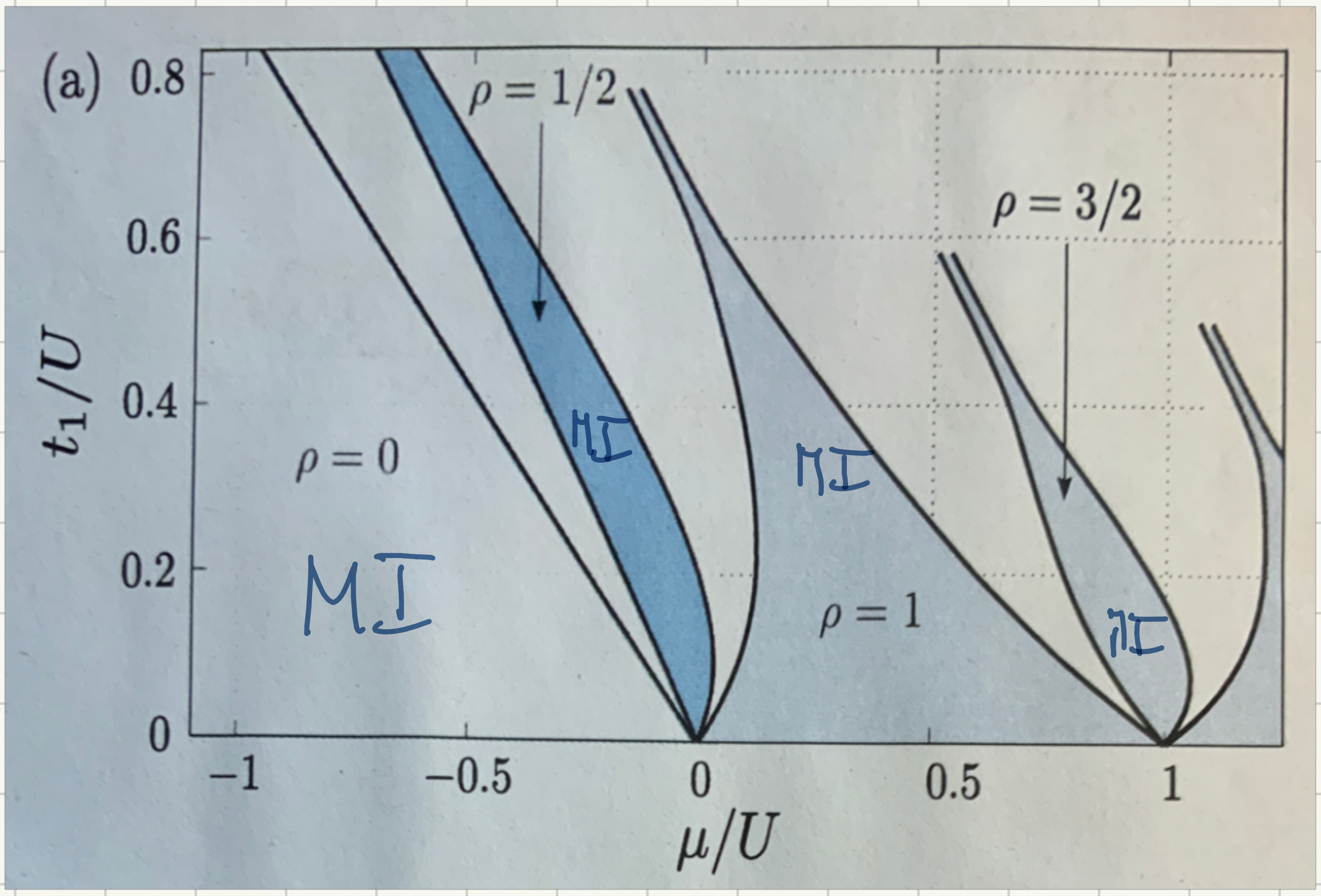
$\nu = 1/2$



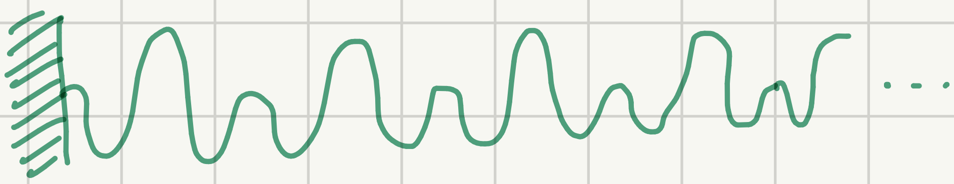
hopping costs energy $2t_2$ gains energy $\sim t_1 < t_2$

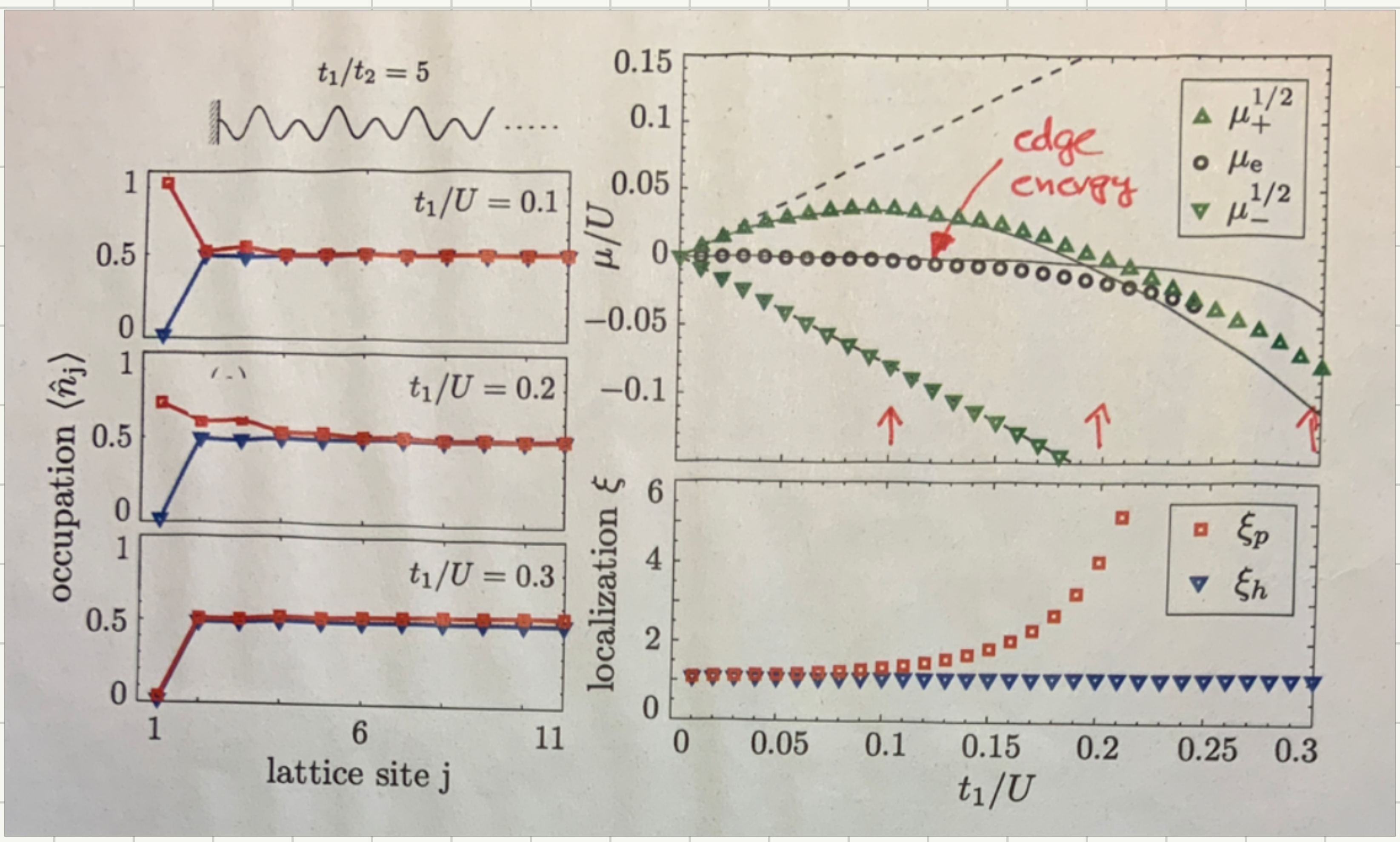
- large t/U motion allowed superfluid phase (SF)

phase diagram for $\Delta = 0$



edge state in topologically nontrivial phase of SSH





→ edge state no longer at zero energy ⇒ absence of PHS!

(442) $$\mu_e \simeq -2t_2^2 \frac{U-2t_1}{(U+t_1)(U-3t_1)}$$

chemical potential of edge

$\mu < \mu_e$ edge empty

$\mu > \mu_e$ edge occupied

10.2 Many-body topological invariant

For finite interactions the single particle lattice momentum is no longer a good quantum number

$\Rightarrow$ Bloch function Zak phase not meaningful

Q. Niu, D. Thouless J. Phys A 17, 2453 (1984)

For noninteracting gapped fermion systems a topological invariant is the winding of the Zak phase related to particle transport and a winding of the polarization

(443) $$\Delta n = \nu = \frac{1}{2\pi} \oint d\lambda \frac{\partial \phi_{ZAK}}{\partial \lambda} = \Delta P$$

(444) $$P = \frac{1}{2\pi} \operatorname{Im} \ln \langle \exp(\tfrac{2\pi i}{L} \hat{x}) \rangle$$

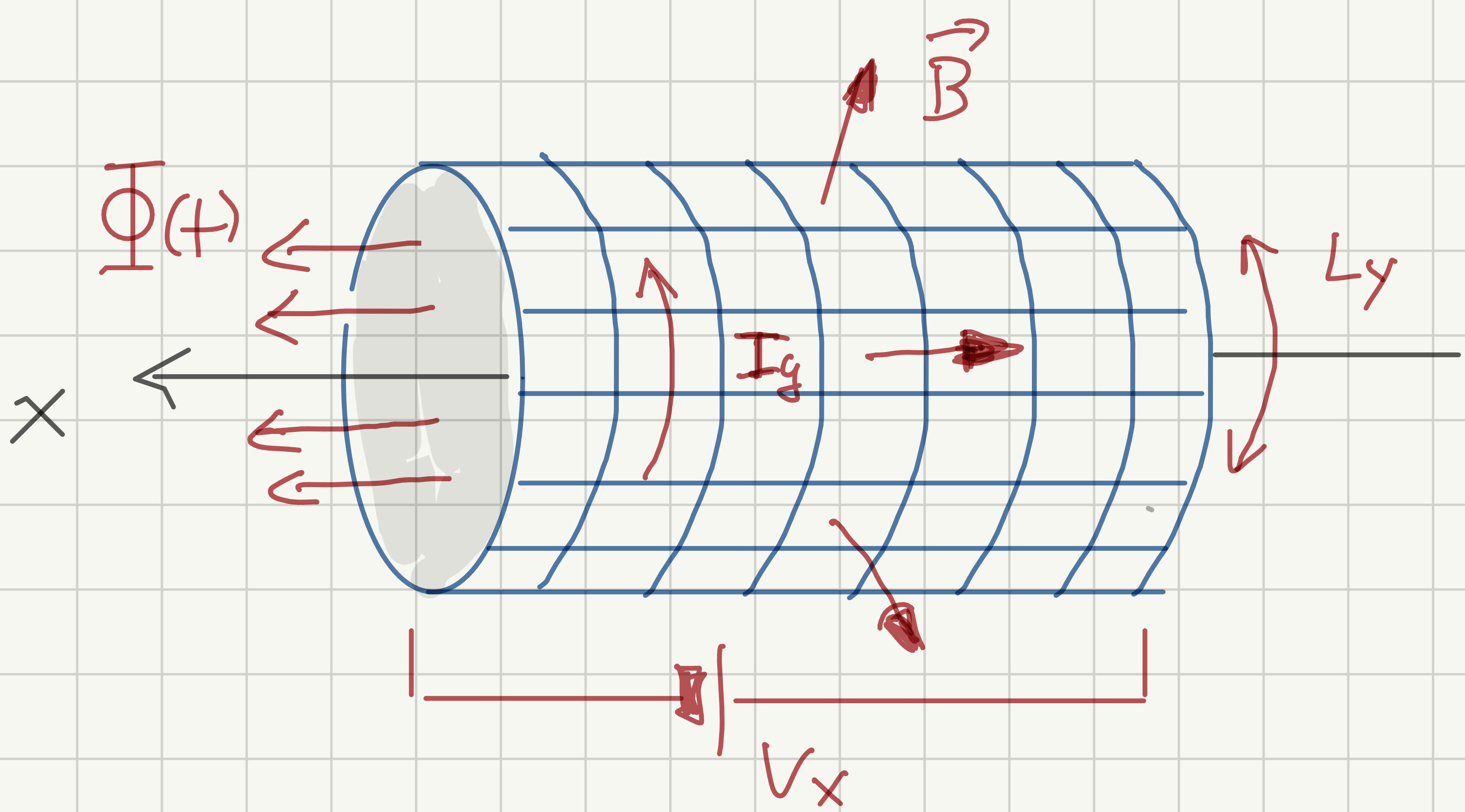
Δn and ΔP remain well defined also for interacting particles. Following Niu and Thouless one can define a

many-body Zak phase

To understand this many-body generalization let us first look at non-interacting systems again and discuss an argument by Laughlin

(A) Laughlins flux insertion

Consider a lattice wrapped to a cylinder i.e. with periodic boundary conditions



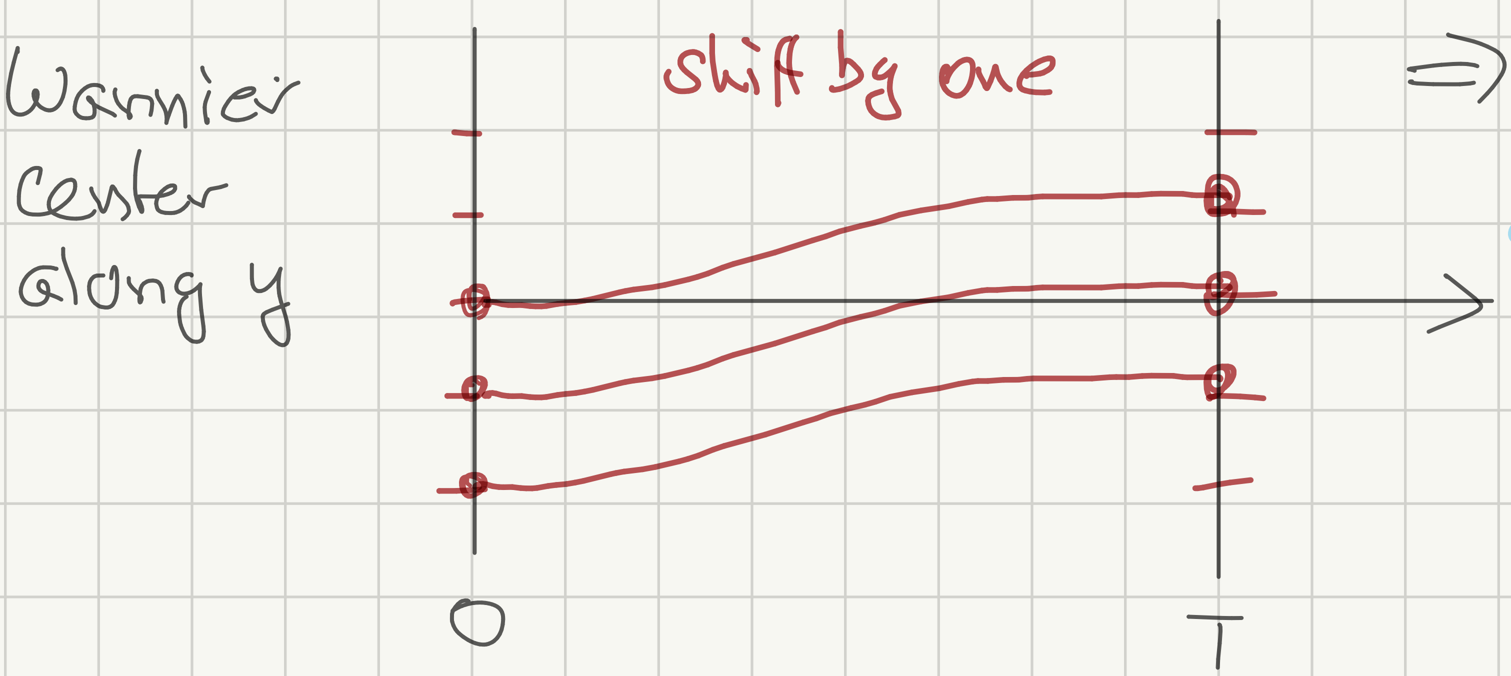
- periodic in y $\quad k_y$ quantized $\quad k_y = m \frac{2\pi}{L_y}$
- adiabatic flux insertion through the tube

(445) $\vec{A} \to \vec{A} + \frac{\Phi(t)}{L_y}\vec{e}_y$

so $k_y \to$

(446) $$\boxed{k_y(t) = k_y + \frac{\Phi(t)\, e}{L_y \hbar c} = k_y + \frac{2\pi}{L_y}\frac{\Phi(t)}{\Phi_0}}$$

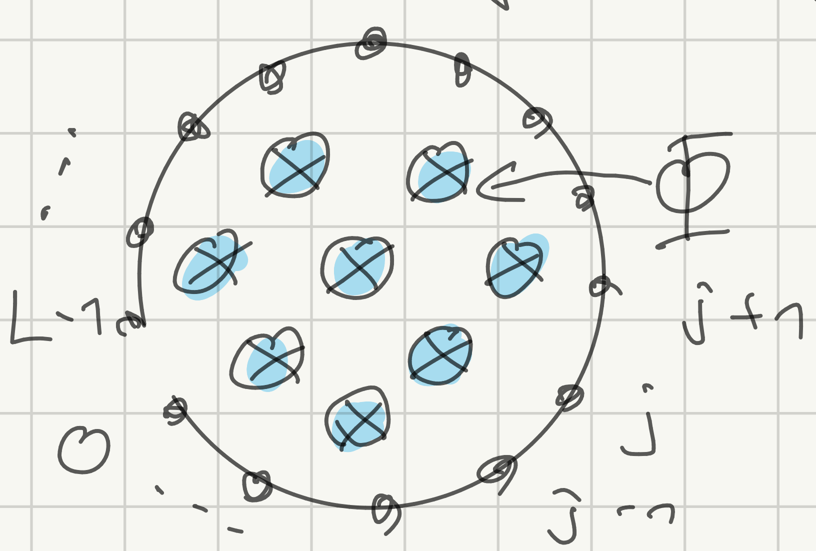
i.e. if we insert one flux quantum all lattice momenta increase by one unit



$\Rightarrow$ Hall response by flux insertion

Flux insertion and boundary twist along y

The gauge potential (445) corresponds to a Peierls phase along the y-direction



(447) $$c^\dagger_{j+1}c_j \to c^\dagger_{j+1}c_j\, e^{i\alpha}$$

$$\alpha = \frac{2\pi}{L}\frac{\Phi}{\Phi_0}$$

Hamiltonian with periodic boundary conditions

(448) $$H_y = -t \sum_{j=0}^{L-1} c_{j+1}^\dagger c_j e^{i\frac{2\pi}{L}\frac{\Phi}{\Phi_0}} + h.c.$$

(449) $$c_L = c_0 \qquad \text{PBC}$$

With a conical transformation

(450) $$\tilde{c}_j = c_j e^{i\alpha j}$$

the Peierls phase in (448) can be removed

(451) $$\tilde{H}_y = -t \sum_{j=0}^{L-1} \tilde{c}_{j+1}^\dagger \tilde{c}_j$$

but the boundary conditions become twisted

(452) $$\boxed{\tilde{c}_L = \tilde{c}_0 e^{i2\pi\frac{\Phi}{\Phi_0}}} \qquad \text{TBC}$$

So Laughlin's flux insertion is equivalent to adiabatically changing the twist angle in the twisted BC from 0 to 2π!

A twist of boundary conditions can be transferred to interacting system where the lattice momentum k_y is no good quantum number.

(B) The Niu-Thouless Wu invariant

consider interacting N-particle Hamiltonian

(453) $$H(t) = \underbrace{\sum_{j=1}^{N}\left(-\frac{1}{2}\frac{\partial^2}{\partial x_j^2} + V(x_j,t)\right)}_{\text{free part}} + \underbrace{\sum_{i>j}^{N} U(x_i - x_j)}_{\text{interactions}}$$

where $V(x_j,t)$ is a time periodic potential.

Twisted boundary conditions for the N-particle wavefunction would read

(454) $$\psi(x_1, x_2, \dots x_j + L, \dots x_N) = e^{i\alpha L}\,\psi(x_1, \dots x_j \dots x_N) \quad \forall j$$

- instantaneous eigenstate of $H(t)$ with TBC
  $$|E_n(\alpha;t)\rangle$$
- assume that there is finite many body gap in the period of $V(t)$ in $t = 0 \dots T$
  $\Rightarrow$ many body Thouless pump

Density matrix of many-body system

(455) $$\rho(t) = \rho_{\text{adiab}}(t) + \delta\rho(t)$$

(456) $\rho_{ad}(t) = |E_0(\alpha;t)\rangle\langle E_0(\alpha;t)|$

start in ground state at $t=0$ $\quad |E_0(\alpha, t=0)\rangle$

$$i\dot{\rho} = [H(t), \rho(t)] = [H(t), \delta\rho(t)]$$

Now $\qquad\qquad\qquad\qquad\qquad\qquad n \neq 0$

$$\langle E_0 | [H, \delta\rho] | E_n\rangle = (E_0 - E_n)(\delta\rho)_{on}$$

$$(\delta\rho)_{on} = \frac{\langle E_0| [H, \delta\rho] | E_n\rangle}{E_0 - E_n} = i\frac{\langle E_0 | \dot{\rho} | E_n\rangle}{E_0 - E_n}$$

$$\cong \frac{i\langle E_0 | \dot{\rho}_{ad} | E_n\rangle}{E_0 - E_n} = i\frac{\langle \partial_t E_0 | E_n\rangle}{E_0 - E_n}$$

Transported charge ($\hat{p}$ - particle momentum)

(457)
$$\Delta n = \frac{1}{L}\int_0^T dt\, \langle \hat{p}\rangle = \frac{1}{L}\int_0^T dt\, \mathrm{Tr}\{\rho(t)\hat{p}\}$$

$$= \frac{1}{L}\int_0^T dt\, (1 + \delta\rho_{00}(t)) \underbrace{\langle E_0 | \hat{p} | E_0\rangle}_{=0}$$

$$+ \frac{1}{2L}\int_0^T dt \sum_{n\neq 0}\left(\delta\rho_{on}\langle E_n | \hat{p} | E_0\rangle + c.c.\right)$$

+ other contributions negligible

In the instantaneous eigenstate $\hat{p}$ vanishes, so

(458) $$\Delta n = \frac{1}{2L}\int_0^T dt \sum_{n\neq 0}\left(i\frac{\langle \partial_t E_0 | E_n\rangle}{E_0 - E_n} P_{n0} + c.c.\right)_\alpha$$

The index "α" denotes the chosen boundary conditions.

For the following discussion we remove the TBC by a unitary trafo

(459) $$H(t) \rightarrow H(t,\alpha) = T(\alpha)\, H(t)\, T^{-1}(\alpha)$$

$$|E_n\rangle \rightarrow |\tilde{E}_n(t;\alpha)\rangle = T(\alpha)\, |E_n(t;\alpha)\rangle$$

For the new real-space wavefunctions ϕ we find

(460) $$\phi(x_1, \dots x_N) = e^{-i\alpha(x_1 + x_2 + \dots + x_N)}\, \psi(x_1 \dots x_N)$$

which now fulfill periodic BC

(461) $$\phi(x_1, \dots x_j + L, \dots x_N) = \phi(x_1, \dots, x_j, \dots, x_N)$$

The momentum operator gets a gauge term

$$\tilde{p} = T(\alpha)\, p\, T(\alpha)^{-1} = T(\alpha) \sum_{j=1}^{N} \left(-i\frac{\partial}{\partial x_j}\right) T^{-1}(\alpha)$$

$$= \sum_{j=1}^{N} -i\frac{\partial}{\partial x_j} + \alpha$$

so

(462) $$\tilde{p} = \frac{\partial H(\alpha)}{\partial \alpha}$$

Then

$$\langle \tilde{E}_n | \hat{\tilde{p}} | \tilde{E}_0 \rangle = \langle \tilde{E}_n | \frac{\partial H(t,\alpha)}{\partial \alpha} | \tilde{E}_0 \rangle \tag{463}$$

Using

eigenvalues are invariant!

$$\frac{\partial}{\partial \alpha}\Big| \qquad H(\alpha) | \tilde{E}_0 \rangle = E_0 | \tilde{E}_0 \rangle$$

$$\left(\frac{\partial}{\partial \alpha} H(\alpha)\right) | \hat{\tilde{E}}_0 \rangle + H(\alpha) \partial_\alpha | \hat{\tilde{E}}_0 \rangle = E_0 \partial_\alpha | \hat{\tilde{E}}_0 \rangle$$

$$\Rightarrow \quad \langle \tilde{E}_n | \frac{\partial H(\alpha)}{\partial \alpha} | \hat{\tilde{E}}_0 \rangle = (E_0 - E_n) \langle \tilde{E}_n | \partial_\alpha \hat{\tilde{E}}_0 \rangle \tag{464}$$

Thus

$$\Delta n(\alpha) = \frac{1}{2L} \int_0^T dt \underbrace{\sum_{n \neq 0} i \Big(\langle \partial_t \hat{\tilde{E}}_0 | \underbrace{\hat{\tilde{E}}_n \rangle \langle \hat{\tilde{E}}_n}}_{=1} | \partial_\alpha \hat{\tilde{E}}_0 \rangle + c.c. \Big)$$

So

$$\Delta n(\alpha) = \frac{1}{2L} \int_0^T dt \; i \Big(\langle \partial_t \hat{E}_0 | \partial_\alpha \hat{\tilde{E}}_0 \rangle - \langle \partial_\alpha \hat{E}_0 | \partial_t \hat{\tilde{E}}_0 \rangle \Big) \tag{465}$$

Now in a large system it is reasonable to assume that Δn should not depend of α in the small interval

$$-\frac{\pi}{L} \le \alpha \le \frac{\pi}{L} \tag{466}$$

So we can average eq. (465) over this intervall

(467)
$$\overline{\Delta n} = \frac{i}{4\pi L}\int_{-\pi/L}^{\pi/L} d\alpha \int_0^T dt \left[\left\langle \frac{\partial \hat{E}_0}{\partial t}\middle|\frac{\partial \hat{E}_0}{\partial \alpha}\right\rangle - \left\langle \frac{\partial \hat{E}_0}{\partial \alpha}\middle|\frac{\partial \hat{E}_0}{\partial t}\right\rangle\right]$$

Rescaling $\beta = \alpha L$ finally leads to an expression which looks similar to the non-interacting invariant

(468)
$$\boxed{\overline{\Delta n} = \frac{1}{4\pi}\int_{-\pi}^{\pi} d\beta \int_0^T dt \; i \underbrace{\left[\left\langle \frac{\partial \hat{E}_0}{\partial t}\middle|\frac{\partial \hat{E}_0}{\partial \beta}\right\rangle - \left\langle \frac{\partial \hat{E}_0}{\partial \beta}\middle|\frac{\partial \hat{E}_0}{\partial t}\right\rangle\right]}_{\text{Berry curvature}}}$$

NTW invariant

(469)
$$\boxed{\overline{\Delta n} = \nu = \frac{1}{2\pi}\oint dt \, \frac{\partial \phi_{MB}(t)}{\partial t}}$$

winding of a many-body Zak phase

(470)
$$\boxed{\phi_{MB} = \mathrm{Im}\int_{-\pi}^{\pi} d\beta \left\langle E_0 \middle| \frac{\partial E_0}{\partial \beta}\right\rangle}$$