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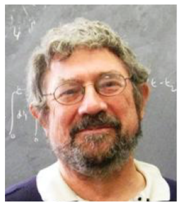
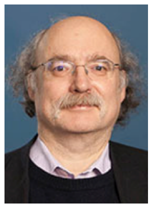
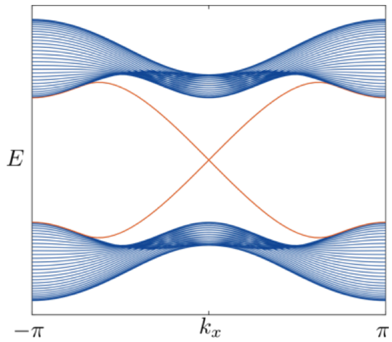
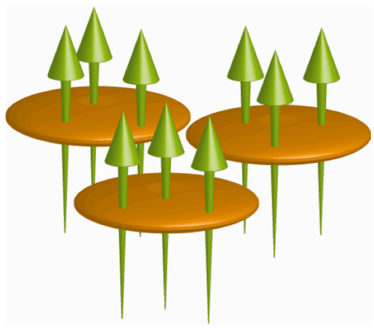




Topological quantum systems
SS 2021 4SWS





Prof. Dr. Michael Fleischhauer

Tue 10:00- 12:00
Thu 14:00-15:30

zoom

https://uni-kl-
de.zoom.us/j/98445830776?pwd=djFFQWJreldVNDlaaklCNjVDcC9SQT09

Topo21**

Literature

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0. Introduction

0.1 Phases of Matter and spontaneous symmetry breaking

Most phases of matter consisting of the same constituents can be distinguished by spontaneously broken symmetries

- example: Ferromagnet $\longleftrightarrow$ Paramagnet

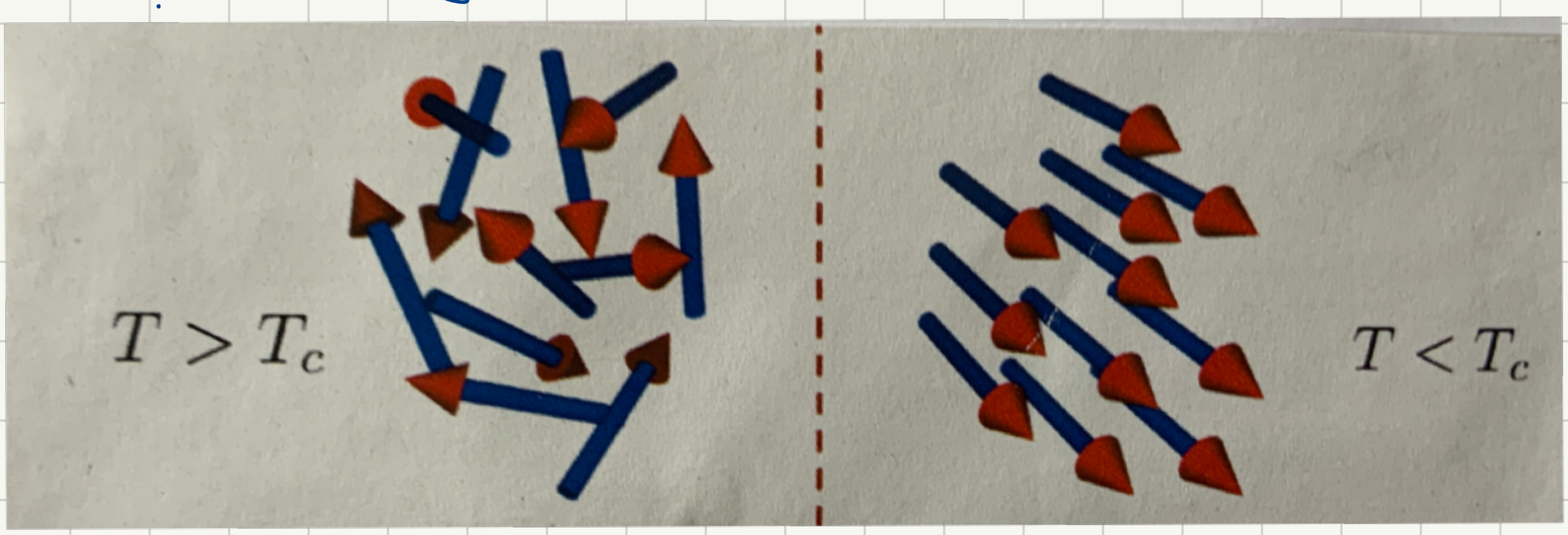
Heisenberg model $\quad H = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j \qquad J>0$

H has $O(3)$ symmetry; i.e.

$$D^{-1} H D = H \qquad D = e^{i\theta \vec{n} \cdot \vec{S}_{tot}}$$

still H describes two different phases

paramagnet $\qquad\qquad$ ferromagnet



$\uparrow$ O(3) symmetric $\qquad\qquad$ $\uparrow$ spontaneously broken O(3) symmetry

$$[\hat{\vec{S}}_{tot}, H] = 0$$

but

$$\langle \psi_0 | \vec{S}_{tot} | \psi_0 \rangle = 0 \qquad \langle \psi_0 | \vec{S}_{tot} | \psi_0 \rangle \neq 0$$

$$T > T_c \qquad\qquad T < T_c$$

Phase transitions of this type can be described by a Ginzburg-Landau theory:

- free energy $\quad F(\vec{M}) = U - TS$

$$F(\vec{M}) = F_0 + \alpha_1(T)\vec{M}^2 + \alpha_2(T)\vec{M}^4$$

$$\alpha_1 > 0 \qquad \alpha_2 = \alpha(T - T_c)$$

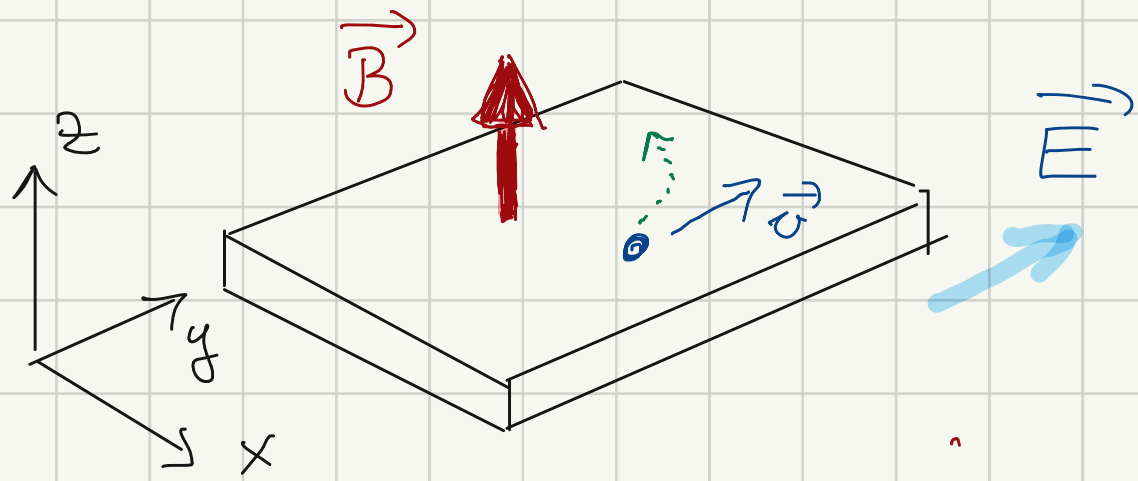
$$F(\vec{M}) \overset{!}{=} \min \Rightarrow \quad T > T_c \quad \vec{M} = 0$$

$$T < T_c \quad |\vec{M}| = \sqrt{\frac{-\alpha_1}{\alpha_2}} \neq 0$$

0.2 Beyond symmetry breaking: Quantum Hall effect

(A) Classical Halleffect of 2D electron gas

free electrons in 2D metal / SC



Lorentz force $\quad \vec{F} = -e\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right) = m\dot{\vec{v}}$

stationarity condition $\quad \dot{\vec{v}} = 0$

$$\vec{B} = (0, 0, B) \qquad \vec{E} = (E_x, E_y, 0)$$

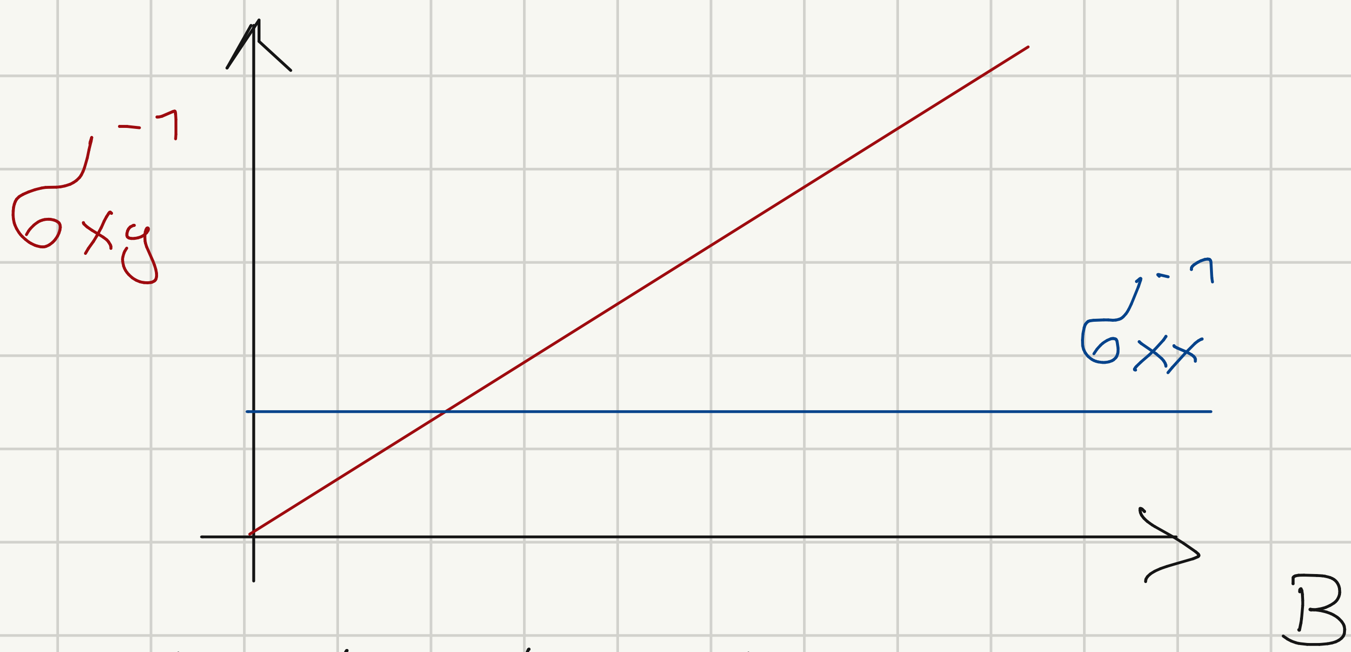
$$\vec{E} = -\frac{\vec{v}}{c} \times \vec{B}$$

current $\quad \vec{j} = -e\rho\vec{v}$

$$j_x = -\frac{e\rho c}{B} E_y \qquad j_y = \frac{e\rho c}{B} E_x$$

Hall resistivity $\quad \vec{j} = \underline{\underline{\sigma}}\,\vec{E}$

$$\sigma_{xy}^{-1} = \frac{E_x}{j_y} = \frac{B}{e\rho c}$$



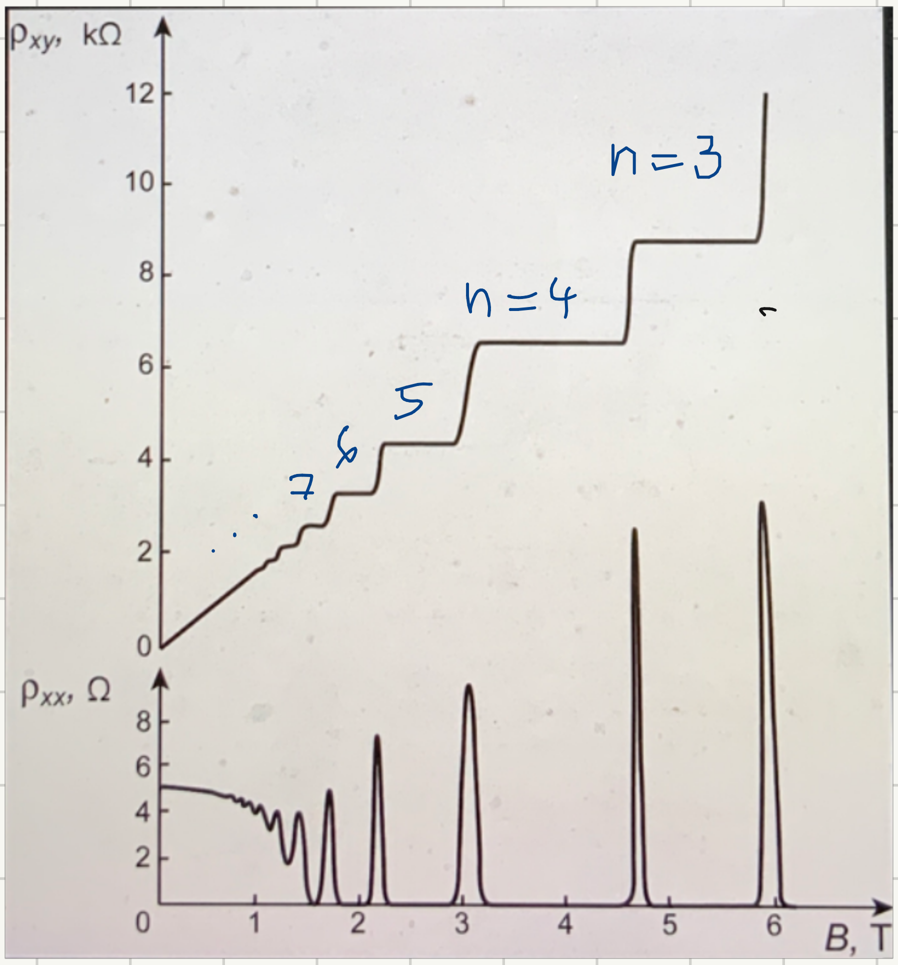
after adding a damping term $-m\frac{\vec{v}}{\tau}$

consistent with observations in classical regime, i.e. low B high temperature

$\rightarrow$ quantum regime?

(B) the discoveries of von Klitzing (1980) and Tsui, Störmer and Gossard (1982)

- Klaus v. Klitzing: resistivity in high magnetic field at low T



Wikipedia

integer quantum Hall effect

Nobel prize 1985

- plateaus at

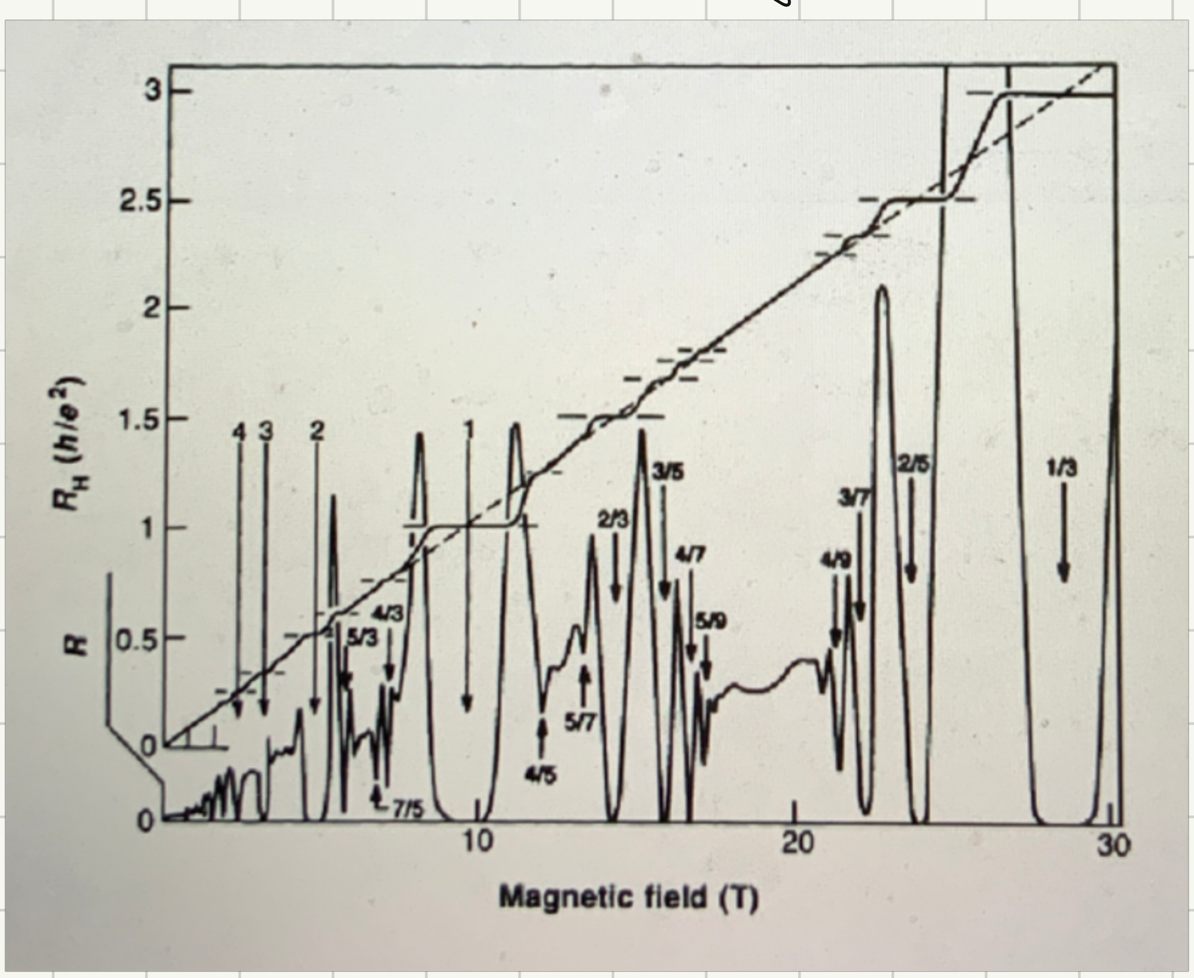
$$\boxed{\rho_{xy} = \frac{1}{n}\frac{h}{e^2}} \qquad n \in \mathbb{N}$$

$$h/e^2 = 25813.807\ \Omega$$

- and

$$\boxed{\rho_{xx} \longrightarrow 0}$$

- H.L. Störmer, D.C. Tsui, A.G. Gossard experiment
  R. Laughlin theory



Nobel prize
Störmer, Tsui, Laughlin 1998

- plateaus at fractional numbers

$$\boxed{\rho_{xy} = \frac{1}{f}\frac{h}{e^2} \qquad f = \frac{p}{q}}$$

fractional quantum Hall effect

phenomena robust against disorder
$\rightarrow$ stable phases

<u>problem:</u> phase transitions but no spontaneously broken symmetries !?

$\Rightarrow$ $\boxed{\text{beyond Landau Ginzburg}}$

- another system with different phases w/o spontaneously broken symmetries $\Rightarrow$ spin liquids

But phases can be characterized by <u>discrete quantum numbers</u>; we will see that these are

$\boxed{\text{topological quantum numbers}}$

How does topology come into the game?
$\rightarrow$ quantum states have a phase!

0.3 Berry phase

- Consider a Hamiltonian that depends smoothly on a parameter $\vec{\lambda}$

(1)
$$H = H(\vec{\lambda})$$

define instantaneous eigenvalues / eigenstates

(2)
$$H(\vec{\lambda})\,|n(\vec{\lambda})\rangle = E_n(\vec{\lambda})\,|n(\vec{\lambda})\rangle$$

Adiabatic theorem

> If $\vec{\lambda} = \vec{\lambda}(t)$ changes sufficiently slowly in time, a system prepared at time $t=0$ in an eigenstate $|n(t=0)\rangle$ will remain in that eigenstate at later times t

proof: (3) $|\psi(t)\rangle = \sum_n c_n e^{i\alpha_n(t)} |n(t)\rangle \qquad \alpha_n = -\frac{1}{\hbar}\int_0^t d\tau\, E_n(\tau)$

from SE $\quad \frac{d}{dt}|\psi(t)\rangle = -\frac{i}{\hbar} H(t)|\psi(t)\rangle$ follows

(4) $$\sum_n \left(\dot{c}_n e^{i\alpha_n(t)} |n(t)\rangle + c_n e^{i\alpha_n(t)} |\dot{n}(t)\rangle \right) = 0$$

$$\Rightarrow \quad \dot{c}_m(t) = -\sum_n c_n(t)\, e^{i(\alpha_n(t) - \alpha_m(t))} \langle m|\dot{n}\rangle$$

need $\langle m|\dot{n}\rangle = ?$ $\qquad \langle m|n\rangle = \delta_{mn}$

$$\frac{d}{dt} \Big| \qquad H(t)|n(t)\rangle = E_n(t)|n(t)\rangle$$

$$\langle m| \Big| \qquad \dot{H}|n\rangle + H|\dot{n}\rangle = \dot{E}_n|n\rangle + E_n|\dot{n}\rangle$$

$|m\rangle \neq |n\rangle \qquad \langle m|\dot{H}|n\rangle + E_m\langle m|\dot{n}\rangle = E_n\langle m|\dot{n}\rangle$

(5) $$\Rightarrow \quad \langle m|\dot{n}\rangle = \frac{\langle m|\dot{H}|n\rangle}{E_n - E_m}$$

i.e.

$$\dot{c}_m = -\sum_{n\neq m} c_n(t)\, e^{i(\alpha_n(t)-\alpha_m(t))} \frac{\langle m|\dot{H}|n\rangle}{E_n - E_m}$$

now at $t=0$ system in state $|n(t=0)\rangle$, i.e.

$$c_n(0) = 1 \qquad c_{m\neq n}(0) \equiv 0$$

then

$$C_m(\delta t) = -i\hbar \frac{\langle m|\dot{H}|n\rangle}{(E_m-E_n)^2}\left(e^{i(E_m-E_n)\frac{\delta t}{\hbar}} - 1\right)$$

so $|C_m(\delta t)| \ll 1$ if

$$(6) \qquad \boxed{\langle m|\dot{H}|n\rangle \ll \frac{|E_m-E_n|^2}{\hbar}} \qquad m \neq n$$

adiabatic condition

The characteristic rate of change of the Hamiltonian must be small compared to energy separation from other states

$\Longrightarrow$ system with energy gap

Knowing that the system remains in the eigenstate is not sufficient since there is a phase ambiguity!

adiabatic limit $\quad C_n(0) = 1 \Longrightarrow |C_n(t)| = 1$

Ansatz $\quad C_n(t) = e^{i\gamma(t)}$

$$\dot{C}_n = i\dot{\gamma}\, C_n \underset{\substack{\uparrow \\ \text{eq. (4)}}}{=} - C_n \langle n|\dot{n}\rangle$$

$$(7) \qquad \boxed{\gamma(t) = i\int_0^t d\tau\, \langle n|\frac{d}{d\tau}|n\rangle} \qquad \text{Berry phase}$$

now

$$\dot{\gamma}(t) = i \langle n | \frac{\partial}{\partial \vec{\lambda}} | n(\vec{\lambda}) \rangle \dot{\vec{\lambda}}$$

(8) $$\gamma(t) = i \int d\vec{\lambda} \, \langle n(\vec{\lambda}) | \frac{\partial}{\partial \vec{\lambda}} | n(\vec{\lambda}) \rangle$$

Berry phase is not unique. It depends on gauge choice

$$|n(\vec{\lambda})\rangle \longrightarrow e^{i\phi(\vec{\lambda})} |n(\vec{\lambda})\rangle = |\tilde{n}(\vec{\lambda})\rangle$$

(9) $$\gamma' = \gamma - \int d\vec{\lambda} \cdot \vec{\nabla}_\lambda \phi$$

However if we choose a closed path i.e. $\vec{\lambda}(0) = \vec{\lambda}(T)$

(10) $$\boxed{\gamma = i \oint_C d\vec{\lambda} \, \langle n | \vec{\nabla}_\lambda | n \rangle}$$

is invariant under gauge transformations!

if a smooth gauge exists!

$$\boxed{\partial_j \equiv \frac{\partial}{\partial \lambda_j}}$$

Berry connection

(11) $$\boxed{A_j = i \langle n | \partial_j n \rangle = A_j^*}$$

(like a gauge potential in Edyn)

note $\partial_j \langle n | n \rangle = 0 \Rightarrow \langle \partial_j n | n \rangle + \langle n | \partial_j n \rangle = 0$

depends on gauge

(12) $$\vec{A}' = \vec{A} - \vec{\nabla}_\lambda \phi$$

Berry curvature

$$\boxed{\begin{aligned}\Omega_{jk} &= i\left(\partial_j A_k - \partial_k A_j\right) \\ &= i\left(\langle\partial_j n|\partial_k n\rangle - \langle\partial_k n|\partial_j n\rangle\right)\end{aligned}} \tag{12}$$

(like field tensor in Edyn)

$$\boxed{\Omega_i = \varepsilon_{ijk}\Omega_{jk} = i\,\varepsilon_{ijk}\left(\langle\partial_j n|\partial_k n\rangle - \langle\partial_k n|\partial_j n\rangle\right)} \tag{13}$$

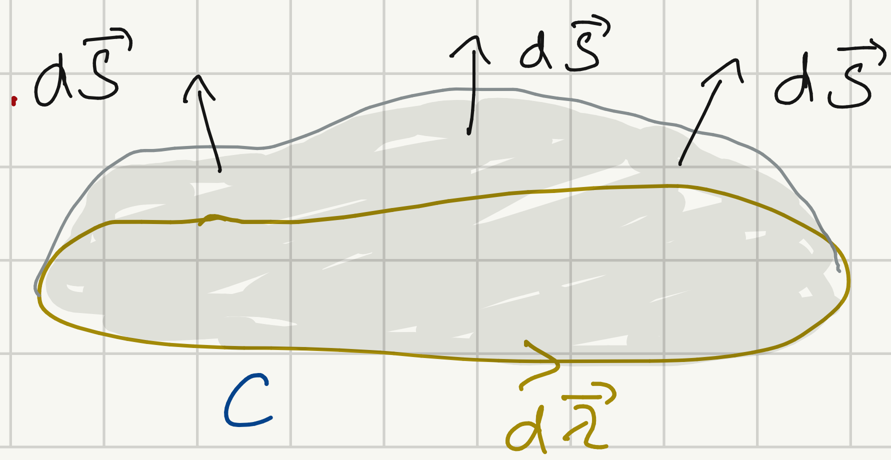
(like magnetic field in Edyn)

In a 3-dimensional parameter space $\vec{\lambda} \in \mathbb{R}^3$

$$\vec{\Omega} = \vec{\nabla}_\lambda \times \vec{A} \tag{14}$$

and using Stokes theorem

$$\boxed{\gamma = \int_C d\vec{\lambda}\cdot\vec{A} = \iint_{S(C)} d\vec{S}\cdot\vec{\Omega}} \tag{15}$$



note: S is not a closed surface!

There is an important general relation for $\lambda \in \mathbb{R}^3$

$$\vec{B} = \vec{B}_n = \operatorname{Im}\left(\vec{\nabla} \times \langle n | \vec{\nabla} n \rangle\right) = \dots$$

$$= \operatorname{Im} \sum_{m \neq n} \langle \vec{\nabla} n | m \rangle \times \langle m | \vec{\nabla} n \rangle$$

$$= -\operatorname{Im} \sum_{m \neq n} \frac{\langle n | \vec{\nabla} H | m \rangle \times \langle m | \vec{\nabla} H | n \rangle}{(E_m - E_n)^2}$$

$$\Rightarrow \quad \boxed{\sum_n \gamma_n = 0} \qquad (14)$$

sum of Berry phases over all eigenstates vanishes

- Generalization to degenerate eigenstates ?

So far we have assumed non-degenerate eigenstates. What happens if there are degeneracies (even only for few values of λ) ?

$$d\text{-fold degeneracy} \quad \{ |n_1\rangle, \dots, |n_d\rangle \}$$

Due to degeneracy (at some values of λ) no adiabatic following possible and transition between states possible $\Rightarrow$ consider whole subspace

$$(15)\quad \begin{bmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ c_d(t) \end{bmatrix} = \underline{\underline{U}}(t) \begin{bmatrix} c_1(0) \\ c_2(0) \\ \vdots \\ c_d(0) \end{bmatrix} = T e^{i\int_0^t d\tau\, \underline{\underline{A}}(\tau)} \begin{bmatrix} c_1(0) \\ c_2(0) \\ \vdots \\ c_d(0) \end{bmatrix}$$

i.e.

$$A_j \longrightarrow \underline{\underline{A}}_j \qquad \text{matrix in } d\text{-dimensional space}$$

$$(16)\quad \boxed{\begin{aligned} \underline{\underline{A}}_j \Big|_{ke} &= i \langle n_k | \partial_j n_e \rangle \\ \underline{\underline{A}}_j &= \underline{\underline{A}}_j^{+} \end{aligned}}$$

matrix-valued Berry connection

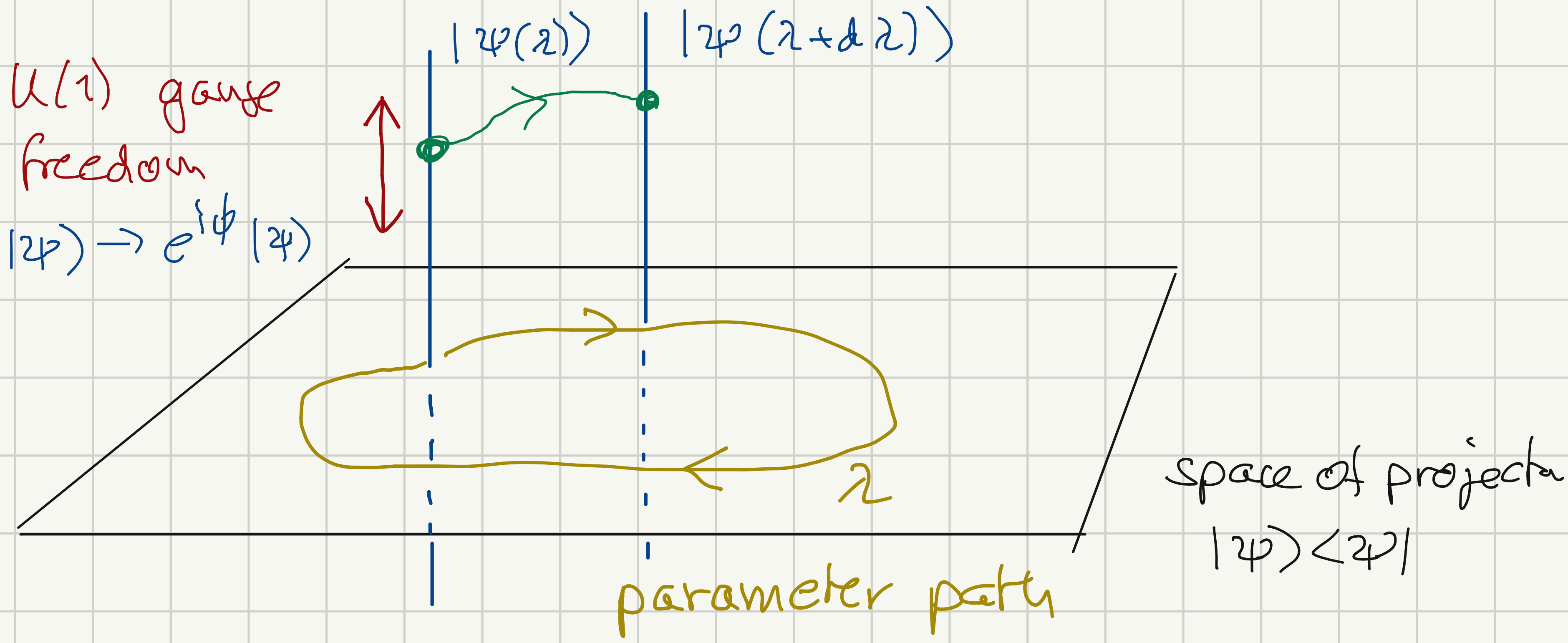
may be non-Abelian if e.g.

$$(17)\quad \left[\underline{\underline{A}}(\vec{\lambda}_1), \underline{\underline{A}}(\vec{\lambda}_2) \right] \neq 0$$

non-Abelian Berry connection

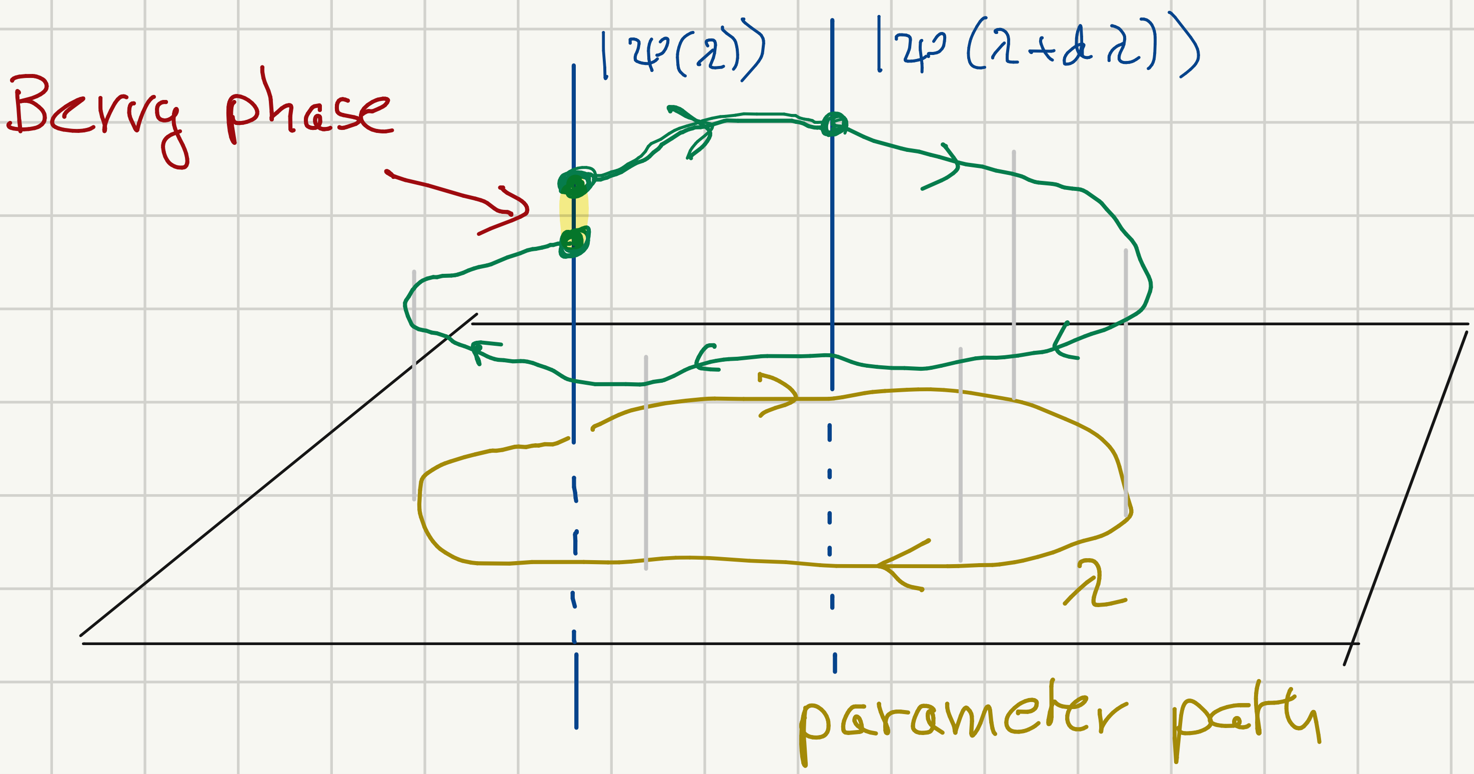
For (partially) degenerate states we have to consider (non-Abelian) Berry connections.

Berry phase and parallel transport



parallel transport of state $|\psi(\lambda)\rangle \to |\psi(\lambda+d\lambda)\rangle$

$$\left\| \, |\psi(\lambda)\rangle - |\psi(\lambda+d\lambda)\rangle \, \right\| \overset{!}{=} \min \tag{18}$$



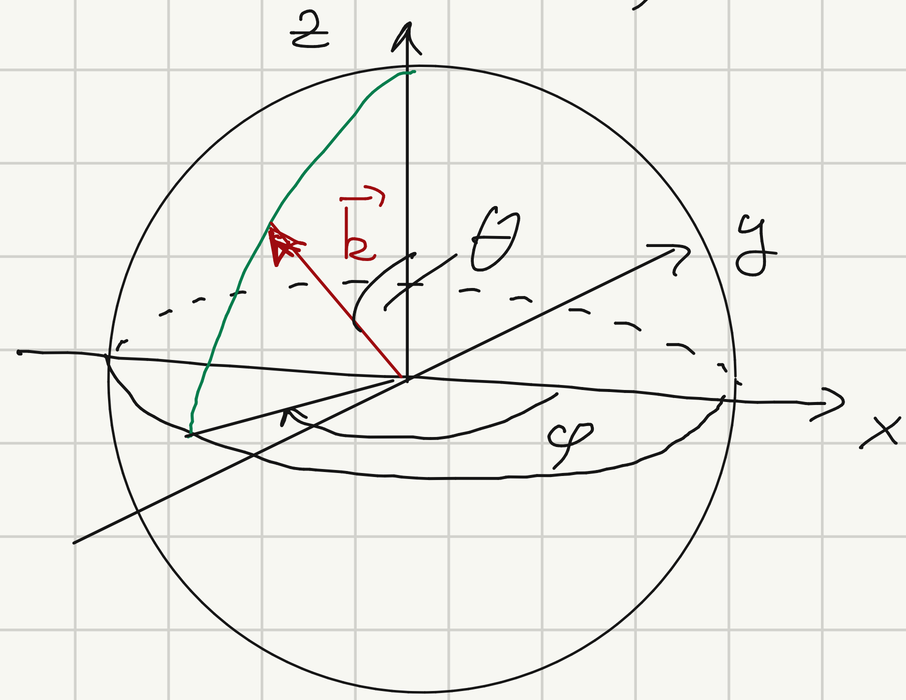
0.4 Absence of smooth gauge, Chern number

instructive example

(19) $$H = \vec{h}(\vec{k})\cdot\vec{\sigma} \qquad \vec{b}\in\mathbb{R}^3$$

where $$\vec{h}(\vec{b}) = \vec{e}_k = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

eigenstates



$E_-=-1$ $$|E_-\rangle = \begin{pmatrix} \sin\frac{\theta}{2}e^{-i\varphi} \\ -\cos\frac{\theta}{2} \end{pmatrix}$$

(20)

$E_+=+1$ $$|E_+\rangle = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\varphi} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

- Berry connections in θ and φ direction

(21) $$A_\theta^{(-)} = i\langle E_-|\partial_\theta E_-\rangle = 0 \qquad A_\varphi^{(-)} = i\langle E_-|\partial_\varphi E_-\rangle = \sin^2\theta/2$$

$\vec{A}$ defined everywhere except at south pole since for $\theta=\pi$

$$|E_-\rangle = \begin{pmatrix} e^{-i\phi} \\ 0 \end{pmatrix}$$ ϕ not defined!

• o.k. so lets choose a different gauge (phase of state)

$$|E'_\pm\rangle \equiv e^{i\phi}|E_\pm\rangle$$

i.e.

$$(22)\quad |E'_-\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}\, e^{-i\varphi} \end{pmatrix} \qquad |E'_+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{-i\varphi} \end{pmatrix}$$

now

$$(23)\quad A_\theta^{(-)'} = 0 \qquad A_\varphi^{(-)'} = -\cos^2\frac{\theta}{2}$$

but now $\vec{A}'$ is undefined at north pole since for $\theta = 0$

$$|E'_-\rangle = \begin{pmatrix} 0 \\ -e^{i\varphi} \end{pmatrix} \qquad \phi \text{ not defined}$$

For this example we cannot find a gauge where the Berry connection is well defined in the whole parameter space!

$$\Rightarrow \quad \nexists \text{ globally smooth gauge} \qquad (i)$$

Now let us calculate the Berry curvature

$$\Omega_{\theta\varphi}^{(-)} = i\left(\partial_\theta A_\varphi^{(-)} - \partial_\varphi A_\theta^{(-)}\right).$$

in gauge (21) <u>or</u> (23) (Ω is gauge invariant)

$$(24)\quad \Omega_{\theta\varphi}^{(-)} = \Omega_{\theta\varphi}^{(+)} = \frac{1}{2}\sin\theta$$

If we calculate the surface integral over the closed surface of the sphere

(25) $$\boxed{\frac{1}{2\pi}\int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \, \Omega^{(-)}_{\theta\varphi} = 1 = C}$$

we obtain an integer, called Chern number

$\Rightarrow$ The Chern number is non-zero (ii)

Summarizing (i) and (ii)

> If on a closed surface one cannot define a globally smooth Berry connection the corresponding surface integral over the Berry connection, the Chern number is nonzero.

remark: The fact that the Chern number is an integer has an instructive interpretation using Gauß law ($\mathbb{R}^3$)

(26) $$\iint_{S(V)} d\vec{S} \cdot \vec{\Omega}^{(-)} = \iiint dV \, \mathrm{div}\, \vec{\Omega}^{(-)}$$

For our example $\vec{\Omega}_\pm = \mp \frac{1}{2} \frac{\vec{k}}{k^3}$

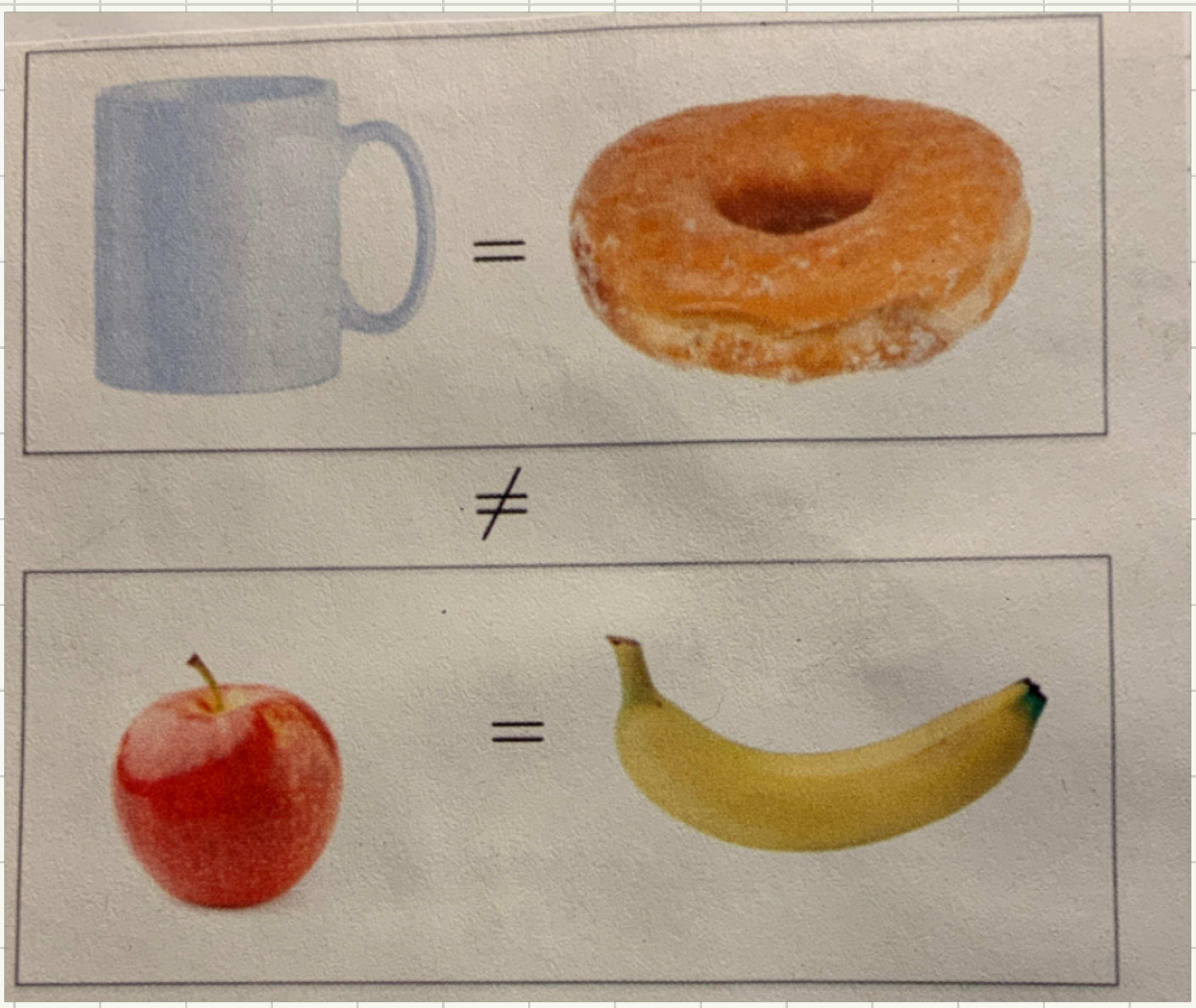
corresponds to a monopol at the origin

$$\mathrm{div}\, \vec{\Omega}_\pm = \pm 2\pi\, \delta^{(3)}(\vec{k})$$

$$(27) \quad C_\pm = \frac{1}{2\pi} \iint d\vec{S} \cdot \vec{\Omega}^{(\pm)} = \frac{1}{2\pi} \int dV\, \mathrm{div}\, \vec{\Omega}^{(\pm)} = \mp 1$$

0.5 Some elementary notions from topology

"A topologist is a mathematician which cannot distinguish a coffee cup from a donut but from a banana or an apple"



" topology is the mathematical study of the properties that are preserved through smooth deformations twisting and stretching of objects. Tearing, however is not allowed "

Eric W. Weisstein "topology"

objects that can smoothly be transformed into each other are called homotop

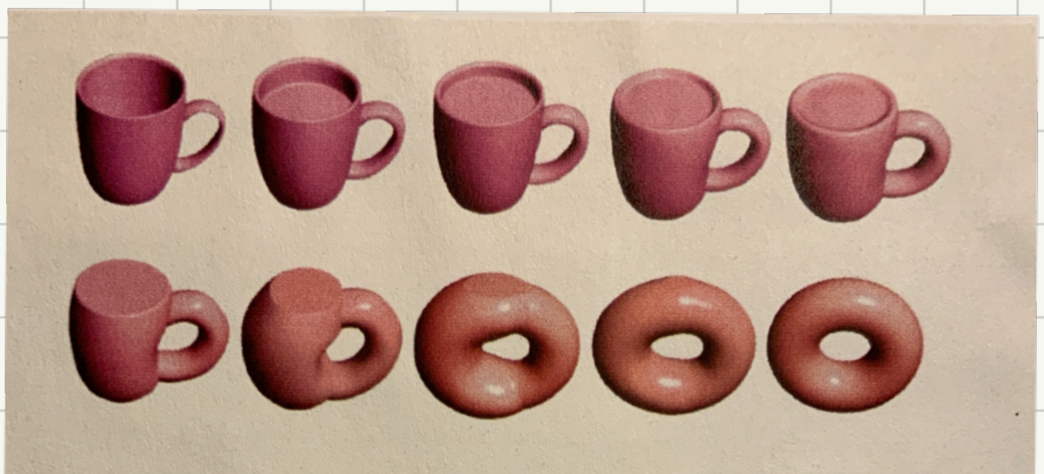
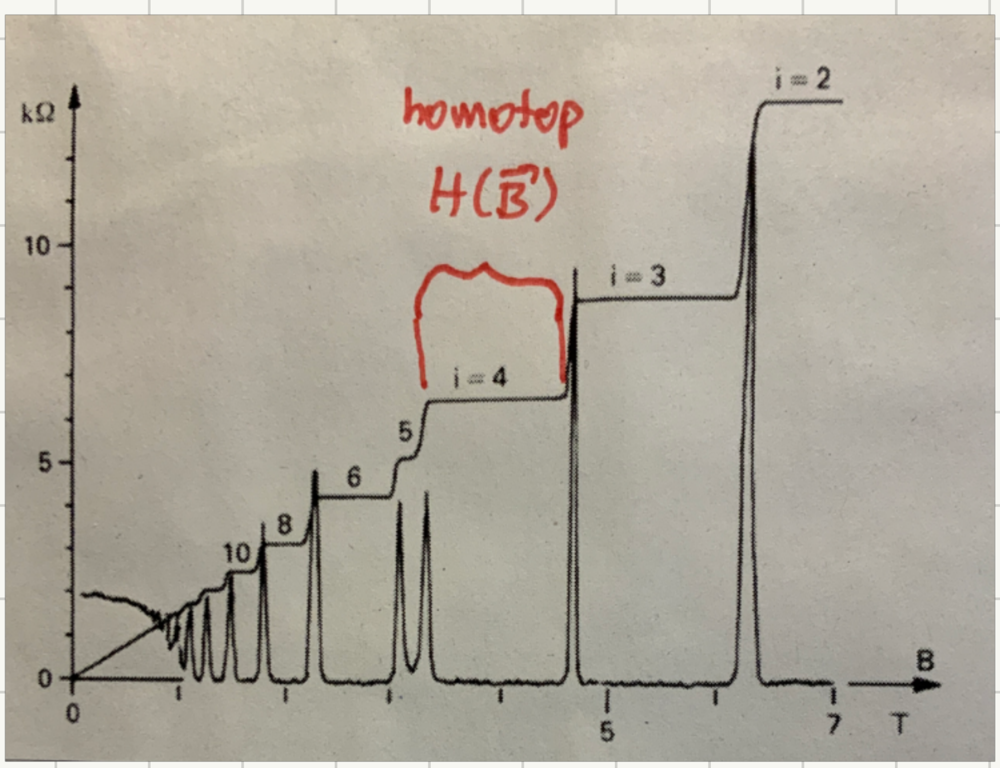
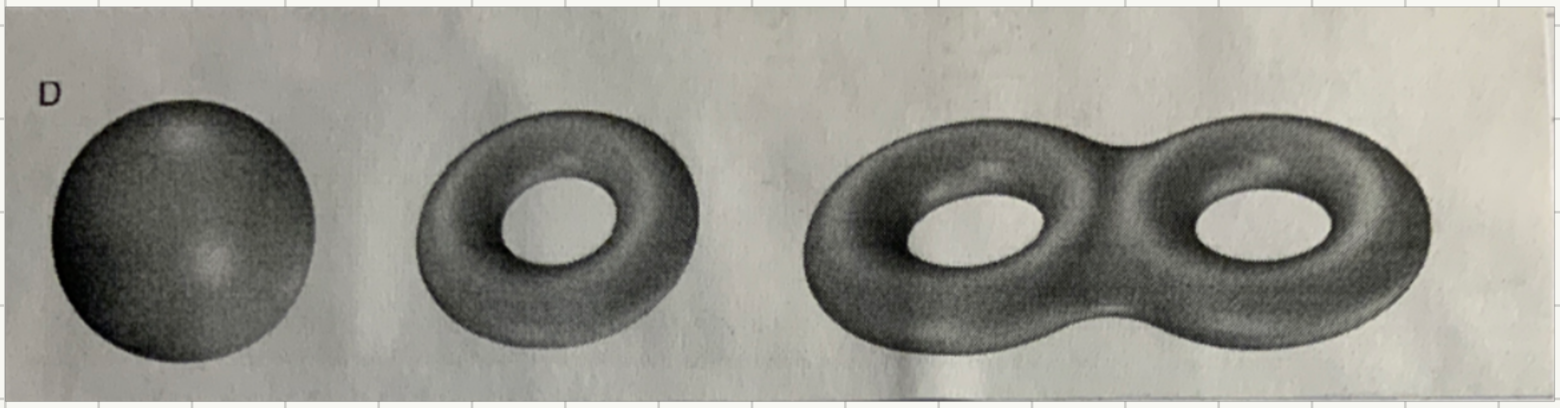


Figure: Homotopy between mug and torus.

We will see that all Hamiltonians of a quantum Hall system for magnetic fields corresponding to the same Hall plateau are homotop, i.e. belong to the same topological phase



topologically distinct surfaces in $\mathbb{R}^3$

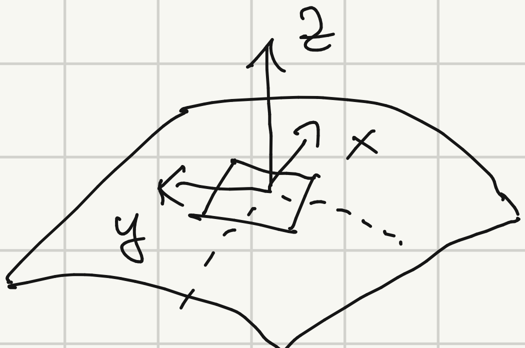


can be distinguished by integer numbers

topological invariants

<u>example of topological invariant</u>: Gauss-Bonnet theorem

2d compact surface embedded in $\mathbb{R}^3$



$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0 \quad \text{at } (0,0)$$

surface in neighborhood of $(0,0)$

$$z(x,y) \simeq \frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix}^T \underbrace{\begin{pmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial x \partial y} \\ \frac{\partial^2 z}{\partial x \partial y} & \frac{\partial^2 z}{\partial y^2} \end{pmatrix}}_{\text{Hesse matrix } H} \begin{pmatrix} x \\ y \end{pmatrix}$$

eigenvalues of H = inverse radii of curvature

$\lambda_1, \lambda_2 = r_1^{-1}, r_2^{-1}$ $\qquad \det(H) = (r_1 r_2)^{-1}$

example: sphere $\lambda_1 = \lambda_2 = r^{-1}$

Gauß Bonnet

(28) $$\boxed{\int dA\ \lambda_1 \lambda_2 = 2\pi\chi = 2\pi(2-2g)}$$

χ : Euler characteristics
g : genus
$\Big\} \in \mathbb{Z}$

topological invariants

topological phase transitions

- lets modify the example from chapter 0.4

(29) $$H = \underline{h}(\vec{k}) \cdot \vec{\sigma} \qquad \vec{h}(\vec{k}) = \vec{k}$$

$$E_\pm = \pm|\vec{k}| \qquad |E_-\rangle = \begin{pmatrix} \sin\frac{\theta}{2} e^{-i\varphi} \\ -\cos\frac{\theta}{2} \end{pmatrix} \qquad |E_+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\varphi} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

$$C_\pm = \pm 1$$

- all Hamiltonians (29) are topologically equivalent for all $|\vec{k}|$

- now lets add a shift

(30) $$\vec{h}(\vec{k}) \longrightarrow \vec{h}'(\vec{k}) = \vec{k} + \vec{e}_x$$

then one finds:

(31) $$C_\pm(|\vec{b}| < 1) = \pm 1$$

but $$C_\pm(|\vec{b}| > 1) = 0$$

So something happens at $|\vec{b}| = 1$!

Here

(32) $$E_+(|\vec{b}| = 1) = E_-(|\vec{b}| = 1)$$

i.e. there is a degeneracy

(31), (32) characterize what we call a topological phase transition