9. Fractional Quantum Hall Effect and Composite Fermions

So far we have discussed topological systems without interactions or where interactions play only a perturbative role. Now we want to discuss systems where interactions either strongly modify or even induce topological properties.

Historically the first interacting topological system is the

Fractional Quantum Hall Effect (FQHE)

Shortly after the discovery of the IQHE, Tsui, Störmer and Gossard observed in a 2D electron gas subject to even stronger magnetic fields quantized Hall plateaus

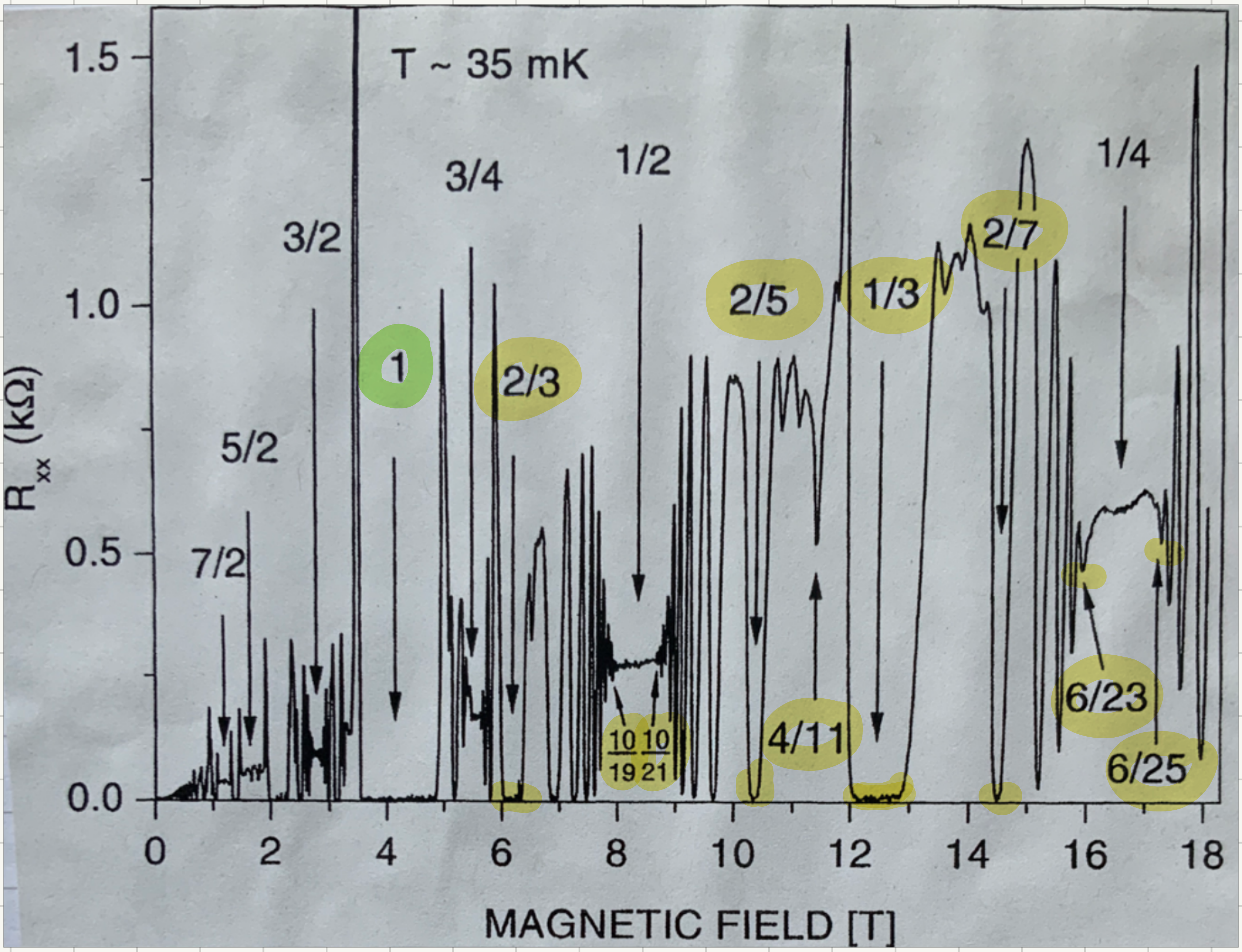
$$\sigma_H = f \frac{e}{h} \tag{349}$$

which correspond to fractional fillings f mostly less than unity.

Most prominent fractional plateaus had the structure

$$\boxed{f = \frac{n}{2np \pm 1}} \qquad n, p \in \mathbb{N} \tag{350}$$

These could only be understood as resulting from interactions.



9.1 Haldane's pseudopotentials

In order to see what properties the many-body state must have to minimize the total energy including the interaction it is useful to consider the projection of the interaction to the relevant sub-space of states

for the very high magnetic fields (see figure) it is sufficient to

(i) consider only one spin (Zeeman effect)

(ii) consider only LLL (lowest Landau level)

2-particle problem in 2D in external magnetic field

(351) $$H = H_1 + H_2 + V = H_0 + V$$

where

$$H_i = \frac{1}{2m}\left(\vec{p}_i + \frac{e}{c}\vec{A}(\vec{r}_i)\right)^2$$

(352)

$$V = V(|\vec{r}_1 - \vec{r}_2|)$$

complex coordinates in 2D $\quad x_i, y_i \to z_i, z_i^*$

Center of mass (COM) and relative coordinates

(353) $$X = \frac{1}{2}(z_1 + z_2) \qquad z = z_1 - z_2$$

$$H = \frac{1}{2}\left[-4\ell_R^2 \frac{\partial^2}{\partial X \partial X^*} + \frac{1}{4\ell_R^2} X X^* - X\frac{\partial}{\partial X} + X^* \frac{\partial}{\partial X^*}\right] \text{ COM}$$

(354) $$+\frac{1}{2}\left[-4\ell_r^2 \frac{\partial^2}{\partial z \partial z^*} + \frac{1}{4\ell_r^2} z z^* - z\frac{\partial}{\partial z} + z^*\frac{\partial}{\partial z^*}\right] \text{ rel. motion}$$

$$+ V(z, z^*)$$

where

(355) $$\ell_R = \frac{1}{\sqrt{2}}\ell_0 \qquad \ell_r = \sqrt{2}\,\ell_0 \qquad \ell_0 = \sqrt{\frac{\hbar c}{eB}}$$

As usual we can separate COM and relative motion

See Sec. 2 $$\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{z}{2} + 2\frac{\partial}{\partial z^*}\right) \qquad \hat{a}^+ = \frac{1}{\sqrt{2}}\left(\frac{z^*}{2} - 2\frac{\partial}{\partial z}\right)$$

$$\hat{b} = \frac{1}{\sqrt{2}}\left(\frac{z^*}{2} + 2\frac{\partial}{\partial z}\right) \quad \hat{b}^+ = \frac{1}{\sqrt{2}}\left(\frac{z}{2} - 2\frac{\partial}{\partial z^*}\right)$$

and similarly for COM motion with $z \to X$

COM $\quad \hat{a}_R, \hat{a}_R^+, \hat{b}_R, \hat{b}_R^+$

relative $\quad \hat{a}, \hat{a}^+, \hat{b}, \hat{b}^+$

lowest Landau level (see Sec. 2)

(356) $$\boxed{\hat{a}_R^+ \hat{a}_R |\psi\rangle = \hat{a}^+ \hat{a} |\psi\rangle = 0}$$

Thus within the LLL a complete set of states is

(357) $$|M,m\rangle = \frac{(\hat{b}_R^+)^M (\hat{b}^+)^m}{\sqrt{M!\, m!}} |0,0\rangle$$

b, b_R describe angular momentum degrees of freedom (see Sect. 2). We have seen there that a polynomial z^m has angular momentum projection m $\Longrightarrow$

- two-particle wavefunctions in LLL

(358) $$\psi^{LLL}_{M,m}(z,X) \sim X^M z^m \exp\left\{-\frac{|z|^2}{4\ell_r^2} - \frac{|X|^2}{4\ell_R^2}\right\}$$

Let us assume that the interaction $\bar{V}$ does not mix Landau levels i.e. is weak enough

$\Rightarrow$ We need to consider only the projection of V to the LLL

$$PVP = \sum_{m,M} \sum_{m',M'} |M',m'\rangle\langle M',m'|V|M,m\rangle\langle M,m|$$

Since $V = V(|z|)$

$$\langle M'm'|V|M\,m\rangle = \delta_{MM'}\delta_{mm'}\langle m|V|m\rangle$$

(359) $$\boxed{V_m \equiv \langle m|V|m\rangle}$$

Haldane's pseudopotential

for Coulomb interaction

(360) $$\boxed{V_m^C = \frac{1}{2\ell_0}\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)}}$$

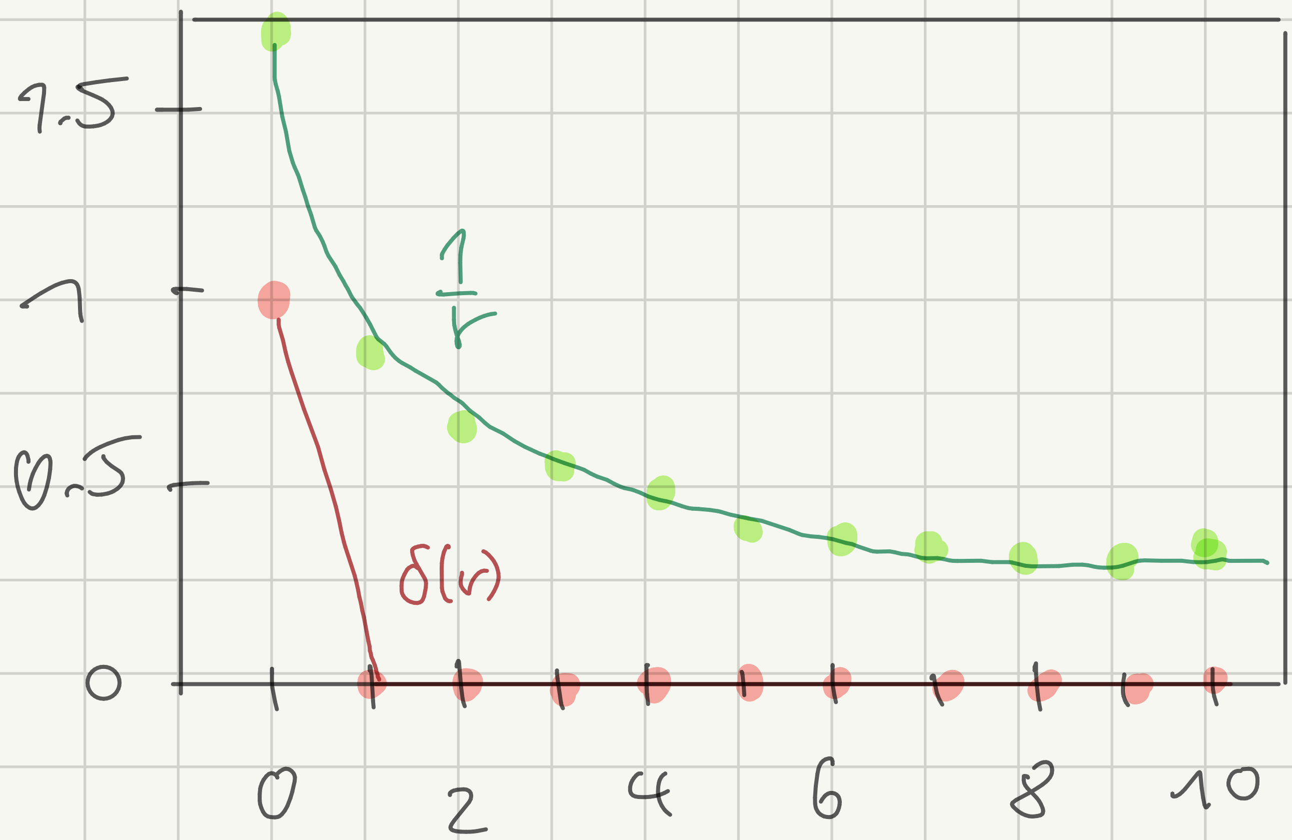
$V(r) = \frac{1}{r}$

for point interaction (2D)

$V(r) = \delta^{(2)}(r)$

(361) $$\boxed{V_0 = 1 \qquad V_2 = V_3 = V_4 = \cdots = 0}$$

(362) $$\boxed{PVP = \sum_{M,m} |M,m\rangle\langle M,m|\, V_m}$$



Haldane's pseudopotentials

bosons: only even m

fermions: only odd m

In a variational construction the best wavefunction is that in which the lowest pseudopotentials are vanishing.

9.3 Laughlin's wavefunction

Robert Laughlin constructed an ansatz wavefunction for the simplest fractions like 1/3.

The single-particle wavefunctions in the LLL with angular momentum (in z direction) ℓ are

$$\psi_\ell(z) = (2\pi 2^\ell \ell!)^{-1/2} z^\ell e^{-\frac{1}{4}|z|^2} \qquad (\ell_0 = 1) \tag{363}$$

The many-body wavefunction must thus have the structure

$$\Psi = F_A\left[\{z_j\}\right] \exp\left\{-\frac{1}{4}\sum_{j=1}^{N} |z_j|^2\right\} \tag{370}$$

F_A must be anti-symmetric (for fermions) and only a function of the difference between coordinates.

One can make an educated guess;

(i) Motivated by the success of so-called Jastrow wavefunctions in the description of superfluid He we set

$$F_A\left[\{z_j\}\right] = \prod_{j<k} f(z_j - z_k)$$

(ii) Ψ should be an eigenfunction of total angular momentum

$\Rightarrow \prod_{j<k} f(z_j - z_k)$ polynomial of z_j ~~of~~ of order L

[rotation in x-y plane by angle θ

$$z_j \to z_j e^{i\theta}$$

if Ψ eigenfunction of $\hat{L}_z$: $\Psi \to \Psi e^{-iL\theta}$

i.e. $f(z_j - z_k)$ must have fixed angular momentum]

(iii) $f(z)$ must be anti-symmetric (fermions) [or symmetric (bosons)]

$$f(-z) = f(z)$$

(371) $\Rightarrow$ $$\boxed{f(z) = z^m} \qquad m = \begin{cases} \text{odd} & \text{fermions} \\ \text{even} & \text{bosons} \end{cases}$$

This given as an Ansatz wavefunction

(372) $$\boxed{\Psi_{1/m} = \mathcal{N} \prod_{j<k} (z_j - z_k)^m \exp\left\{-\frac{1}{4}\sum_{j=1}^{N} |z_j|^2\right\}}$$

Laughlin wavefunction

One finds that $\Psi_{1/m}$ has a uniform density distribution with fractional filling

(373) $$\boxed{\nu = \frac{1}{m} = \frac{N}{N_\Phi}}$$

For Coulomb interactions $\Psi_{1/3}$ and $\Psi_{1/5}$ (i.e. $m=3,5$) are exact ground states with filling $\nu = 1/3$; $1/5$

but: $\exists$ other fractions at which there are quantum Hall plateaus!

9.3 Jain's composite fermions

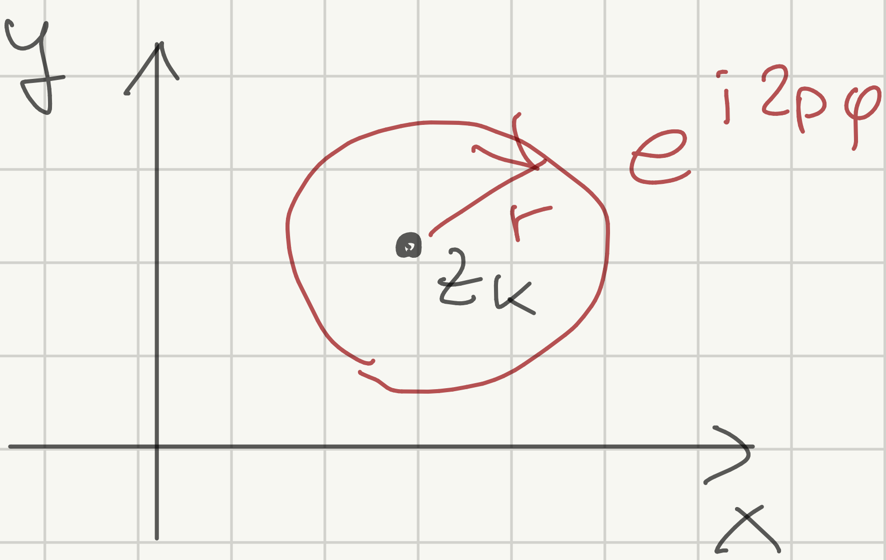
- Lets have a look again at Laughlin's wavefunction

$$\Psi_{1/2p+1} = N \prod_{j<k} (z_j - z_k)^{2p} \underbrace{(z_j - z_k) e^{-\frac{1}{4}\sum_j |z_j|^2}}_{\text{like in IQHE}} \quad (374)$$

$$\boxed{\nu = \frac{1}{2p+1}} \quad (375)$$

The prefactor looks like a vortex for the jth particle at the position of the kth particle

$$(z_j - z_k)^{2p} = (r e^{i\varphi})^{2p} = r^{2p} e^{i2p\varphi}$$

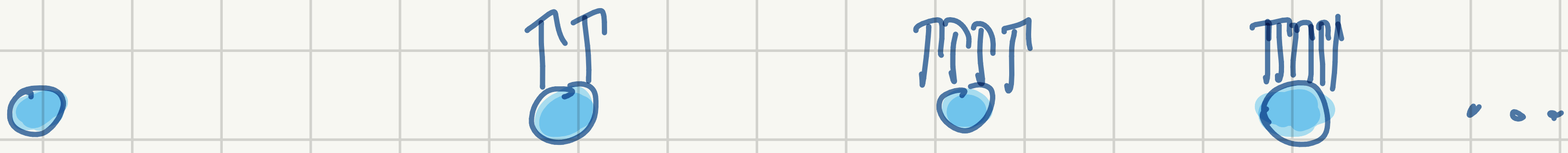


vortex of strength $2p$ at position z_k of every other particle

interpretation

electrons in LLL with $2p$ vortices (flux quanta) attached to them

composite fermions



bare fermion $\quad {}^{2}CF \quad {}^{4}CF \quad {}^{6}CF$

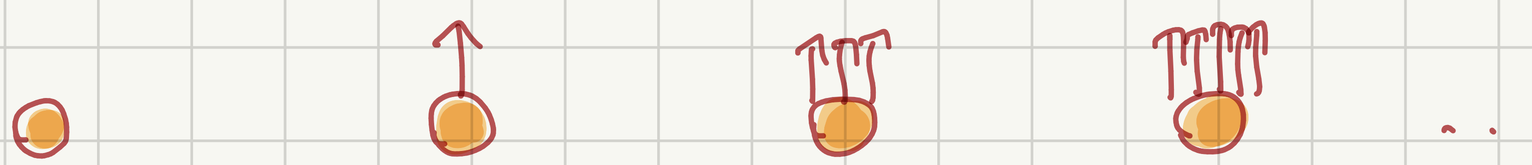
$$\boxed{{}^{2p}CF \quad \text{bound state of fermion and even number } (2p) \text{ of flux quanta attached to them}}$$

- If the original particles are bosons, the total wave function must be symmetric

$$\Psi_{1/2p} = \mathcal{N} \prod_{j<k} (z_j - z_k)^{2p-1} \underbrace{(z_j - z_k) e^{-\frac{1}{4}\sum_l |z_l|^2}}_{\text{as in IQHE}} \tag{376}$$

These wavefunctions describe fractional quantum Hall states at even filling fractions

$$\boxed{\nu = \frac{1}{2p}} \tag{377}$$



bare boson $\quad {}^{1}CF \quad {}^{3}CF \quad {}^{5}CF$

➔ $\Psi_{1/2p+1}$ and $\Psi_{1/2p}$ describe non-interacting fermions with attached flux quanta

(A) Why are non-interacting composite fermions good candidates for (approximate) ground-state wavefunctions?

- Slater determinants of composite fermions fulfill all symmetry requirements and are eigenfunctions of H_0 (kinetic energy)

- how about interactions?

$$PVP = \sum_{m,M} V_m \; |Mm\rangle\langle Mm|$$

$$\langle \Psi_{1/2p+1} | PVP | \Psi_{1/2p+1} \rangle \sim$$

(378) $$\sim \sum_m V_m \Big(\underbrace{\sum_{n=0}^{m} c_n \int d^2 z \, z^{*n} \, z^{2p+1} e^{-|z|^2/4}}_{= |\langle Mm | \Psi_{1/2p+1}\rangle|^2} \Big)^2$$

(angular momentum $\hat{L}_z \sim -b^+ b$; $|Mm\rangle$ has all z^n with maximum m)

carry out angular integration $z = re^{i\varphi}$

$$\int_0^{2\pi} d\varphi \; e^{i\varphi(2p+1-n)} \sim \delta_{n,2p+1}$$

$\Rightarrow$ in eq. (378) only terms survive with

$$m \geq 2p+1$$

$$\boxed{\text{CF wavefunctions screen all pseudopotentials with } m \leq 2p}$$

i.e. all V_m with $m \leq 2p$ do not contribute to the interaction energy

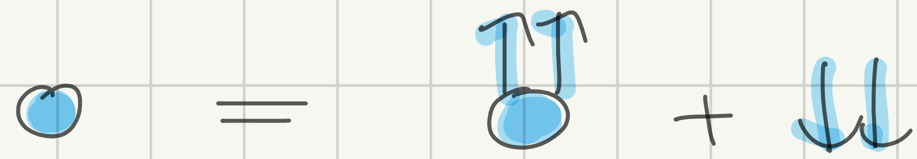
coupling of $2p$ flux quanta to a fermion leads to formation of a ^{2p}CF quasi-particles which eliminates all pseudopotential energies with $m \leq 2p$ and thus can be approximately treated as non interacting

(B) FQHE as integer QHE of composite fermions: idea

(i) fermions at filling ν at magnetic field B

(379) $$\nu = \frac{\rho \Phi_0}{|B|} \qquad \vec{B}: \uparrow$$

(ii) write fermion wavefunction as CF plus flux quanta in opposite direction



F = ^{2p}CF + 2p flux quanta

(iii) distribute the extra flux quanta homogeneously over the whole system

(380) $\Rightarrow$ effective magnetic field $$\boxed{B^* = B - 2p\varrho\Phi_0}$$

This corresponds to an effective filling ν^* in the reduced magnetic field

$$\nu^* = \frac{\varrho\Phi_0}{|B^*|} = \frac{\varrho\Phi_0}{|B - 2p\varrho\Phi_0|}$$

$$= \frac{1}{\left|\frac{B}{\varrho\Phi_0} - 2p\right|} = \frac{1}{|1/\nu - 2p|}$$

So if the ^{2p}CF fill $\nu^* = n$ effective Landau levels $\Rightarrow$

(381) $$\boxed{\nu = \frac{n}{2pn \pm 1}}$$ Jain's principle series of fractions

$\Rightarrow$ FQHE $\hat{=}$ IQHE for composite fermions in a reduced magnetic field

9.4* Chern-Simons field theory of composite fermions

(A) bare composite particles

Our goal is to construct a field theory of composite fermions which arise from bare fermions by attachment of flux quanta

- bare fermion field $\hat{\psi}(z)$ $\qquad z = x + iy$

(382) $$\{\hat{\psi}(z), \hat{\psi}^{+}(z')\} = \delta^{(2)}(z - z')$$

- define bare composite particle

(383) $$\boxed{\hat{\phi}_m(z) \equiv e^{-im\hat{\Theta}(z)}\,\hat{\psi}(z)}$$

where the angle operator $\hat{\Theta}(z)$ is defined as

(384) $$\boxed{\hat{\Theta}(z) = \int d^2z'\, \varphi(z - z')\, \hat{\psi}^{+}(z')\,\hat{\psi}(z')}$$