

## 4. Classification of topological systems of noninteracting fermions by generalized symmetries: 10-fold way

Topological features are robust against local perturbations. Thus the key difference between topological and nontopological systems must be rooted in some global, non-local properties. Generalized symmetries define such properties and we will discuss them in this chapter. They will be both of unitary and of anti-unitary character

### 4.1 Wigner theorem and invariance transformations

A linear transformation  $|\psi\rangle \rightarrow G|\psi\rangle$  is called an invariance transformation if for all  $|\psi\rangle, |\phi\rangle \in \mathcal{H}$  holds

$$(192) \quad |\langle G\psi | G\phi \rangle| = |\langle \psi | \phi \rangle|$$

Wigner (1931)

invariance transformations are either

$$(193) \quad \text{unitary} \quad \langle G\psi | G\phi \rangle = \langle \psi | \phi \rangle$$

or

$$(194) \quad \text{anti-unitary} \quad \langle G\psi | G\phi \rangle = \langle \psi | \phi \rangle^*$$

Def: invariance transformations are called (generalized) symmetry transformations of a quantum system if and only if

$$(195) \quad \boxed{G H G^{-1} = H} \quad \text{or} \quad \boxed{[H, G] = 0}$$

• properties of anti-unitary operators

(i) if  $G$  is anti-unitary  $\Rightarrow G^2$  is unitary

(ii) every anti-unitary operator can be written as product of a unitary operator  $U$  and the complex conjugation  $K$

$$(196) \quad \boxed{G = UK} \quad U^\dagger = U^{-1} \quad Kf = f^*$$

$$(197) \text{ (iii) inverse } G^{-1} = KU^\dagger \quad (K^2 = 1)$$

(iv) transformation of operators

$$(198) \quad A' = G A G^{-1} = UK A K U^{-1} = U A^* U^{-1}$$

since  $KA = A^*K$

(v) transformation of matrix elements

$$A' = G A G^{-1} \quad |\psi'\rangle = G |\psi\rangle$$

$$\begin{aligned} \Rightarrow \langle \psi' | A' | \phi' \rangle &= \langle G\psi | GAG^{-1} | G\phi \rangle \\ (199) \quad &= \langle G\psi | G A \phi \rangle = \langle \psi | A \phi \rangle^* \end{aligned}$$

## 4.2 time reversal of spinless particles

How can we define a "time reversal" operation?

$$T: t \rightarrow -t$$

Consider quantum mechanics  $\hat{X}, \hat{P}$

$$(200) \quad T \hat{X} T^{-1} \stackrel{!}{=} \hat{X} \quad T \hat{P} T^{-1} \stackrel{!}{=} -\hat{P}$$

$$\Rightarrow T [\hat{X}, \hat{P}] T^{-1} \stackrel{!}{=} T i\hbar T^{-1} \stackrel{!}{=} -[\hat{X}, \hat{P}] = -i\hbar$$

$\Rightarrow T$  must be anti-unitary!

$$(201) \quad \boxed{T = UK} \quad T_i T = -i$$

Furthermore it holds

$$(202) \quad \boxed{T^2 = \pm 1}$$

proof:  $T^2 = UKUK = UU^*K^2 = UU^*$

$$= U(U^\dagger)^{-1} \equiv A$$

Since  $T^2 |\psi\rangle = e^{i\varphi} |\psi\rangle$  the operator  
(for any  $|\psi\rangle$ )

$A$  must be diagonal, i.e.  $A = A^T$

$$\Rightarrow U = A U^T \leadsto U^T = U A^T = U A$$

$$\Rightarrow U = A U A \quad A U A = A^2 U A^2 \quad A^2 = 1 \quad A = \pm 1$$

$$(203) \quad \begin{array}{ll} T^2 = 1 & \Rightarrow U = U^T \text{ symmetric} \\ T^2 = -1 & \Rightarrow U = -U^T \text{ antisymmetric} \end{array}$$

### (A) action of $T$ in Fock space on a lattice

From similarity between boson annihilation operator and ladder operator in quantum mechanics

$$\hat{a}_e = \alpha \hat{x}_e + i\beta \hat{p}_e \quad \begin{array}{l} e \text{ position index} \\ \alpha, \beta \in \mathbb{R} \end{array}$$

$$(204) \quad \Rightarrow \quad T \hat{a}_e T^{-1} = T (\alpha \hat{x}_e + i\beta \hat{p}_e) T^{-1} = \hat{a}_e$$

time reversal of real-space operators

So here

$$(205) \quad \boxed{U = \mathbb{I} \quad \text{and} \quad T = K} \quad \boxed{T^2 = +1}$$

Now let's construct the action of  $T$  on momentum-

space operators:

$$\begin{aligned}\hat{a}_e &\stackrel{!}{=} T \hat{a}_e T^{-1} = T \frac{1}{\sqrt{L}} \sum_{\vec{k}} e^{i\vec{k} \cdot \vec{R}_e} \tilde{a}_{-\vec{k}} T^{-1} \\ &= \frac{1}{\sqrt{L}} \sum_{\vec{k}} e^{-i\vec{k} \cdot \vec{R}_e} T \tilde{a}_{-\vec{k}} T^{-1} \stackrel{!}{=} \frac{1}{\sqrt{L}} \sum_{\vec{k}} e^{+i\vec{k} \cdot \vec{R}_e} \tilde{a}_{\vec{k}}\end{aligned}$$

Thus  
(206)

$$T \tilde{a}_{-\vec{k}} T^{-1} = \tilde{a}_{-\vec{k}}$$

Let's make a sanity check  $\hat{\vec{p}} = \sum_{\vec{k}} \hbar \vec{k} \tilde{a}_{-\vec{k}}^{\dagger} \tilde{a}_{\vec{k}}$

$$\begin{aligned}T \hat{\vec{p}} T^{-1} &= \sum_{\vec{k}} \hbar \vec{k} T \tilde{a}_{-\vec{k}}^{\dagger} \tilde{a}_{\vec{k}} T^{-1} \\ &= \sum_{\vec{k}} \hbar \vec{k} \tilde{a}_{-\vec{k}}^{\dagger} \tilde{a}_{-\vec{k}} = - \sum_{\vec{k}} \hbar \vec{k} \tilde{a}_{\vec{k}}^{\dagger} \tilde{a}_{\vec{k}} \\ &= - \hat{\vec{p}} \quad \checkmark\end{aligned}$$

- Under what conditions is a single-particle Hamiltonian

$$(207) \quad H = \sum_{\vec{k}, \mu, \nu} \tilde{C}_{\vec{k}, \mu}^{\dagger} h_{\mu\nu}(\vec{k}) \tilde{C}_{\vec{k}, \nu}$$

time-reversal symmetric (TR invariant)?  
For this

$$T H T^{-1} = \sum_{\vec{k}, \mu, \nu} T h_{\mu\nu}(\vec{k}) T^{-1} T \tilde{C}_{\vec{k}, \mu}^{\dagger} \tilde{C}_{\vec{k}, \nu} T^{-1}$$

$$= \sum_{\vec{k}_{\mu\nu}} T h_{\mu\nu}(\vec{k}) T^{-1} \tilde{C}_{-\vec{k}\mu} \tilde{C}_{-\vec{k}\nu} \stackrel{!}{=} H$$

$$(208) \Leftrightarrow \boxed{T \underline{h}(\vec{k}) T^{-1} \stackrel{!}{=} \underline{h}(-\vec{k})}$$

Since for spinless particle  $T = K$

$$(209) \quad \boxed{\underline{h}^*(\vec{k}) \stackrel{!}{=} \underline{h}(-\vec{k})}$$

TR invariance of single-particle Hamiltonian of spinless particles

## (B) Vanishing Chern number for TR invariant Hamiltonians

- assume TR invariance

$$h(\vec{k}) |\psi(\vec{k})\rangle = T^{-1} h(-\vec{k}) T |\psi(\vec{k})\rangle = \epsilon_{\vec{k}} |\psi(\vec{k})\rangle$$

i.e. if  $|\psi(\vec{k})\rangle$  is eigenfunction of  $h(\vec{k})$  then  $T|\psi(\vec{k})\rangle$  is eigenfunction of  $h(-\vec{k})$

- Bloch functions

$$u_{-\vec{k}} = T u_{\vec{k}} = u_{\vec{k}}^*$$

$\Rightarrow$  Berry curvature

$$\begin{aligned} F_{em}(\vec{k}) &= -i \left( \langle \partial_e u_{\vec{k}} | \partial_m u_{\vec{k}} \rangle - \langle \partial_m u_{\vec{k}} | \partial_e u_{\vec{k}} \rangle \right) \\ &= -i \left( \langle \partial_e u_{-\vec{k}}^* | \partial_m u_{-\vec{k}}^* \rangle - "e \leftrightarrow m" \right) \\ &= -i \left( \langle \partial_m u_{-\vec{k}} | \partial_e u_{-\vec{k}} \rangle - "e \leftrightarrow m" \right) \\ &= -F_{em}(-\vec{k}) \end{aligned}$$

integral of  $F_{em}$  over Brillouin zone

$$\int_{BZ} d\vec{k} F_{em}(\vec{k}) = 0$$

(210)

$$C = 0 \quad \text{i.e.} \quad \sigma_{xy} = 0$$

There is no QHE for systems with TR symmetry  
The Chern number vanishes

We will see in the course of this lecture that also systems with TR symmetry can be topologically nontrivial, but the topological invariant is not the Chern number.

These topological insulators with TR invariance require spin-ful particles.

### 4.3 Time reversal for particles with spin

For particles with spin we have to think about how the T-reversal acts in the spin space. To this end we first look at the orbital angular momentum

$$\vec{L} = \vec{r} \times \vec{p} \Rightarrow T \vec{L} T^{-1} = -\vec{L}$$

Thus we impose

$$(211) \quad T \vec{S} T^{-1} = -\vec{S}$$

The sign change can be achieved by rotation about arbitrary axis by  $\pi$ . Convention: y-axis

$$(212) \quad T = e^{-i\pi \hat{S}_y} K \quad \text{i.e. } U = e^{-i\pi \hat{S}_y}$$

$$T^2 = e^{-i\pi \hat{S}_y} K e^{-i\pi \hat{S}_y} K = e^{-i\pi \hat{S}_y} e^{i\pi \hat{S}_y^*}$$

$$\text{since } \hat{S}_y^* = -\hat{S}_y$$

$$(213) \quad T^2 = e^{-2\pi i \hat{S}_y}$$

integer spin

half-integer spin

$$(214) \quad T^2 = 1$$

a (single) boson

$$T^2 = -1$$

a (single) fermion

Rem.: It is important that (214) holds for single particle states

# time-reversal of lattice models of spinfull fermions (for simplicity no sub-bands)

$$(215) \quad H = \sum_{\vec{e}, \vec{e}', \sigma} \tilde{C}_{\vec{e}\sigma}^\dagger h_{\vec{e}\vec{e}'}(\vec{e}) \tilde{C}_{\vec{e}'\sigma} \quad \sigma = \pm 1$$

Time reversal rotates spin  $|\uparrow\rangle \leftrightarrow |\downarrow\rangle$ , so

$$(216) \quad \Rightarrow \quad \boxed{\begin{aligned} T C_{j\uparrow} T^{-1} &= C_{j\downarrow} \\ T C_{j\downarrow} T^{-1} &= -C_{j\uparrow} \end{aligned}}$$

The minus sign in the second line seems strange. In fact where put a "-" in the upper or lower equation is a matter of convention, but one "-" must be there. It follows

$$T^2 \hat{C}_{j\uparrow} T^{-2} = -\hat{C}_{j\uparrow} \quad ?$$

but  $T^2 = -1$  so  $T^2$  and  $T^{-2}$  given  $+1$ ?

No! We need to remember that  $\hat{C}_{j\uparrow}$  reduces the number of fermions by one

$$T^2 \hat{C}_{j\uparrow} T^{-2} |\psi\rangle = T^2 \hat{C}_{j\uparrow} (-|\psi\rangle)$$

e.g. state with odd number of fermions  $\Rightarrow T^{-2} = -1$

state with even number  $\Rightarrow T^2 = +1$

$$= -\hat{C}_{j\uparrow} |\psi\rangle \quad \square$$

compact form using Pauli matrix

$$-i\sigma_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$(217) \quad \begin{aligned} T \hat{C}_{j\lambda} T^{-1} &= -i\sigma_y_{\lambda\lambda'} \bar{C}_{j\lambda'} \\ T \underline{\bar{C}}_j T^{-1} &= -i\underline{\sigma_y} \underline{\bar{C}}_j \end{aligned}$$

Time reversal of real-space fermion operators  
with spin

- transformation to momentum space (as for spin-less particle)

$$(218) \quad T \underline{\tilde{C}}(\vec{k}) T^{-1} = -i\underline{\sigma_y} \underline{\tilde{C}}(-\vec{k})$$

- Let ask also here under what conditions a single particle Hamiltonian (here unit cell of 1 for simplicity) for spinful fermions is TR invariant

$$(219) \quad \begin{aligned} H &= \sum_{\vec{k}, \lambda, \lambda'} h^{\lambda\lambda'}(\vec{k}) \tilde{C}_\lambda^\dagger(\vec{k}) \tilde{C}_{\lambda'}(\vec{k}) \\ &= \sum_{\vec{k}} \left( \underline{\tilde{C}}^\dagger(\vec{k}) \right)^T \underline{h}(\vec{k}) \underline{\tilde{C}}(\vec{k}) \end{aligned}$$

where  $\underline{\tilde{C}}(\vec{k}) = \begin{pmatrix} \tilde{C}_\uparrow \\ \tilde{C}_\downarrow \end{pmatrix}$

TR invariance:

$$THT^{-1} = H$$

$$\begin{aligned} H &\stackrel{!}{=} THT^{-1} = \sum_{\vec{k}} T \underline{\tilde{C}}^{\dagger}(\vec{k}) T^{-1} T \underline{h}(\vec{k}) T^{-1} T \underline{\tilde{C}}(\vec{k}) T^{-1} \\ &= \sum_{\vec{k}} \left( \underline{i\sigma^y} \underline{\tilde{C}}^{\dagger}(-\vec{k}) \right)^T \underline{h}^*(\vec{k}) \left( \underline{i\sigma^y} \underline{\tilde{C}}(-\vec{k}) \right) \\ &= \sum_{\vec{k}} \left( \underline{\tilde{C}}^{\dagger}(-\vec{k}) \right)^T \underline{i\sigma^y}^T \underline{h}^*(\vec{k}) \left( \underline{i\sigma^y} \underline{\tilde{C}}(-\vec{k}) \right) \\ &= \sum_{\vec{k}} \left( \underline{\tilde{C}}^{\dagger}(\vec{k}) \right)^T \left( \underline{i\sigma^y}^T \underline{h}^*(-\vec{k}) \underline{i\sigma^y} \underline{\tilde{C}}(\vec{k}) \right) \\ &\stackrel{!}{=} \sum_{\vec{k}} \left( \underline{\tilde{C}}^{\dagger}(\vec{k}) \right)^T \underline{h}(\vec{k}) \underline{\tilde{C}}(\vec{k}) \end{aligned}$$

$$\Rightarrow (220) \quad \underline{h}(\vec{k}) = - \underline{\sigma^y}^T \underline{h}(-\vec{k})^* \underline{\sigma^y}$$

$$\text{since } \underline{\sigma^y}^T = - \underline{\sigma^y}$$

$$(221) \quad \underline{h}(\vec{k}) = \underline{\sigma^y} \underline{h}(-\vec{k})^* \underline{\sigma^y}$$

condition for TR invariant single-particle  
Hamiltonian for spin 1/2 particle

## Kramers theorem

For a TR invariant fermionic Hamiltonian every (single-particle) eigenstate is at least doubly degenerate

proof:  $H|\psi\rangle = E|\psi\rangle \quad [T, H] = 0$

$\Rightarrow$  also  $T|\psi\rangle$  is eigenstate of  $H$  with energy  $E$  but it may not be orthogonal to  $|\psi\rangle$ ?

$$\begin{aligned} \langle \psi | T \psi \rangle &= \sum_{mn} \psi_m^* U_{mn} \psi_n \\ &= \sum_{mn} \psi_m^* U_{mn} \psi_n^* \quad \text{remember } T^2 = -1 \\ &\quad \Rightarrow U^T = -U \\ &= \sum_{mn} \psi_m^* (-U_{nm}) \psi_n^* = - \sum_{mn} \psi_n^* U_{nm} \psi_m^* \\ &= - \sum_{m,n} \psi_n^* U_{nm} \psi_m^* = - \langle \psi | T \psi \rangle \end{aligned}$$

$$(2.22) \Rightarrow \langle \psi | T \psi \rangle = 0 \quad \text{for } T^2 = -1$$

□

## 4.4 Charge conjugation

We will now introduce a second fundamental transformation in fermion Fock space: charge conjugation. To motivate this operation let us consider the

BCS Hamiltonian which is the foundation of the Bardeen-Cooper-Schrieffer theory of superconductivity where a fermion-fermion interaction is replaced by a mean-field term describing the pairing of fermions of opposite spin and momentum

$$(223) \quad C_{\uparrow}^{\dagger}(k) C_{\uparrow}(k) C_{\downarrow}^{\dagger}(-k) C_{\downarrow}(-k) \rightarrow \langle C_{\uparrow}^{\dagger}(k) C_{\downarrow}^{\dagger}(-k) \rangle C_{\uparrow}(k) C_{\downarrow}(-k)$$

The resulting mean-field Hamiltonian, called Bogoliubov-de Gennes Hamiltonian reads

$$(234) \quad H = \frac{1}{2} \sum_{\vec{k}} \begin{pmatrix} C^{\dagger}(k) \\ C(-k) \end{pmatrix}^T \underbrace{\begin{pmatrix} \epsilon_0 & \Delta \\ -\Delta & -\epsilon_0 \end{pmatrix}}_{= h_{\text{BdG}}(\vec{k})} \begin{pmatrix} C(k) \\ C^{\dagger}(-k) \end{pmatrix}$$

Here  $\Psi(k) = \begin{pmatrix} C(k) \\ C^{\dagger}(-k) \end{pmatrix}$  is called Nambu spinor

One recognizes the following property

$$(235) \quad \text{if } C(k) \leftrightarrow C^{\dagger}(-k) \quad \hat{=} \quad H \leftrightarrow -H$$

So we define the charge conjugation in real space of spinless particle

$$(236) \quad \boxed{C \hat{c}_{\mu j} C^{-1} = \sum_{\nu} U_{c \mu \nu}^* \hat{c}_{\nu j}^{\dagger}}$$

where  $U$  is some unitary matrix in the space of internal degrees of freedom of the  $j$ th unit cell

The action in momentum space follows

$$\begin{aligned} C \hat{C}_\mu(k) C^{-1} &= \frac{1}{\sqrt{L}} \sum_j e^{-ijk} C \hat{C}_{\mu j} C^{-1} \\ &= \frac{1}{\sqrt{L}} \sum_{j\nu} e^{-ijk} (U_C^*)_{\mu\nu} \hat{C}_{\nu j}^+ \\ &= \sum_\nu (U_C^*)_{\mu\nu} \hat{C}_\nu^+(-k) \end{aligned}$$

$$\Rightarrow (237) \quad C \hat{C}_\mu(k) C^{-1} = \sum_\nu (U_C^*)_{\mu\nu} \hat{C}_\nu^+(-k)$$

• consider double application of  $C$

$$\begin{aligned} C^2 \hat{C}_A^+ C^{-2} &= \sum_B (U_C)_{AB} C \hat{C}_B C^{-1} \\ &= \sum_B (U_C U_C^*)_{AB} \hat{C}_B^+ \end{aligned}$$

If we consider a single-particle state  $|\varphi\rangle = \hat{C}_A^+ |0\rangle$

$$C^2 |\varphi\rangle = U_C U_C^* |\varphi\rangle \equiv A |\varphi\rangle$$

and also  $C^2 |\varphi\rangle \stackrel{!}{=} e^{i\varphi} |\varphi\rangle$

$A = U_c U_c^* = U_c (U_c^T)^{-1}$  is diagonal  $A = A^T$

$$\Rightarrow U_c = A U_c^T \quad U_c^T = U_c A^T = U_c A$$

$$\Rightarrow U_c = A U_c A \quad \Rightarrow A = \pm 1$$

$$(238) \quad \boxed{C^2 = \pm 1}$$

I.e. also the charge conjugation operator squares to  $\pm 1$ .

### • action of charge conjugation on spin



$$(239) \quad C \hat{\vec{\mu}} C^{-1} = -\hat{\vec{\mu}} \quad \hat{\vec{\mu}} \sim \hat{\vec{S}}$$

$\Rightarrow$  similar to time reversal of spin-full particles  
real space

$$(240) \quad \boxed{\begin{aligned} C \hat{C}_{j\uparrow} C^{-1} &= \hat{C}_{j\downarrow}^+ \\ C \hat{C}_{j\downarrow} C^{-1} &= -\hat{C}_{j\uparrow}^+ \end{aligned}}$$

plus some possible unitary transformation in the space of internal state

• momentum space

(241)

$$\begin{aligned} C \tilde{c}_{\uparrow}(k) C^{-1} &= \tilde{c}_{\downarrow}^{\dagger}(-k) \\ C \tilde{c}_{\downarrow}(k) C^{-1} &= -\tilde{c}_{\uparrow}^{\dagger}(-k) \end{aligned}$$

(242)

$$C \tilde{c}_{\pm}(k) C^{-1} = -i\sigma_y \tilde{c}_{\mp}^{\dagger}(-k)$$

Invariance of a Hamiltonian under charge conjugation

(243)

$$C H C^{-1} \doteq H$$

$$(244) \quad H = \sum_{\vec{k}} (\tilde{c}_{\pm}^{\dagger}(\vec{k}))^T \underline{h}(\vec{k}) \tilde{c}_{\pm}(\vec{k})$$

$\underline{h}(\vec{k})$  matrix in internal space (spin and Bloch bands)

(A) spinless particles

$$\begin{aligned} C \sum_{\vec{k}} (\tilde{c}_{\pm}^{\dagger}(\vec{k}))^T \underline{h}(\vec{k}) \tilde{c}_{\pm}(\vec{k}) C^{-1} &= \\ &= \sum_{\vec{k}} (C \tilde{c}_{\pm}^{\dagger}(\vec{k}) C^{-1})^T C \underline{h}(\vec{k}) C^{-1} C \tilde{c}_{\pm}(\vec{k}) C^{-1} \\ &= \sum_{\vec{k}} (\tilde{c}_{\pm}^{\dagger}(-k))^T C \underline{h}(\vec{k}) C^{-1} \tilde{c}_{\pm}^{\dagger}(-k) \\ &= - \sum_{\vec{k}} (\tilde{c}_{\pm}^{\dagger}(-k))^T \underline{h}^T(\vec{k}) \tilde{c}_{\pm}(-k) \end{aligned}$$

anticommute

$$= - \sum_{\vec{k}} (\tilde{\underline{C}}^{\dagger}(\vec{k}))^T \underline{h}^T(-\vec{k}) \tilde{\underline{C}}(\vec{k})$$

$$\stackrel{!}{=} \sum_{\vec{k}} \tilde{\underline{C}}^{\dagger}(\vec{k})^T \underline{h}(\vec{k}) \tilde{\underline{C}}(\vec{k})$$

$$\Rightarrow (245) \quad \underline{h}(\vec{k}) = -\underline{h}^T(-\vec{k})$$

(B) spin full particles

$$(246) \quad \underline{h}(\vec{k}) = -\underline{\mathcal{O}}_g \underline{h}^T(-\vec{k}) \underline{\mathcal{O}}_g$$

## 4.5 Combination of time-reversal and charge conjugation: Chiral transformation

Let's consider the combined trf

$$(247) \quad S \equiv T \circ C$$

$S$  is anti-unitary

$$(248) \quad S; S^{-1} = -i$$

$$(249) \quad S^2 = (TC)^2 = +1 \quad \text{always } +1$$

consider spin-full fermions

$$\begin{aligned}
 S \hat{c}_j S^{-1} &= T C \hat{c}_j C^{-1} T^{-1} \\
 &= T (-i \gamma_j \hat{c}_j^{\dagger}) T^{-1} = -i \gamma_j T \hat{c}_j^{\dagger} T^{-1} \\
 &= -i \gamma_j i \gamma_j \hat{c}_j^{\dagger} = \hat{c}_j^{\dagger}
 \end{aligned}$$

(250)

$$S \hat{c}_j S^{-1} = \hat{c}_j^{\dagger}$$

$$S \hat{c}^{\dagger}(\bar{b}) S^{-1} = \hat{c}^{\dagger}(\bar{b})$$

Invariance of Hamiltonian under chiral transformation

(251)

$$\underline{h}(k) = -\underline{\sigma}_y \underline{h}(k) \underline{\sigma}_y$$

## 4.6 Summary of Fock-space transformations

(252)

|                           |           |                                                     |                 |
|---------------------------|-----------|-----------------------------------------------------|-----------------|
| <u>unitary</u>            | $U$       | $U \hat{c}_j U^{-1} = U_{je} \hat{c}_e$             | $U U^{-1} = i$  |
| <u>time reversal</u>      | $T$       | $T \hat{c}_j T^{-1} = U_{je}^t \hat{c}_e$           | $T T^{-1} = -i$ |
| <u>charge conjugation</u> | $C$       | $C \hat{c}_j C^{-1} = U_{je}^c \hat{c}_e^{\dagger}$ | $C C^{-1} = i$  |
| <u>chiral</u>             | $S = T C$ | $S \hat{c}_j S^{-1} = U_{je}^s \hat{c}_e^{\dagger}$ | $S S^{-1} = -i$ |

• This is an exhaustive list.

• The unitary matrices contain in the case of spin 1/2 particles

(253)  $-i \underline{\underline{G}}_y$  in spin space

• In momentum space

$$T \tilde{C}(k) T^{-1} \quad C \tilde{C}(k) C^{-1} \quad k \rightarrow -k$$

► There are 10 different classes under Fock-space transformations

not invariant

"0"

• invariant and  $G^2 = +1$

"+1"

invariant and  $G^2 = -1$

"-1"

(254)

|   |   |    |    |
|---|---|----|----|
| T | 0 | +1 | -1 |
| C | 0 | +1 | -1 |
| S | 0 | +1 |    |

} 3x3 = 9 classes

only non-trivial if

Ten-fold way

$T=0$  and  $C=0$   
 $\Rightarrow$  1 new class

single particle Hamiltonians in momentum space are invariant if

(255)

$$\begin{aligned} T \quad \underline{h}(k) &= U_t \underline{h}^*(-k) U_t^{-1} \\ C \quad \underline{h}(k) &= -U_c \underline{h}^T(-k) U_c^{-1} \\ S \quad \underline{h}(k) &= -U_s \underline{h}(k) U_s \end{aligned}$$

$$U_{t,c} = -i \sigma_y U_{\pm,c} \quad \text{for spin } 1/2$$

periodic table of topological insulators

(256)

| Class | T | C | S | 1              | 2              | 3              | 4              |
|-------|---|---|---|----------------|----------------|----------------|----------------|
| A     | 0 | 0 | 0 | 0              | $\mathbb{Z}$   | 0              | $\mathbb{Z}$   |
| AIII  | 0 | 0 | + | $\mathbb{Z}$   | 0              | $\mathbb{Z}$   | 0              |
| AI    | + | 0 | 0 | 0              | 0              | 0              | $\mathbb{Z}$   |
| BDI   | + | + | + | $\mathbb{Z}$   | 0              | 0              | 0              |
| D     | 0 | + | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}$   | 0              | 0              |
| DIII  | - | + | + | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$   | 0              |
| AII   | - | 0 | 0 | 0              | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$   |
| CII   | - | - | + | $\mathbb{Z}$   | 0              | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| C     | 0 | - | 0 | 0              | $\mathbb{Z}$   | 0              | $\mathbb{Z}_2$ |
| CI    | + | - | + | 0              | 0              | $\mathbb{Z}$   | 0              |

← intrinsic topology

symmetry protected topology

We will see that all classes with  $T \neq 0$  have vanishing Chern number and must be characterized with a different topological invariant

## example

### (I) SST model

$$(257) \quad \underline{h}(k) = \begin{bmatrix} 0 & -t_1 - t_2 e^{-ik} \\ -t_1 - t_2 e^{ik} & 0 \end{bmatrix} = -(t_1 + t_2 \cos k) \sigma_x - t_2 \sin k \sigma_y$$

• time reversal  $\sigma_y^* = -\sigma_y \quad \sigma_y^T = -\sigma_y$

$$\underline{h}^*(-k) = -(t_1 + t_2 \cos k) \sigma_x - t_2 \sin k \sigma_y = \underline{h}(k)$$

$T = +1$

• charge conjugation

$$\underline{h}^T(-k) = -(t_1 + t_2 \cos k) \sigma_x - t_2 \sin k \sigma_y = -\sigma_z \underline{h}(k) \sigma_z$$

$C = +1$

• chiral transformation

$$\underline{h}(k) = -(t_1 + t_2 \cos k) \sigma_x - t_2 \sin k \sigma_y = -\sigma_z \underline{h}(k) \sigma_z$$

$S = 1$

BDI

## (II) Rice-Mele model (RMM)

$$(258) \quad \underline{h}(k) = \begin{bmatrix} \Delta & -t_1 - t_2 e^{-ik} \\ -t_1 - t_2 e^{ik} & -\Delta \end{bmatrix} = -(t_1 + t_2 \cos k) \sigma_x - t_2 \sin k \sigma_y + \Delta \sigma_z$$

- time reversal

$$\underline{h}^* (-k) = \underline{h}(k)$$

$$T = +1$$

- charge conjugation

$$\underline{h}^T (-k) \neq -\sigma_z \underline{h}(k) \sigma_z$$

$$C = 0$$

AI

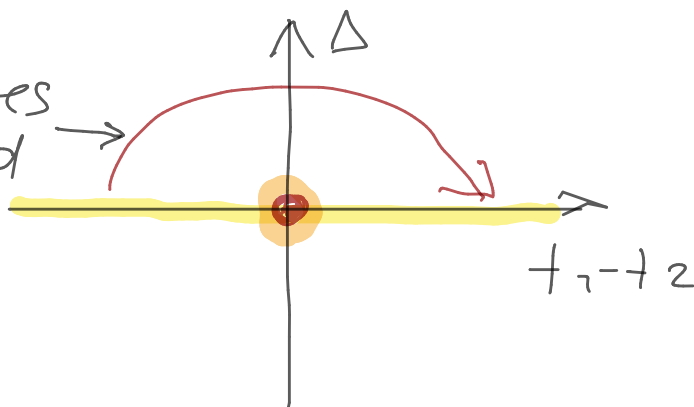
- chiral transformation

$$\underline{h}(k) \neq -\sigma_z \underline{h}(k) \sigma_z$$

$$S = 0$$

not topological  
in 1D

topological phases  
can be connected  
without closing  
the gap



## (III) reinstalling topology in the RMM

$$\Delta \rightarrow \Delta(t) \quad t_1 - t_2 \rightarrow (t_1 - t_2)(t)$$

- time reversal

$$T=0$$

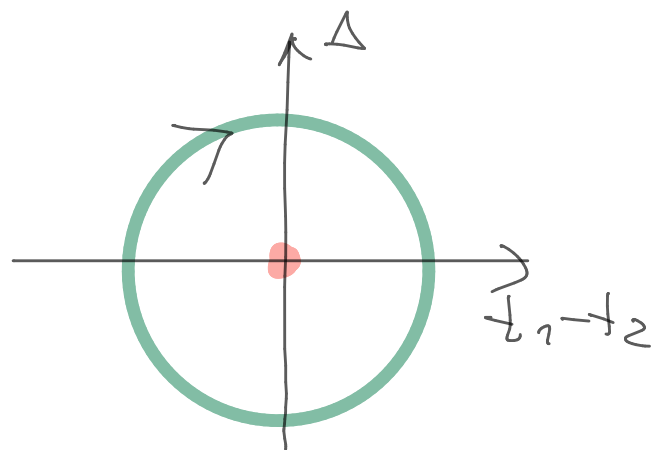
- charge conjugation

$$C=0$$

- chiral symmetry

$$S=0$$

A



topological in 2D  
 $\hat{=}(k, t)$

### (IV) Qi-Wu-Zhang model

$$(259) \quad \underline{h}(\vec{k}) = \sin k_x \sigma_x + \sin k_y \sigma_y + [\mu + \cos k_x + \cos k_y] \sigma_z$$

- time reversal

$$\underline{h}(-\vec{k}) = -\sin k_x \sigma_x - \sin k_y \sigma_y + [\mu + \cos k_x + \cos k_y] \sigma_z$$

$$\underline{h}^*(\vec{k}) = +\sin k_x \sigma_x - \sin k_y \sigma_y + [\mu + \cos k_x + \cos k_y] \sigma_z$$

$$T=0$$

- charge conjugation

$$\underline{h}^T(-\vec{k}) = -\sin k_x \sigma_x + \sin k_y \sigma_y + [\mu + \cos k_x + \cos k_y] \sigma_z$$

$$C=0$$

- chiral symmetry

$$\sigma_x \underline{h}(\vec{k}) \sigma_x = \sin k_x \sigma_x - \sin k_y \sigma_y - [u + \cos k_x + \cos k_y] \sigma_z$$

$$\sigma_y \underline{h}(\vec{k}) \sigma_y = -\sin k_x \sigma_x + \sin k_y \sigma_y - [u + \cos k_x + \cos k_y] \sigma_z$$

$$\sigma_z \underline{h}(\vec{k}) \sigma_z = -\sin k_x \sigma_x - \sin k_y \sigma_y + [u + \cos k_x + \cos k_y] \sigma_z$$

$$S = 0$$



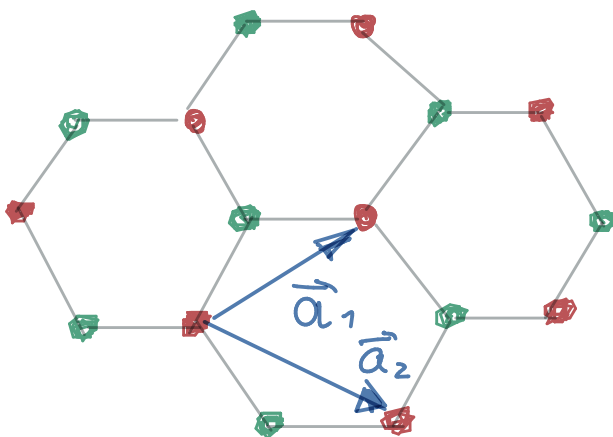
topological  
in 2D

## 5. Graphene and Haldane model

An important and interesting class of Chern insulators is based on honeycomb lattices

### 5.1 Graphene and Dirac fermions

#### (A) Lattice structure and Hamiltonian



unit cell has two lattice points (A, B)

lattice vectors

$$\vec{a}_1 = \frac{a}{2} (3, \sqrt{3}) \quad (260)$$

$$\vec{a}_2 = \frac{a}{2} (3, -\sqrt{3})$$