

12. Floquet topological insulators

We have seen in the discussion of TR invariant topological insulators that a proper bandstructure and sufficiently strong spin-orbit coupling is needed to generate a non-trivial topology. In particular in solid-state systems it is very difficult to find appropriate systems and the manipulation of Hamiltonian parameter is hard. Here periodic driving, e.g. with an external laser field can offer a new twist.

12.1 The principle of Floquet engineering: Effective Hamiltonians

Consider e.g. a 2 band Bloch Hamiltonian in 2D

$$(523) \quad \underline{H}_0(\vec{k}) = \varepsilon_0 \mathbb{1} + \vec{d}(\vec{k}) \cdot \underline{\vec{G}}$$

and add a time-periodic potential

$$(524) \quad \underline{H}(\vec{k}, t) = \underline{H}_0(\vec{k}) + \underline{U}(\vec{k}, t) \quad \underline{U}(\vec{k}, t+T) = \underline{U}(\vec{k}, t)$$

whose periodicity T is short compared to all other time scales of the system, then at stroboscopic times the system is described by the Floquet Hamiltonian

$$(525) \quad H_F \text{ defined as } \boxed{e^{i \underline{H}_F(\vec{k}) T} = \underline{U}_{\vec{k}}(t_0+T, t_0)}$$

where

$$(526) \quad U_{\vec{b}}(t_0+T, t_0) = T \exp \left\{ -i \int_{t_0}^{t_0+T} \underline{h}(\vec{b}, \tau) d\tau \right\}$$

$\underline{h}_f(\vec{b})$ can have different symmetries and topological properties than the original static Hamiltonian.

In the above case $\underline{h}_f(\vec{b})$ is again a 2 band Bloch Hamiltonian

$$(527) \quad \underline{h}_f(\vec{b}) = \varepsilon_f(\vec{b}) \mathbb{1} + \vec{n}(\vec{b}) \cdot \vec{\sigma}$$

and one can define a Chern number

$$(528) \quad C_f = \frac{1}{4\pi} \int_{BZ} d^2k \left(\frac{\partial \hat{n}(\vec{k})}{\partial k_x} \times \frac{\partial \hat{n}(\vec{k})}{\partial k_y} \right) \cdot \hat{n}(\vec{k})$$

where $\hat{n}(\vec{k}) = \vec{n}(\vec{k}) / |\vec{n}(\vec{k})|$ which may be completely different from that of the original static Hamiltonian

$$(529) \quad C = \frac{1}{4\pi} \int_{BZ} d^2k \left(\frac{\partial \hat{d}(\vec{k})}{\partial k_x} \times \frac{\partial \hat{d}(\vec{k})}{\partial k_y} \right) \cdot \hat{d}(\vec{k})$$

where $\hat{d} = \vec{d} / |\vec{d}|$. Note that (529) is equivalent to a description in terms of Bloch states.

We will see later that the Floquet Hamiltonian does not capture all topological properties of the periodically driven system.

(A) Mathematical basis of Floquet theory

Consider a time-periodic Hamiltonian

$$(530) \quad H(t+T) = H(t)$$

Then the time-evolution operator can be written as

$$(531) \quad U(t+T, 0) = U(t, 0) U(T, 0)$$

So it is sufficient to know $U(t, 0)$ for $t \in (0, T)$ and $U(T, 0)$, which is called the one-cycle evolution operator, which we write as $(\hbar=1)$

$$(532) \quad U(T, 0) = \exp(-i H_{\text{eff}} T) \quad H_{\text{eff}}^\dagger = H_{\text{eff}}$$

With this we define

$$(533) \quad P(t) := U(t, 0) e^{+i H_{\text{eff}} t}$$

$$\begin{aligned} \Rightarrow P(t+T) &= U(t+T, 0) e^{i H_{\text{eff}} (t+T)} \\ &= U(t, 0) \cancel{U(T, 0)} e^{i H_{\text{eff}} T} e^{i H_{\text{eff}} t} = P(t) \end{aligned}$$

Thus

$$(534) \quad U(t, 0) = P(t) e^{-i H_{\text{eff}} t}$$

Micromotion
(T periodic)

effective evolution

where the unitary operator P is T -periodic

(535)

$$P(t+T) = P(t)$$

Floquet states and quasenergies

The eigenstates of the time independent hermitian operator H_{eff} are and form a complete set

(536)

$$H_{eff} |n\rangle = \epsilon_n |n\rangle$$

(assume finite Hilbert space for simplicity)

ϵ_n : quasi energies

An initial state

(537)

$$|\psi(0)\rangle = \sum_n a_n |n\rangle$$

then evolves in time according to

(538)

$$\begin{aligned} |\psi(t)\rangle &= U(t,0) |\psi(0)\rangle = \sum_n a_n P(t) e^{-iH_{eff}t} |n\rangle \\ &= \sum_n a_n \underline{P(t) |n\rangle} e^{-i\epsilon_n t} \end{aligned}$$

with time-independent a_n and ϵ_n

(539)

$$|\epsilon_n(t)\rangle \equiv P(t) |n\rangle$$

Floquet states

They are periodic in time

(540)

$$|\epsilon_n(t+T)\rangle = |\epsilon_n(t)\rangle$$

from time-periodic to time-independent problem

let us consider a unitary transformation

$$(541) \quad |\psi(t)\rangle = P(t) |\tilde{\psi}(t)\rangle$$

Then

$$\begin{aligned} i \frac{d}{dt} |\tilde{\psi}(t)\rangle &= i \dot{P}(t)^{-1} |\psi(t)\rangle + i P(t)^{-1} \frac{d}{dt} |\psi(t)\rangle \\ &= i \frac{d}{dt} \left(e^{-iH_0 t} U^{-1}(t, 0) \right) |\psi(t)\rangle + P(t)^{-1} H(t) \underbrace{P(t) |\tilde{\psi}(t)\rangle}_{|\psi(t)\rangle} \\ &= H_{\text{eff}} \underbrace{e^{-iH_0 t} U^{-1}(t, 0) |\psi(t)\rangle}_{P^{-1}} - \underbrace{e^{-iH_0 t} U^{-1} H(t) P(t) |\tilde{\psi}(t)\rangle}_{P^{-1}} \\ &\quad + \cancel{P^{-1}(t) H(t) P(t) |\tilde{\psi}(t)\rangle} \\ &= H_{\text{eff}} |\tilde{\psi}(t)\rangle \end{aligned}$$

J.e.

$$(542) \quad i \frac{d}{dt} |\tilde{\psi}(t)\rangle = H_{\text{eff}} |\tilde{\psi}(t)\rangle$$

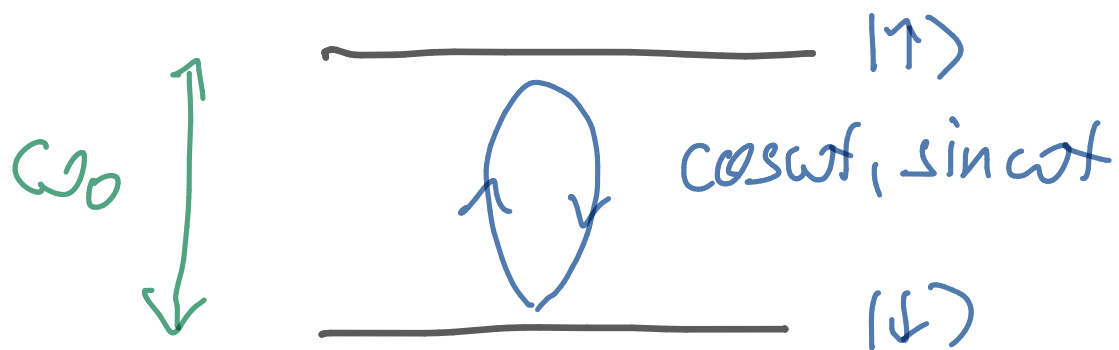
time-independent
problem in
rotating frame

This allows the solution of certain integrable time-periodic problems.

example

(i) two-level system in time-periodic
circular polarized light
(without rotating wave approximation.)

$$(543) \quad H(t) = \frac{\omega_0}{2} \sigma_z + \frac{\mu E}{2} (\sigma_x \cos \omega t + \sigma_y \sin \omega t)$$



choose

$$(544) \quad P(t) = e^{-i \frac{\omega_0}{2} \sigma_z t} e^{i \frac{\omega}{2} t} = \exp\left(i \frac{\omega t}{2} (1 - \sigma_z)\right)$$

with

$$i \frac{d}{dt} | \psi(t) \rangle = i \dot{P}(t) | \tilde{\psi}(t) \rangle + i P \frac{d}{dt} | \tilde{\psi}(t) \rangle$$

$\Rightarrow$

$$i \frac{d}{dt} | \tilde{\psi}(t) \rangle = \left(P^{-1}(t) H(t) P(t) - i P^{-1}(t) \dot{P}(t) \right) | \tilde{\psi}(t) \rangle$$

so

$$G = P^{-1}(t) \left[H(t) - i \frac{d}{dt} P(t) \right] P(t)$$

(545)

$$= \frac{\omega}{2} \mathbb{1} + \frac{1}{2} (\omega_0 - \omega) \sigma_z + \frac{\mu E}{2} \sigma_x$$

quasi energies:

$$(546) \quad \underline{\underline{\varepsilon_{\pm} = \frac{1}{2} (\omega \pm \Omega)}}$$

$$\underline{\underline{\Omega = \sqrt{(\omega_0 - \omega)^2 + (\mu E)^2}}}$$

This sounds all too simple! So where is the catch?

(i) the transformation $P(t)$ can in general only be found approximately
 $\Rightarrow$ high frequency expansion

(ii) the quasi energy spectrum has a Brillouin zone structure and is not unique

Let's look at the limit of $\mathcal{E}_{\pm}$ if $E \rightarrow 0$

$$\omega < \omega_0 \quad \begin{aligned} \mathcal{E}_+ &\rightarrow + \frac{\omega_0}{2} \\ \mathcal{E}_- &\rightarrow - \frac{\omega_0}{2} + \omega \end{aligned}$$

?

$$\omega > \omega_0 \quad \begin{aligned} \mathcal{E}_+ &\rightarrow + \frac{\omega_0}{2} + \omega \\ \mathcal{E}_- &\rightarrow - \frac{\omega_0}{2} \end{aligned}$$

In fact we realize from (535) that a transformation

$$(547) \quad P(t) \rightarrow P(t) e^{\pm i n \omega t} \quad \omega = \frac{2\pi}{T}$$

does not change the results. So the quasienergies are only defined modulo ω

$$(548) \quad \boxed{\mathcal{E}_n \rightarrow \mathcal{E}_{n,m} = \mathcal{E}_n + m\omega} \quad m = 0, \pm 1, \pm 2$$

$\Rightarrow$ Brillouin zone structure of quasi-energy spectrum

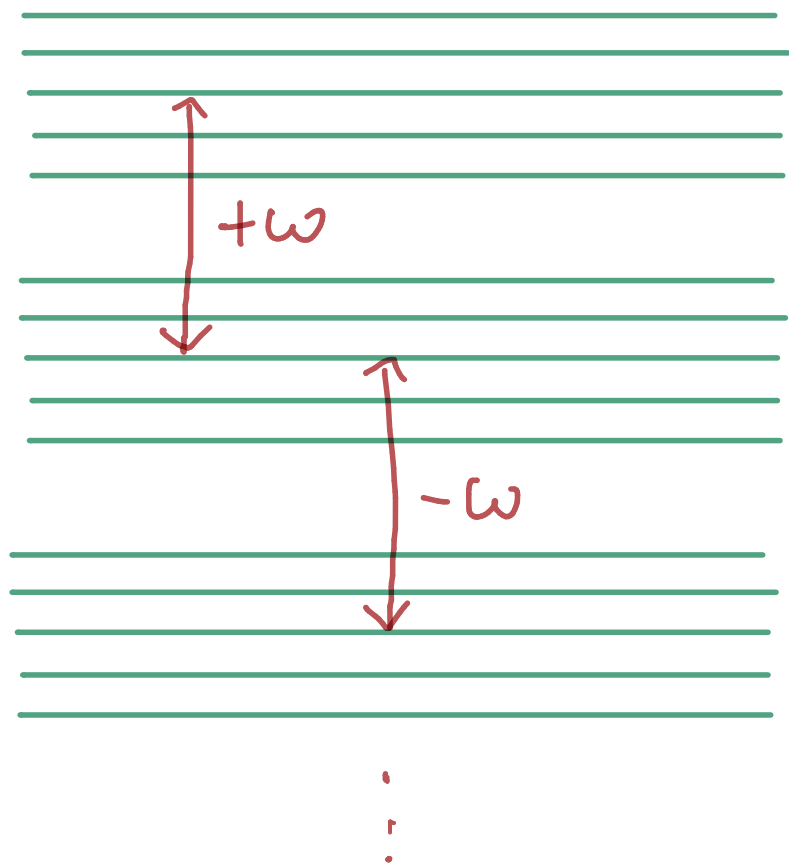
1. Floquet band

(549)

$$-\frac{\omega}{2} < \varepsilon \leq \frac{\omega}{2}$$

⋮

"energetically
degenerate"
periodicity



} ε_n 1st Floquet band

(B) Approximate solutions: high-frequency expansion

Let's return to the two-level example in Sec. (A) but now with linear polarized light

$$(550) \quad H(t) = \frac{\omega_0}{2} \sigma_z + \mu E \sigma_x \cos \omega t$$

Here we do not find a simple analytic function $P(t)$. However if we use $P(t)$ from eq. (544) we would get

$$(557) \quad P^{-1}(t) \left[H(t) - i \frac{d}{dt} \right] P(t) =$$

$$= H_{\text{eff}} + \frac{\mu E}{2} \left(\cancel{b_x \cos 2\omega t} \approx 0 - \cancel{b_y \sin 2\omega t} \approx 0 \right)$$

with H_{eff} from (545), i.e. the circular polarized case.

We realize that the remaining time dependent terms are fast oscillating and one can neglect

than hoping that their effect cancels out when

time averaging over times $\Delta t \gg T$.

This is the famous rotating wave approximation.

small parameter

$$\frac{\epsilon_{\text{typical}}}{\omega}$$

General goal:

- find a unitary such that

$$P(t) = e^{-iK}$$

$$(552) \quad P^{-1}(t) \left[H(t) - i \frac{d}{dt} \right] P(t) \simeq H_{\text{eff}} = \text{constant}$$

procedure:

expansion in ω^{-1}

$$(553) \quad H_{\text{eff}} = \sum_{n=0}^{\infty} \frac{1}{\omega^n} H_{\text{eff}}^{(n)}$$

$$(554) \quad K(t) = \sum_{n=0}^{\infty} \frac{1}{\omega^n} K^{(n)}$$

$$\begin{aligned}
 P^{-1} H P &= e^{ik} H e^{-ik} && \text{Baker Hausdorff} \\
 &= H + i \underbrace{[k, H]}_{\omega^{-1}} - \frac{1}{2!} \underbrace{[k, [k, H]]}_{\omega^{-2}} + \dots && \text{etc.}
 \end{aligned}$$

$$\begin{aligned}
 -i P^{-1} \frac{d}{dt} P &= i \left(\frac{d}{dt} P^{-1} \right) P \\
 &= - \underbrace{\frac{\partial k}{\partial t}}_{\omega^{-1}} - \frac{i}{2} \underbrace{[k, \frac{\partial k}{\partial t}]}_{\omega^{-2}} + \dots && \text{etc.}
 \end{aligned}$$

- We now apply this expansion to a Hamiltonian

$$(555) \quad H(t) = H_0 + \hat{V}(t)$$

with

$$(556) \quad \hat{V}(t) = \sum_{n=-\infty}^{\infty} \left(\hat{V}^{(n)} e^{in\omega t} + \hat{V}^{(-n)} e^{-in\omega t} \right)$$

- up to zeroth order in $1/\omega$ one finds

$$(556) \quad H_{\text{eff}}^{(0)} = H_0 + \hat{V}(t) - \frac{\partial k^{(1)}(t)}{\partial t}$$

choosing
(557)

$$K^{(1)}(t) = \int_0^t d\tau V(\tau)$$

makes $H_{\text{eff}}^{(0)}$ time independent.

- up to first order in $1/\omega$ one finds for the effective Hamiltonian

$$(558) \quad H_{\text{eff}} = H_0 + \frac{1}{\omega} \sum_{n=1}^{\infty} \frac{1}{n} [\hat{V}^{(n)}, \hat{V}^{(-n)}] + \mathcal{O}\left(\frac{1}{\omega^2}\right)$$

and for the kick operator determining the micromotion

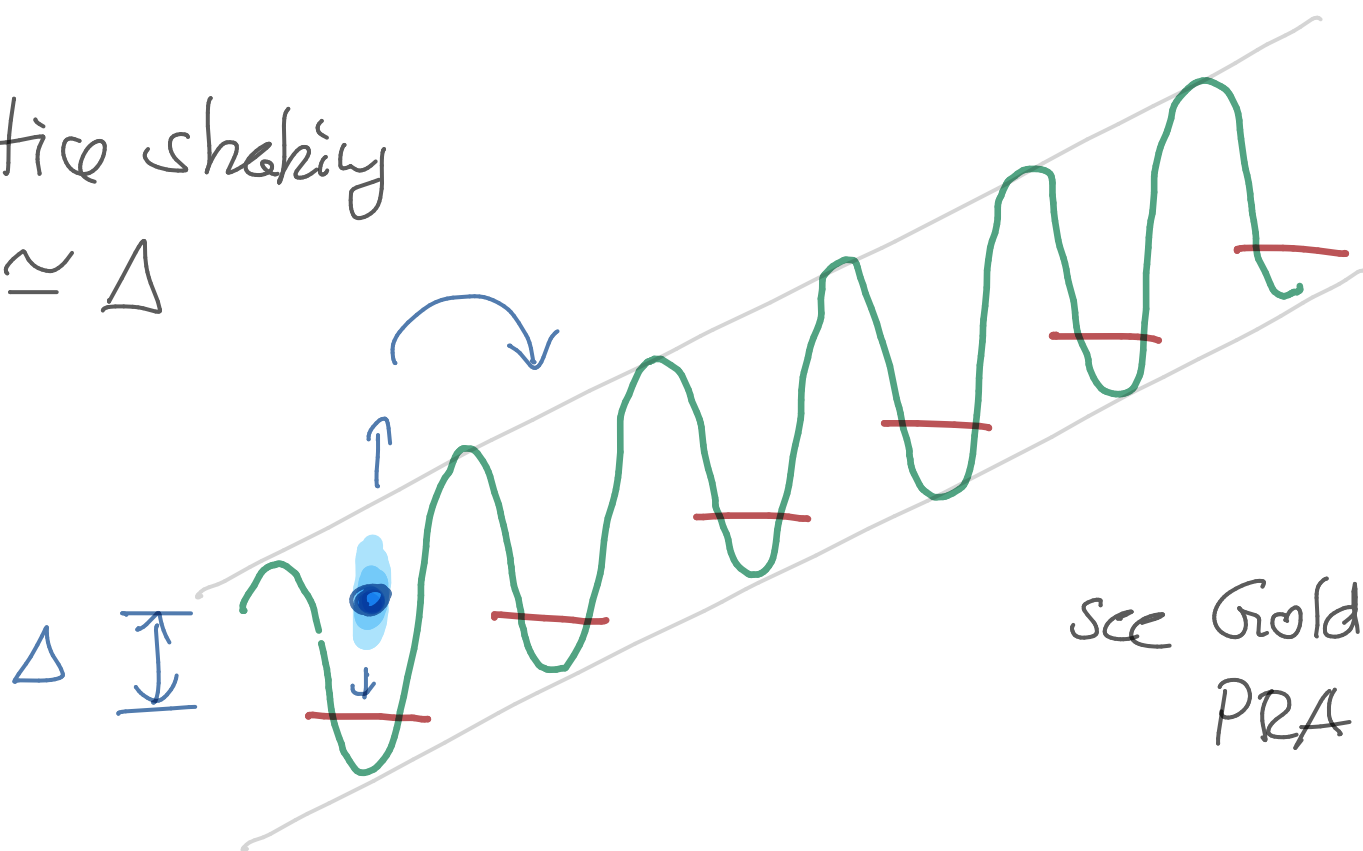
$$(559) \quad K = \frac{1}{i\omega} \sum_{n=1}^{\infty} \frac{1}{n} \left(\hat{V}^{(n)} e^{in\omega t} - \hat{V}^{(-n)} e^{-in\omega t} \right) + \mathcal{O}\left(\frac{1}{\omega^2}\right)$$

(C) Generalization to systems with resonant modulation *

Floquet techniques are used e.g. for optical lattice schemes where a linear potential is added forming a so-called Wannier-Stark ladder

Lattice shaking

$$\omega \simeq \Delta$$



see Goldman et al.
PRA 91, 033632 (2015)

For these systems $H_0 \sim \Delta = \omega$ and the above derivation of the Floquet Hamiltonian must be slightly revised. We assume that the static Hamiltonian H_0 has the form

$$(560) \quad H_0 = \sum_{\alpha, \beta} \hat{\Pi}^\alpha H_{\alpha\beta}^0 \hat{\Pi}^\beta + \omega \sum_{\alpha} \hat{\Pi}^\alpha$$

where $\hat{\Pi}^\alpha$ are orthogonal projectors in some sectors labelled by α . Then

$$(561) \quad \hat{V}^{(\pm n)} = \sum_{\alpha, \beta} \hat{\Pi}^\alpha \hat{V}_{\alpha\beta}^{(\pm n)} \hat{\Pi}^\beta \quad n=1, 2, 3, \dots$$

Now we add a unitary trf

$$(562) \quad R(t) = \exp \left\{ i \sum_{\alpha} \alpha \omega t \hat{\Pi}^\alpha \right\} = R(H + T)$$

i.e.
(563)

$$|2\rangle \rightarrow |2'\rangle = R H |2\rangle$$

and

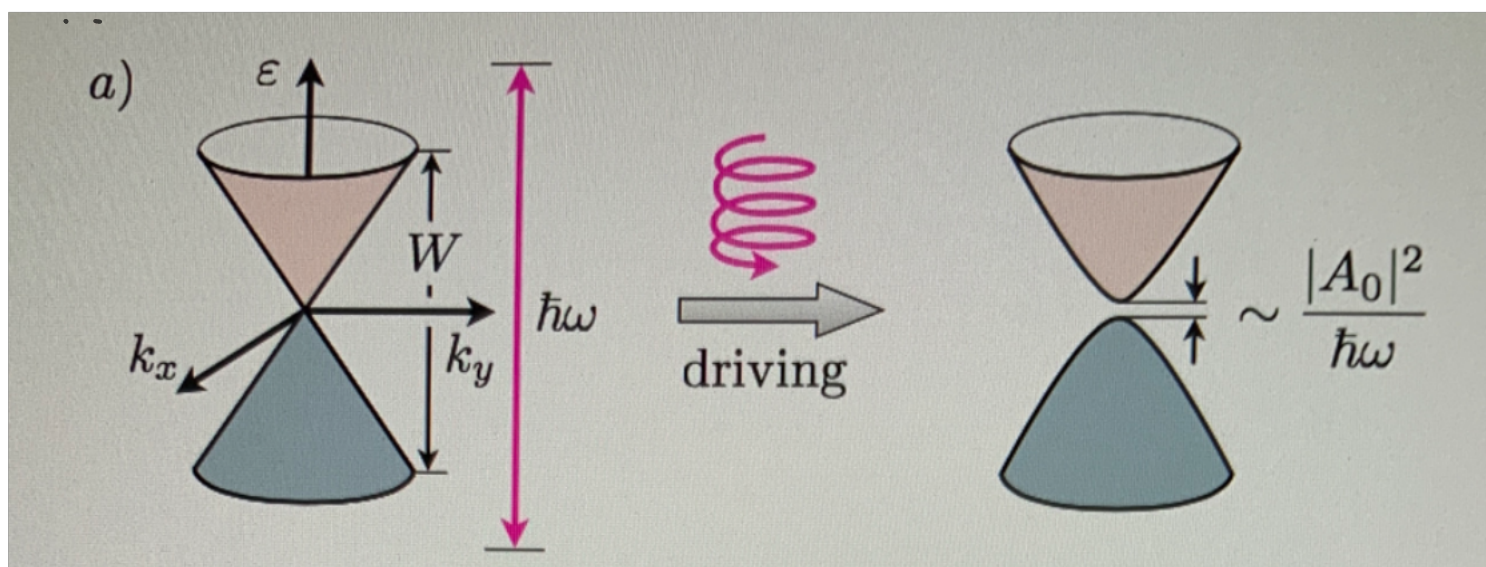
$$(564) \quad H(t) \rightarrow \tilde{H}(t) = R H R^\dagger - i R \frac{d}{dt} R^\dagger$$

In $\tilde{H}$ we now got rid of terms $\sim \omega$ so we can apply the technique from Ser. (B).

12.2 Topology of Floquet bands

(A) Creating topological phases by high-frequency modulation

- Graphene irradiated by circular polarized light



low-energy Hamiltonian of Graphene

$$(565) \quad \underline{h}_0(\vec{k}, t) = v_F (\sigma_x \tau_x q_x + \sigma_y q_y)$$

where $\vec{q} = \vec{k} - \xi \vec{k}'$ $\xi = \pm 1$

σ - sublattice τ - valley $\mathbf{k} \leftrightarrow \mathbf{k}'$

(566) with elm. driving

$$\underline{h}(\vec{k}, t) = v_F (\sigma_x \tau_x (q_x + eA_x(t)) + \sigma_y (q_y + eA_y(t)))$$

where

$$(567) \quad \vec{A} = -A_0 (\sin \omega t \hat{e}_x + \sin(\omega t - \varphi) \hat{e}_y)$$

φ allows to tune polarization.

high-frequency limit $\omega \gg eA_0 v_F$

$$(568) \quad \underline{h}_F(\vec{k}) \approx \underline{h}_0 + \frac{1}{\omega} [\underline{h}^{(-1)}, \underline{h}^{(+1)}]$$

$$= \underline{h}_0 - \frac{(eA_0 v_F)^2}{\omega} \sin \varphi \sigma_z \tau_z$$

Haldane mass term

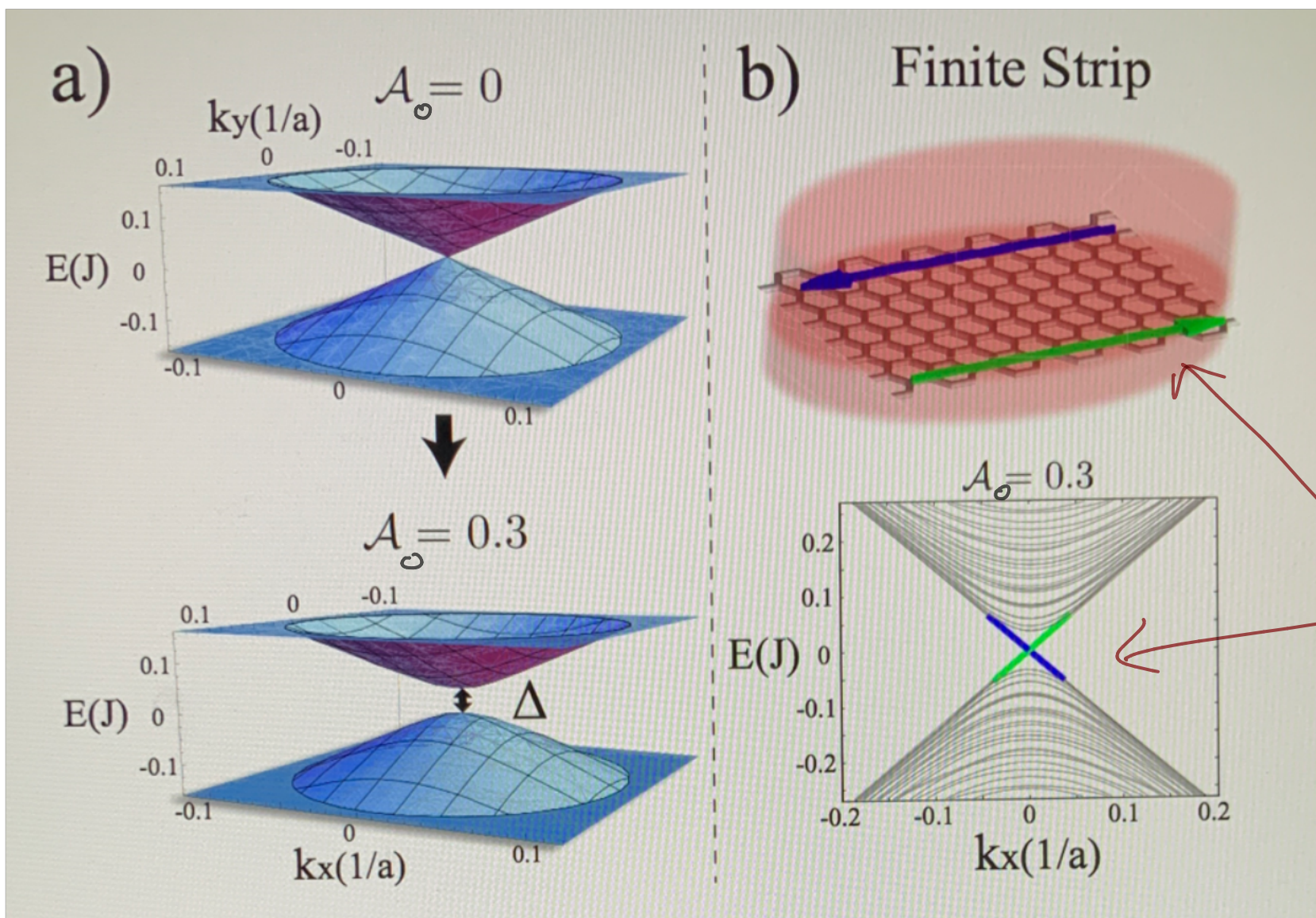
$\Rightarrow$ photoinduced gap ($\hbar = 1$)

$$(569) \quad \Delta = 2 \frac{(eA_0 v_F)^2}{\omega} \sin \varphi$$

vanishes for linear polarization $\varphi = 0, 2\pi$

Chirality generated by circular polarized light!

$\Rightarrow$ Floquet topological insulator

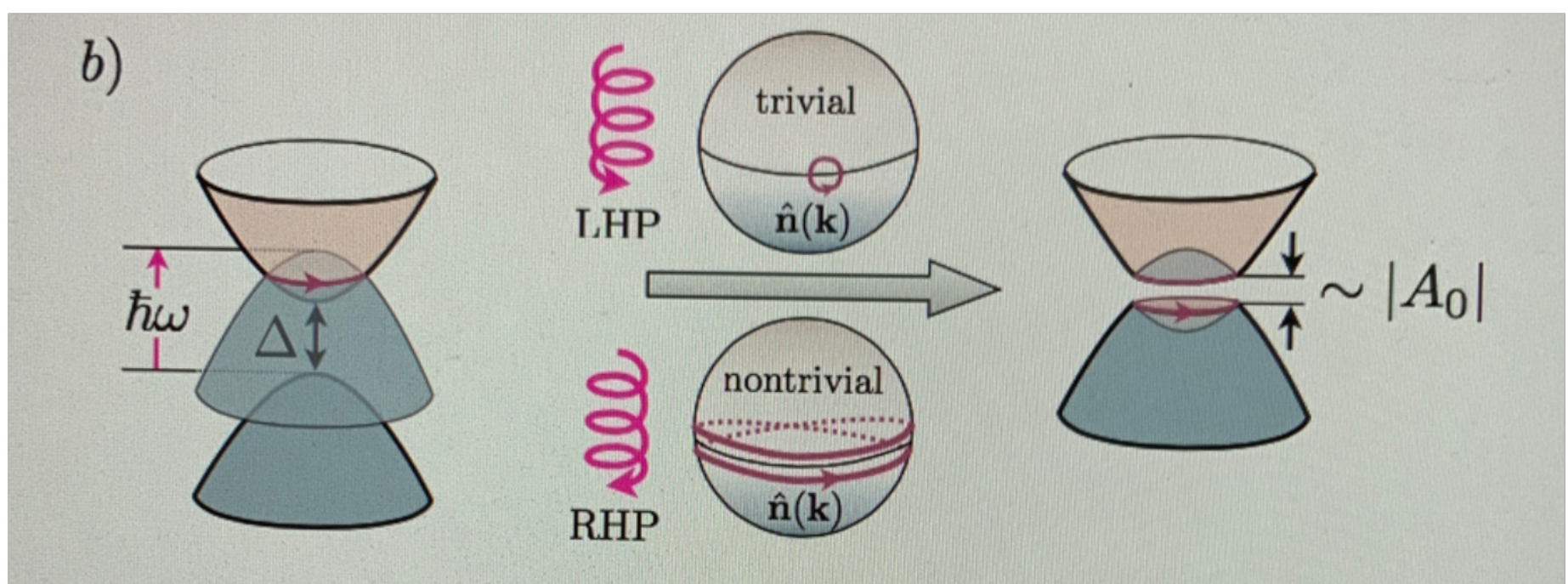


Kitagawa
et al.
PRB 84
235108
(2011)

chiral
edge
modes

• HgTe/CdTe heterostructures with circular pol. light

A second example where fast periodic drive creates a topological insulator are HgTe/CdTe heterostructures



This system can show a QSH (Brenneig, Hughes, Zhang
Science 314, 1757 (2006)) and is TR symmetric

$$(570) \quad \underline{\underline{h}}(\vec{k}) = \begin{pmatrix} \underline{\underline{\tilde{h}}}(\vec{k}) & 0 \\ 0 & \underline{\underline{\tilde{h}}}^*(-\vec{k}) \end{pmatrix} \quad 4 \times 4$$

$$(571) \quad \underline{\underline{h}}(\vec{k}) = \varepsilon(\vec{k}) \underline{\underline{1}} + \vec{d}(\vec{k}) \cdot \underline{\underline{\sigma}} \quad \text{sub-block Hamiltonian}$$

$\vec{d}(\vec{k})$ describes spin-orbit coupling. The spectrum has two doubly-degenerate subbands

$$(572) \quad \varepsilon_{\pm}(\vec{k}) = \varepsilon(\vec{k}) \pm |\vec{d}(\vec{k})| \quad \hat{d} = \frac{\vec{d}}{|\vec{d}|}$$

Within each of the two sub-blocks of $\underline{\underline{h}}(\vec{k})$ one finds (opposite) Chern numbers

$$(573) \quad C_{\pm} = \pm \frac{1}{4\pi} \int d^2\vec{k} \left(\frac{\partial \hat{d}}{\partial k_x} \times \frac{\partial \hat{d}}{\partial k_y} \right) \cdot \hat{d}$$

around the Γ -point

$$(574) \quad \vec{d}(\vec{k}) \simeq (Ak_x, Ak_y, M - Bk^2)$$

where $A < 0$, $B > 0$ and M depend on the thickness of the quantum well. One can show that for the simplified band structure with (574)

$$(575) \quad C_{\pm} = \pm \left[1 + \text{sign} \left(\frac{M}{B} \right) \right] / 2$$

So changing the sign of M modifies the sub-band Chern number by 1.

Note that $C_{\pm} + \overline{C}_{\pm} = 0$ by TR symmetry

• Start in topological trivial state

$$(576) \quad M < 0 \quad C_{\pm} = 0$$

Lindner, Refael, Galitski, Nature Phys 7, 490 (2011):

$\Rightarrow$ topological phase transition from $M < 0$ to $M > 0$
by periodic drive:

$$(576) \quad V = \vec{V} \cdot \vec{\sigma} \cos(\omega t)$$

To conserve TR symmetry: linear polarized light.

• Effective Floquet Hamiltonian for sub-band

$$\tilde{h} \rightarrow \tilde{h}^F(\vec{k}) = E_F(\vec{k}) \mathbb{1}_{2 \times 2} + \vec{n}(\vec{k}) \cdot \vec{\sigma} \quad \hat{n} = \frac{\vec{n}}{|\vec{n}|}$$

Floquet Chern number of sub-band

$$(577) \quad C_{\pm}^F = \pm \frac{1}{4\pi} \int d^2k \left(\frac{\partial \hat{n}}{\partial k_x} \times \frac{\partial \hat{n}}{\partial k_y} \right) \cdot \hat{n}$$

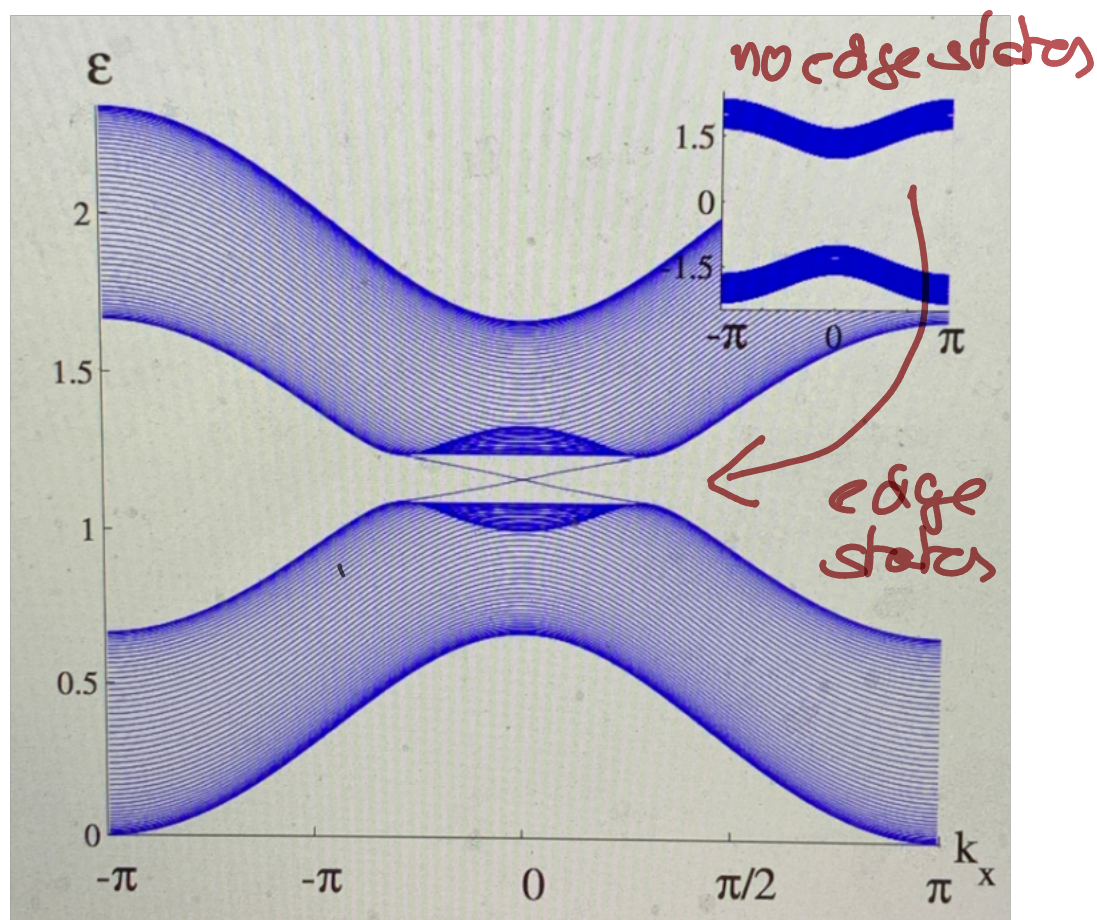
and

$\Rightarrow$

when starting from
 $M < 0$

$$C_{\pm}^F = C_{\pm} \pm 1$$

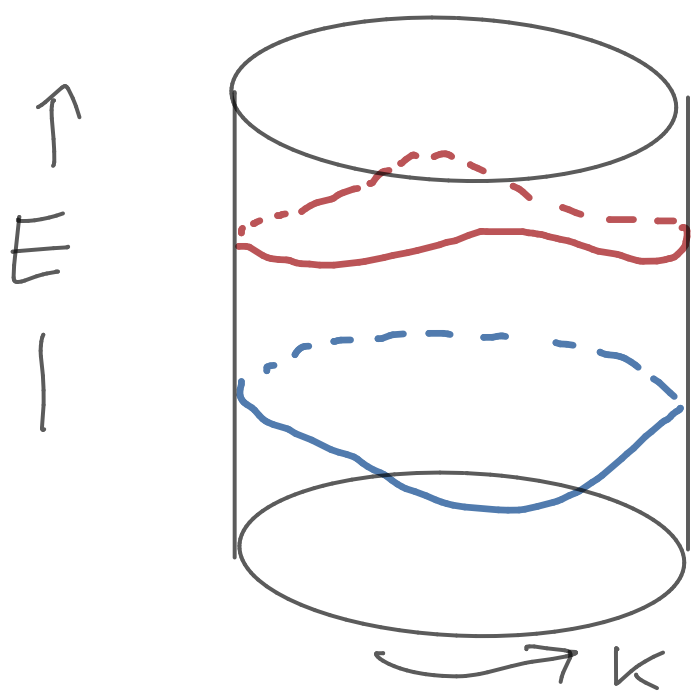
(578)



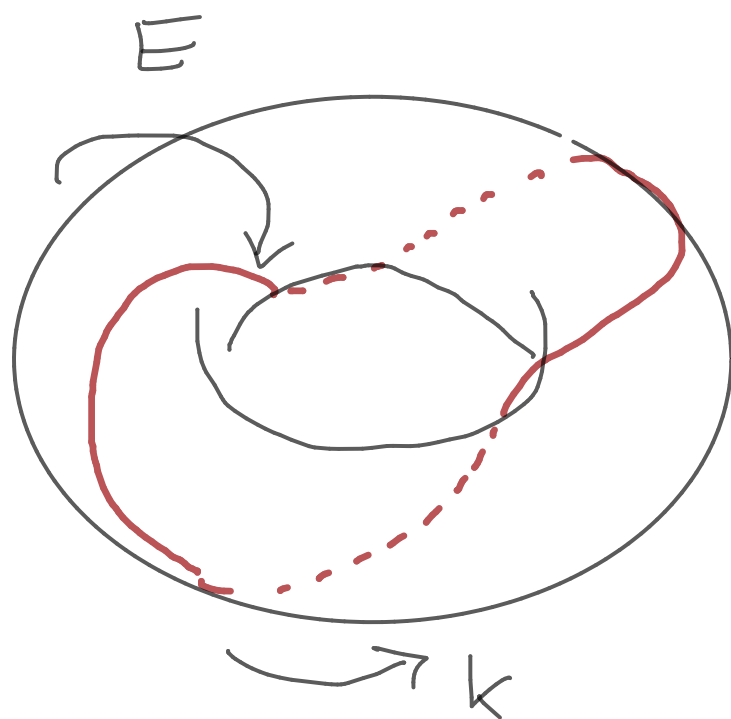
(B) Topology of Floquet bands beyond effective Hamiltonian

There are important generic cases where the Floquet Hamiltonian does not capture all topological properties. This is due to periodicity in quasi-energy

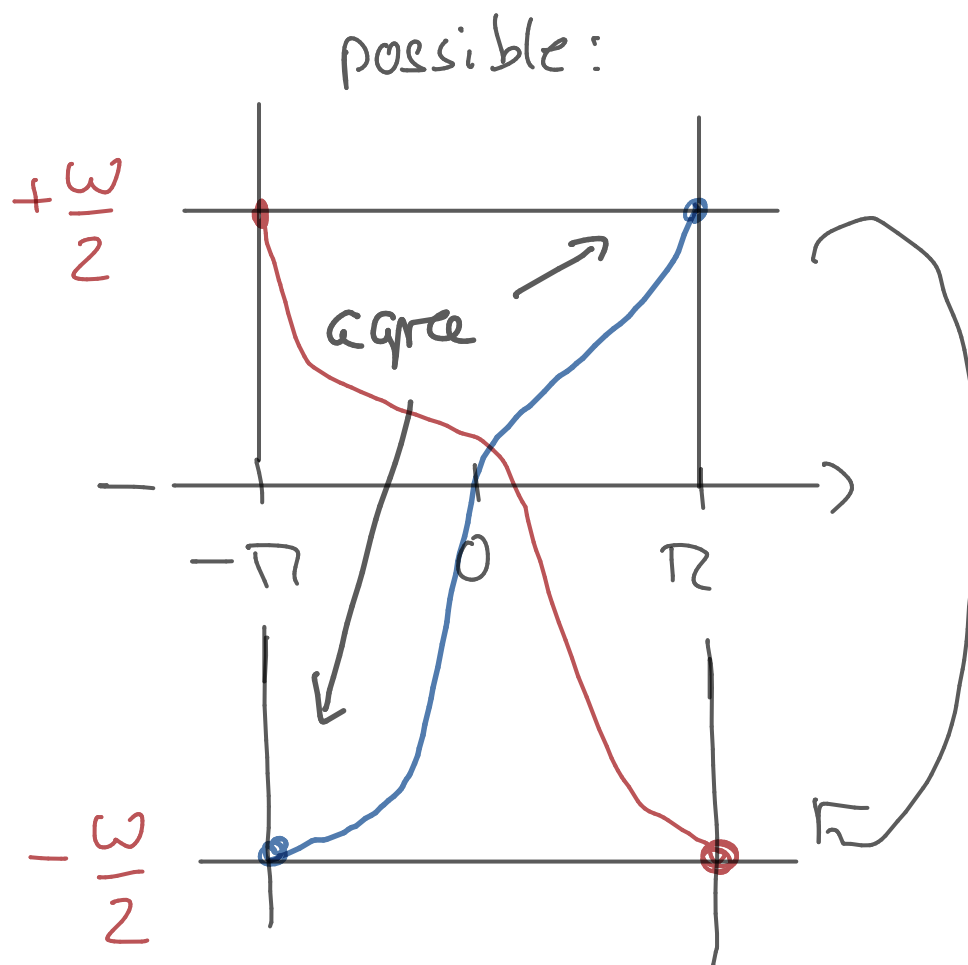
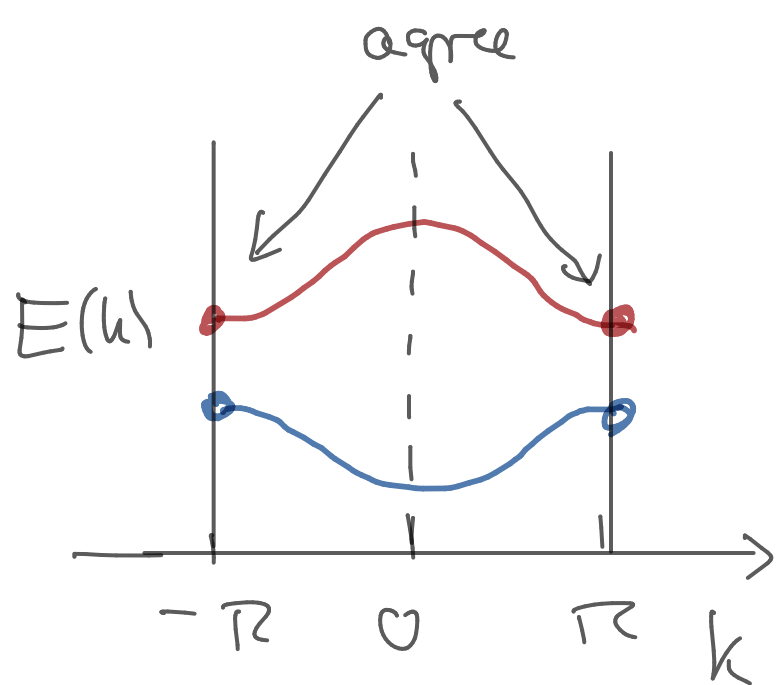
- there is no "top" or "bottom" in Floquet spectrum
- continuous band of quasienergies may wind around the Floquet Brillouin zone an integer number of times when the lattice momentum traverses the Brillouin zone.



non-driven



Floquet



physical consequence of quasienergy winding

(I) quantized bulk transport in 1D Floquet system ($\hat{=}$ Thouless pump) in Floquet cycle

average group velocity of Floquet band

$$(579) \quad \overline{v_g} = \frac{a}{2\pi} \int_{BZ} dk \frac{\partial \varepsilon}{\partial k} = a \frac{\omega}{T} \quad a: \text{lattice constant}$$

$$\Rightarrow \overline{\Delta x} = \overline{v_g} T = \omega a$$

In each cycle ωa units of charge are pumped

Floquet operator

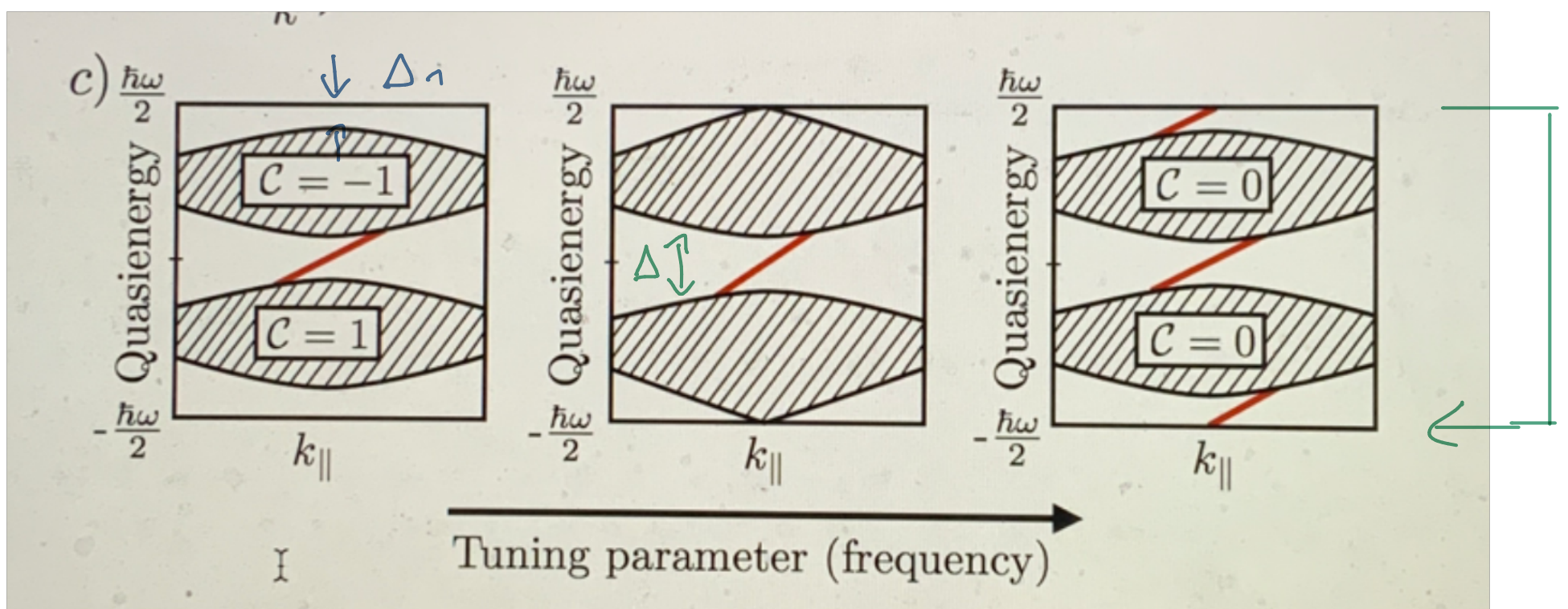
$$U_{10}(k, T) = T \exp(-i \int_0^T dt H_{1D}(k, t))$$

$$(579) \quad \nu_n = \frac{1}{2\pi} \int_{BZ} dk \operatorname{Tr} \left\{ U_{10}^\dagger(k, T) i \frac{\partial}{\partial k} U_{10}(k, T) \right\}$$

GNVW index (Gross, Nesme, Vogh, Werner
Comm. Math. Physics 310, 413 (2012))

In general bands with opposite ν_n 's which cross!
Then transport vanishes. Hybridization gap
 $\sim 1/\omega$. But for $\omega \rightarrow 0$ i.e. in the limit of slow
or quasi-adiabatic drive true crossing $\rightarrow$ quantized
transport.

(II) new topological transitions due to
periodicity in quasi-energy



gap Δ never closes, so edge mode in the middle
cannot disappear but quasienergy gap Δ_n may
close and

$$C = \pm 1 \rightarrow C_{\mp} = 0$$

Spectral flow:

Net chirality of edge modes above and below a band must be equal to Chern number of band

$\Rightarrow$ emergence of novel "edge" modes
at top and bottom i.e. at $\varepsilon \approx \pm \omega/2$
in addition to that at $\varepsilon \approx 0$

anomalous edge states $C_F = 0$

Potential problem:

Many body systems in Floquet systems
can continuously absorb energy and heat up.
Combining periodic drive with reservoir couplings
may overcome this problem
