

9. Fractional Quantum Hall Effect and Composite Fermions

So far we have discussed topological systems without interactions or where interactions play only a perturbative role. Now we want to discuss systems where interactions either strongly modify or even induce topological properties.

Historically the first interacting topological system is the

Fractional Quantum Hall Effect (FQHE)

Shortly after the discovery of the IQHE, Tsui, Stormer and Gossard observed in a 2D electron gas subjected to even stronger magnetic fields quantized Hall plateaus

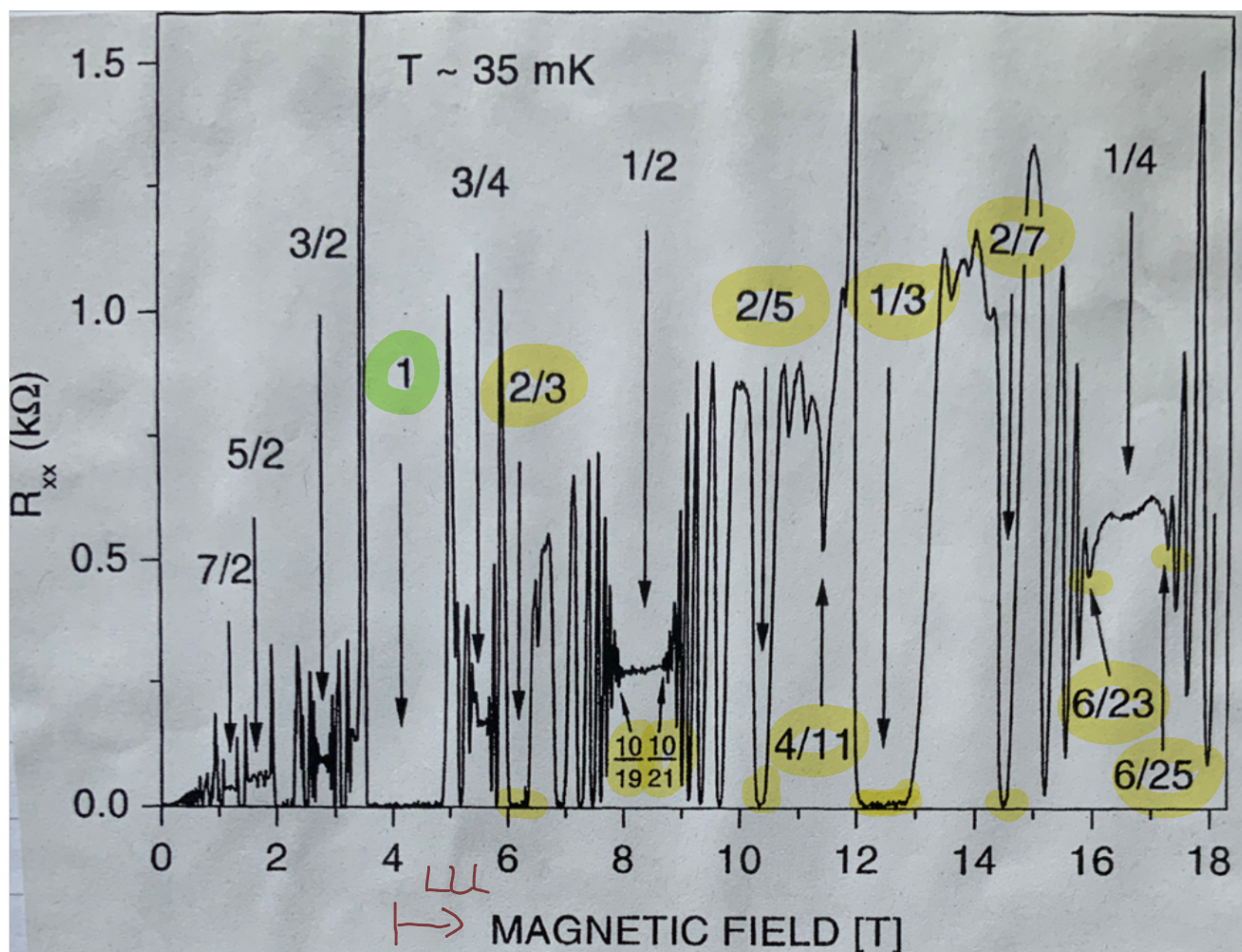
$$(349) \quad \sigma_{xx} = f \frac{e^2}{h}$$

which correspond to fractional fillings f mostly less than unity

Most prominent fractional plateaus had the structure

$$(350) \quad \boxed{f = \frac{n}{2np \pm 1}} \quad n, p \in \mathbb{N}$$

These could only be understood as resulting from interactions.



9.1 Haldane's pseudopotentials

In order to see what properties the many-body state must have to minimize the total energy including the interaction it is useful to consider the projection of the interaction to the relevant sub-space of states

for the very high magnetic fields (see figure) it is sufficient to

- (i) consider only one spin (Zeeman effect)
- (ii) consider only LLL (lowest Landau level)

2-particle problem in 2D in external magnetic field

$$(351) \quad H = H_1 + H_2 + V = H_0 + V$$

where

$$(352) \quad H_i = \frac{1}{2m} \left(\vec{p}_i + \frac{e}{c} \vec{A}(\vec{r}_i) \right)^2$$

$$V = V(|\vec{r}_1 - \vec{r}_2|)$$

complex coordinates in 2D $x_i, y_i \rightarrow z_i, z_i^*$

center of mass (COM) and relative coordinates

$$(353) \quad X = \frac{1}{2}(z_1 + z_2) \quad Z = z_1 - z_2$$

$$(354) \quad H = \frac{1}{2} \left[-4l_R^2 \frac{\partial^2}{\partial X \partial X^*} + \frac{1}{4l_R^2} X X^* - X \frac{\partial}{\partial X} + X^* \frac{\partial}{\partial X^*} \right]_{\text{COM}}$$

$$+ \frac{1}{2} \left[-4l_r^2 \frac{\partial^2}{\partial Z \partial Z^*} + \frac{1}{4l_r^2} Z Z^* - Z \frac{\partial}{\partial Z} + Z^* \frac{\partial}{\partial Z^*} \right]_{\text{rel. motion}}$$

$$+ V(Z, Z^*)$$

where

$$(355) \quad l_R = \frac{1}{\sqrt{2}} l_0 \quad l_r = \sqrt{2} l_0 \quad l_0 = \sqrt{\frac{\hbar c}{eB}}$$

As usual ~~we~~ can separate COM and relative motion

see Sec. 2 $\hat{a} = \frac{1}{\sqrt{2}} \left(\frac{Z}{2} + 2 \frac{\partial}{\partial Z^*} \right) \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}} \left(\frac{Z^*}{2} - 2 \frac{\partial}{\partial Z} \right)$

$$H = \hbar \omega_c (a^\dagger a + \frac{1}{2}) \quad L = \hbar (a^\dagger a - b^\dagger b) \quad \hat{b} = \frac{1}{\sqrt{2}} \left(\frac{z^*}{2} + 2 \frac{\partial}{\partial z} \right) \quad \hat{b}^\dagger = \frac{1}{\sqrt{2}} \left(\frac{z}{2} - 2 \frac{\partial}{\partial z^*} \right)$$

and similarly for COM motion with $z \rightarrow X$

$$\text{COM} \quad \hat{a}_R, \hat{a}_R^\dagger, \hat{b}_R, \hat{b}_R^\dagger$$

$$\text{relative} \quad \hat{a}, \hat{a}^\dagger, \hat{b}, \hat{b}^\dagger$$

lowest Landau level (see Sec. 2)

$$(356) \quad \boxed{\hat{a}_R^\dagger \hat{a}_R |\psi\rangle = \hat{a}^\dagger \hat{a} |\psi\rangle = 0}$$

Thus within the LLL a complete set of states is

$$(357) \quad |M, m\rangle = \frac{(\hat{b}_R^\dagger)^M (\hat{b}^\dagger)^m}{\sqrt{M! m!}} |0, 0\rangle$$

b, b_R describe angular momentum degrees of freedom (see Sec. 2). We have seen there that a polynomial z^m has angular momentum projection m (eq. 120)

• two-particle wavefunctions in LLL

$$(358) \quad \psi_{M,m}^{\text{LLL}}(z, X) \sim X^M z^m \exp\left\{-\frac{|z|^2}{4\ell_r^2} - \frac{|X|^2}{4\ell_R^2}\right\}$$

let us assume that the interaction V does not mix Landau levels i.e. is weak enough

$\Rightarrow$ We need to consider only the projection of V to the LL

$$PVP = \sum_{m,M} \sum_{m',M'} |M',m'\rangle \langle M',m'| V |M,m\rangle \langle M,m|$$

Since $V = V(|z|)$

$$\langle M',m'| V |M,m\rangle = \delta_{MM'} \delta_{mm'} \langle m| V |m\rangle$$

(359)

$$V_m \equiv \langle m| V |m\rangle$$

Haldane's
pseudopotential

for Coulomb interaction

(360)

$$V_m^C = \frac{1}{2\epsilon_0} \frac{\Gamma(m + \frac{1}{2})}{\Gamma(m+1)}$$

$$V(r) = \frac{1}{r}$$

for point interaction (2D)

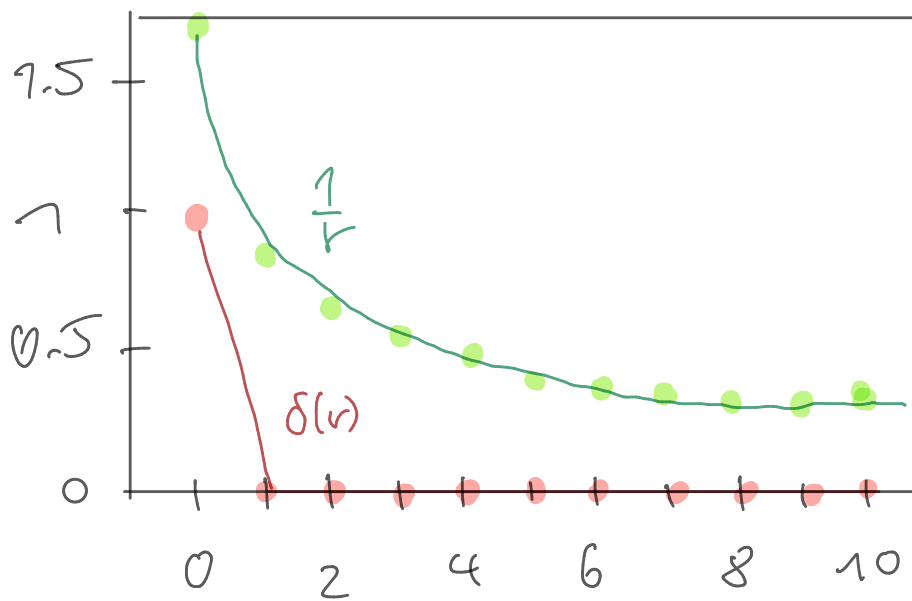
$$V(r) = \delta^{(2)}(r)$$

(361)

$$V_0 = 1 \quad V_2 = V_3 = V_4 = \dots = 0$$

(362)

$$PVP = \sum_{M,m} |M,m\rangle \langle M,m| V_m$$



Haldane's
pseudopotentials

bosons: only even m

fermions: only odd m

In a variational construction the best wavefunction is that in which the lowest pseudopotentials are vanishing.

9.2 Laughlin's wavefunction

Robert Laughlin constructed an ansatz wavefunction for the simplest fractions like $1/3$.

The single-particle wavefunctions in the LLL with angular momentum (in z direction) l are

$$(363) \quad \psi_l(z) = (2\pi 2^l l!)^{-1/2} z^l e^{-\frac{1}{4}|z|^2} \quad (l_0=1)$$

The many-body wavefunction must thus have the structure

$$(370) \quad \Psi = F_A [\{z_j\}] \exp \left\{ -\frac{1}{4} \sum_{j=1}^N |z_j|^2 \right\}$$

F_A must be anti-symmetric (for fermions) and only a function of the difference between coordinates.

One can make an educated guess;

(i) Motivated by the success of so-called Jastrow wavefunctions in the description of superfluid He we set

$$F_A[\{z_j\}] = \prod_{j < k} f(z_j - z_k)$$

(ii) Ψ should be an eigenfunction of total angular momentum

$$\Rightarrow \prod_{j < k} f(z_j - z_k) \text{ polynomial of } z_j \text{ of order } L$$

[rotation in x-y plane by angle θ

$$z_j \rightarrow z_j e^{i\theta}$$

if Ψ eigenfunction of $\hat{L}_z$: $\Psi \rightarrow \Psi e^{-iL\theta}$
i.e. $f(z_j - z_k)$ must have fixed angular momentum]

(iii) $f(z)$ must be anti-symmetric (fermions)
[or symmetric (bosons)]

$$f(-z) = f(z)$$

$$(371) \Rightarrow \boxed{f(z) = z^m} \quad m = \begin{cases} \text{odd fermions} \\ \text{even bosons} \end{cases}$$

This gives as an Ansatz wavefunction

$$(372) \quad \boxed{\Psi_{1/m} = \mathcal{N} \prod_{j < k} (z_j - z_k)^m \exp \left\{ -\frac{1}{4} \sum_{j=1}^N |z_j|^2 \right\}}$$

Laughlin wavefunction

One finds that $\Psi_{1/m}$ has a uniform density distribution with fractional filling

$$(373) \quad \boxed{\nu = \frac{1}{m} = \frac{N}{N\Phi}}$$

For Coulomb interactions $\Psi_{1/3}$ and $\Psi_{1/5}$
(i.e. $m=3, 5$) are exact ground states with filling
 $\nu = 1/3 ; 1/5$

but: $\exists$ other fractions at which there are
quantum Hall plateaus!

9.3 Jain's composite fermions

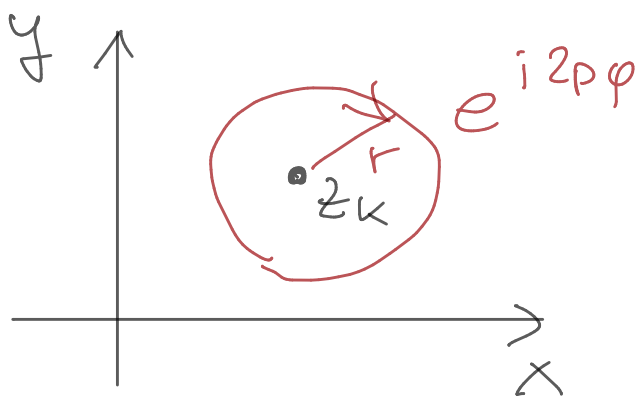
- Let's have a look again at Laughlin's wavefunction

$$(374) \quad \Psi_{1/2p+1} = N \prod_{j < k} \underbrace{(z_j - z_k)^{2p}}_{\text{like in IQHE}} e^{-\frac{1}{4} \sum_j |z_j|^2}$$

$$(375) \quad \boxed{\nu = \frac{1}{2p+1}} \quad \text{like in IQHE}$$

The prefactor looks like a vortex for the j th particle at the position of the k th particle

$$(z_j - z_k)^{2p} = (r e^{i\varphi})^{2p} = r^{2p} e^{i2p\varphi}$$



vortex of strength

$2p$

at position z_k of every other particle

interpretation

electrons in LLL with $2p$ vortices (flux quanta) attached to them

composite fermions



${}^{2p}\text{CF}$ bound state of fermion and even number ($2p$) of flux quanta attached to them

If the original particles are bosons, the total wave function must be symmetric

$$(376) \quad \psi_{1/2p} = N \prod_{j < k} \underbrace{(z_j - z_k)^{2p-1}}_{\text{as in IQHE}} \underbrace{(z_j - z_k) e^{-\frac{1}{4} \sum_e |z_e|^2}}_{\text{as in IQHE}}$$

These wavefunctions describe fractional quantum Hall states at even filling fractions

$$(377) \quad \boxed{\nu = \frac{1}{2p}}$$



$\Rightarrow \psi_{1/2p+1}$ and $\psi_{1/2p}$ describe non-interacting fermions with attached flux quanta

(A) Why are non-interacting composite fermions good candidates for (approximate) ground-state wavefunctions?

- Slater determinants of composite fermions fulfill all symmetry requirements and are eigenfunctions of H_0 (kinetic energy)
- how about interactions?

$$PVP = \sum_{m, M} V_m |Mm\rangle \langle Mm|$$

$$\langle \psi_{1/2p+1} | PVP | \psi_{1/2p+1} \rangle \sim$$

$$(378) \sim \sum_m V_m \left(\sum_{n=0}^m c_n \int d^2z z^{*n} z^{2p+1} e^{-|z|^2/4} \right)^2$$

$$| \langle Mm | \psi_{1/2p+1} \rangle |^2$$

(angular momentum $\hat{L}_z \sim -b^\dagger b$; $|Mm\rangle$ has all z^n with maximum m)

carry out angular integration $z = r e^{i\varphi}$

$$\int_0^{2\pi} d\varphi e^{i\varphi(2p+1-n)} \sim \delta_{n, 2p+1}$$

$\Rightarrow$ in eq. (378) only terms survive with
 $m \geq 2p+1$

CF wavefunctions screen all
pseudopotentials with
 $m \leq 2p$

i.e. all V_m with $m \leq 2p$ do not contribute
to the interaction energy

coupling of $2p$ flux quanta to a fermion
leads to formation of a 2p CF quasi-
particles which eliminates all pseudopotential
energies with $m \leq 2p$ and thus can
be approximately treated as non interacting

(B) FQHE as integer QHE of composite fermions: idea

(i) fermions at filling ν at magnetic field B

(379) $\nu = \frac{S \Phi_0}{|B|} \quad \vec{B} : \uparrow$

(ii) write fermion wavefunction as CF plus flux quanta in opposite direction

$$\textcircled{F} = \textcircled{\uparrow\uparrow}^{2p\text{ CF}} + \textcircled{\downarrow\downarrow}^{2p\text{ flux quanta}}$$

(iii) distribute the extra flux quanta homogeneously over the whole system

(380) $\Rightarrow$ effective magnetic field

$$B^* = B - 2pS\Phi_0$$

This corresponds to an effective filling ν^* in the reduced magnetic field

$$\begin{aligned}\nu^* &= \frac{S\Phi_0}{|B^*|} = \frac{S\Phi_0}{|B - 2pS\Phi_0|} \\ &= \frac{1}{\left|\frac{B}{S\Phi_0} - 2p\right|} = \frac{1}{|1/\nu - 2p|}\end{aligned}$$

So if the ^{2p}CF fill $\nu^* = n$ effective Landau levels
 $\Rightarrow$

(381)

$$\nu = \frac{n}{2pn \pm 1}$$

Jain's
principle
series of
fractions

$\Rightarrow$ FQHE $\hat{=}$ IQHE for composite fermions in a reduced magnetic field

9.4* Chern-Simons field theory of composite fermions

(A) bare composite particles

Our goal is to construct a field theory of composite fermions which arise from bare fermions by attachment of flux quanta

• bare fermion field $\hat{\psi}(z)$ $z = x + iy$

$$(382) \quad \{ \hat{\psi}(z), \hat{\psi}^\dagger(z') \} = \delta^{(2)}(z - z')$$

• define bare composite particle

$$(383) \quad \hat{\phi}_m(z) \equiv e^{-im\hat{\Theta}(z)} \hat{\psi}(z)$$

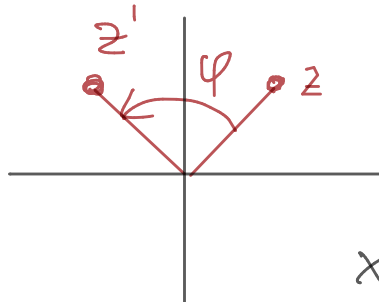
where the angle operator $\hat{\Theta}(z)$ is defined as

$$(384) \quad \hat{\Theta}(z) \equiv \int d^2z' \varphi(z - z') \hat{\psi}^\dagger(z') \hat{\psi}(z')$$

Note that

$$\hat{\psi}^\dagger(z) \hat{\psi}(z) = \hat{\phi}_m^\dagger(z) \hat{\phi}_m(z) = \hat{n}(z)$$

- $\varphi(z-z')$ is the angle between z and z' in the complex plane



Using

x, y also used as complex coordinates

$$\begin{aligned} (385) \quad [\hat{\theta}(z), \hat{\psi}(x)] &= \int d^2 z' \varphi(z-z') [\hat{n}(z'), \hat{\psi}(x)] \\ &= -\varphi(z-x) \hat{\psi}(x) \quad \underbrace{-\delta(x-z')} \end{aligned}$$

$\Rightarrow$

$$(386) \quad e^{im\hat{\theta}(y)} \hat{\psi}(x) e^{-im\hat{\theta}(y)} = e^{-im\varphi(y-x)} \hat{\psi}(x)$$

Fehler im Skript!

With this we can determine the commutation relations of the $\hat{\phi}_m$

$$\begin{aligned} [\hat{\phi}_m(x), \hat{\phi}_m(y)] &= e^{-im\hat{\theta}(x)} \hat{\psi}(x) e^{im\hat{\theta}(y)} \hat{\psi}(y) \\ &\quad - e^{-im\hat{\theta}(y)} \hat{\psi}(y) e^{-im\hat{\theta}(x)} \hat{\psi}(x) \\ &= e^{-im\varphi(y-x)} e^{-im\hat{\theta}(x)} e^{im\hat{\theta}(y)} \hat{\psi}(x) \hat{\psi}(y) \\ &\quad - e^{-im\varphi(x-y)} e^{-im\hat{\theta}(y)} e^{-im\hat{\theta}(x)} \hat{\psi}(y) \hat{\psi}(x) \end{aligned}$$

$$\Rightarrow \text{with } [\hat{\theta}(x), \hat{\theta}(y)] = 0$$

$$(387) \quad \hat{\phi}_m(x) \hat{\phi}_m(y) = (-1)^{m+1} \hat{\phi}_m(y) \hat{\phi}_m(x)$$

Thus

$$(388) \quad \begin{array}{ll} m: \text{odd} & [\hat{\phi}_m(x), \hat{\phi}_m(y)] = 0 \quad \text{bosons} \\ m: \text{even} & \{\hat{\phi}_m(x), \hat{\phi}_m(y)\} = 0 \quad \text{fermions} \end{array}$$

Note that composite bosons ($m: \text{odd}$) constructed from bare fermions still fulfill the hard-core constraint

$$(389) \quad \hat{\phi}_m^2(x) = 0 \quad \text{since} \quad \hat{\psi}^2(x) = 0$$

One can repeat the same calculations to find

$$(390) \quad \begin{array}{ll} [\hat{\phi}_m(x), \hat{\phi}_m^\dagger(y)] = \delta^{(2)}(x-y) & m: \text{odd} \\ \{\hat{\phi}_m(x), \hat{\phi}_m^\dagger(y)\} = \delta^{(2)}(x-y) & m: \text{even} \end{array}$$

• relation between wave-functions

N -particle wave-function for many-body state $|\Psi\rangle$

$$(391) \quad \Phi_m(x_1, \dots, x_N) \equiv \langle 0 | \underbrace{\hat{\Phi}_m(x_1) \dots \hat{\Phi}_m(x_N)}_{\text{composite}} | \psi \rangle$$

$$\psi(x_1, \dots, x_N) = \langle 0 | \underbrace{\hat{\Psi}(x_1) \dots \hat{\Psi}(x_N)}_{\text{bare}} | \psi \rangle$$

or

$$| \psi \rangle = \int d^N x \quad \Phi_m(x_1, \dots, x_N) \hat{\Phi}_m^\dagger(x_1) \dots \hat{\Phi}_m^\dagger(x_N) | 0 \rangle$$

$$(392) \quad = \int d^N x \quad \psi(x_1, \dots, x_N) \hat{\Psi}^\dagger(x_1) \dots \hat{\Psi}^\dagger(x_N) | 0 \rangle$$

Since

$$\hat{\Phi}_m^\dagger(y) \hat{\Phi}_m^\dagger(x) = e^{im\varphi(y-x)} \hat{\Psi}^\dagger(y) \hat{\Psi}^\dagger(x) e^{im\hat{\theta}(y)} e^{im\hat{\theta}(x)}$$

$$\Rightarrow \hat{\Phi}_m^\dagger(y) \hat{\Phi}_m^\dagger(x) | 0 \rangle = e^{im\varphi(y-x)} \hat{\Psi}^\dagger(y) \hat{\Psi}^\dagger(x) | 0 \rangle$$

thus

$$\begin{aligned} \hat{\Phi}_m^\dagger(x_1) \dots \hat{\Phi}_m^\dagger(x_N) | 0 \rangle &= e^{im \sum_{e < j} \varphi(x_e - x_j)} \\ &\cdot \hat{\Psi}^\dagger(x_1) \dots \hat{\Psi}^\dagger(x_N) | 0 \rangle \end{aligned}$$

With this we find the relation between the N-particle wavefunctions from (392)

$$(393) \quad \psi(z_1, \dots, z_N) = e^{im \sum_{e < j} \varphi(z_e - z_j)} \Phi_m(z_1, \dots, z_N)$$

relation to bare composite wavefunction

(B) Effective magnetic field and Chern-Simons field

The Hamiltonian of fermions in the LL in the symmetric gauge reads (in dimensionless units)

$$H_0 = \frac{1}{2m} \int d^2z \, \bar{\psi}^\dagger(z) \left[\underbrace{\left(-i\frac{\partial}{\partial x} - \frac{y}{2}\right)^2}_{\equiv P_x} + \underbrace{\left(-i\frac{\partial}{\partial y} - \frac{x}{2}\right)^2}_{\equiv P_y} \right] \bar{\psi}(z)$$

$$(394) \quad \begin{aligned} &= \frac{1}{2m} \int d^2z \, \bar{\psi}^\dagger(z) [P_x^2 + P_y^2] \bar{\psi}(z) \\ &= \frac{1}{2m} \int d^2z \, \bar{\psi}^\dagger(z) [(P_x + iP_y)(P_x - iP_y)] \bar{\psi}(z) \\ &\quad + \frac{N}{2} \hbar \omega_c \end{aligned}$$

Note that $[P_x, P_y] \neq 0$

We now express H_0 in terms of composite fields using eq. (393)

$$\begin{aligned}
 H = \frac{1}{2m} \int d^2z \, \hat{\Phi}_m^\dagger(z) \left[(\tilde{P}_x + i\tilde{P}_y)(\tilde{P}_x - i\tilde{P}_y) \right] \hat{\Phi}_m(z) \\
 (395) \qquad \qquad \qquad + \frac{N}{2} \hbar \omega_c
 \end{aligned}$$

Where the "covariant" momenta read ($\mu = x, y$)

$$(396) \quad \tilde{P}_\mu = P_\mu + e C_\mu = -i\hbar \frac{\partial}{\partial x_\mu} + e A_\mu^{\text{eff}}$$

Here

($\hbar$ reintroduced)

$$(397) \quad C_\mu(z) = \frac{\hbar m}{e} \partial_\mu \hat{\Theta}(z) = \frac{\hbar m}{e} \int d^2z' \, \underbrace{\partial_\mu \varphi(z-z')}_{\uparrow} \hat{n}(z')$$

is the vector potential of the Chern-Simon field

and the effective total vector potential reads

$$(398) \quad A_\mu^{\text{eff}} = A_\mu + C_\mu$$

Non-interacting fermions in the LLL are identical to composite particles with an additional Chern-Simon field (provided a composite state exists that fulfills the LLL condition).

The Chern Simons field is then given by

$$(399) \quad \begin{aligned} B_{CS} &= \varepsilon_{jk} \partial_j C_k(z) \\ &= \frac{\hbar m}{e} \varepsilon_{jk} \partial_j \partial_k \tilde{\Theta}(z) = m \Phi_0 \delta^{(2)}(z) \end{aligned}$$

I.e. the Chern Simons field of Φ_m -composites corresponds to m Dirac flux quanta at the position of the composite particle. One easily finds that composite particles in an external field B experiences an effective reduced magnetic field

(400)

$$B_{\text{eff}}(z) = B - m \Phi_0 \hat{n}(z)$$

Composite fermions ($m=2p$) with filling

$$(401) \quad \nu = \frac{\Phi_0}{B} = \frac{1}{2p+1}$$

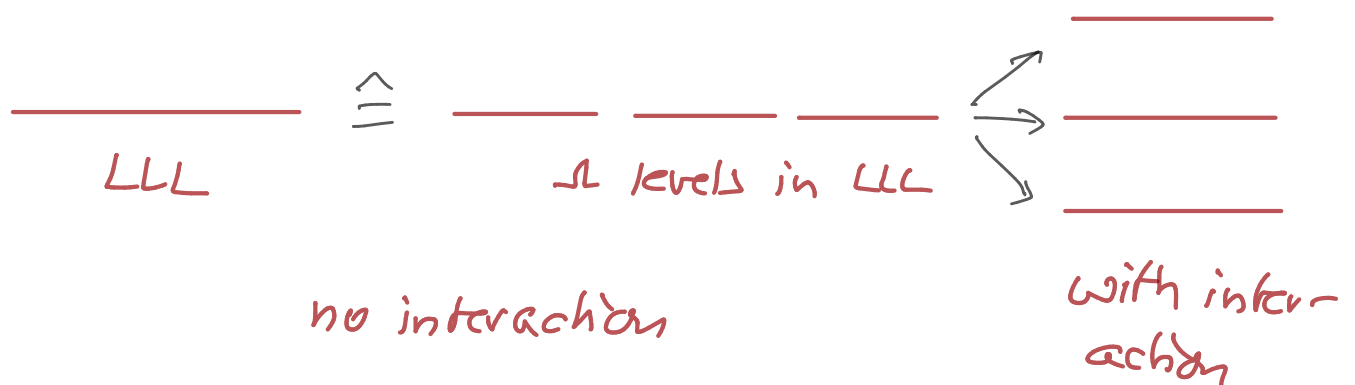
fill all states of the lowest effective Landau level or Λ -level with effective filling

$$(402) \quad \nu^* = \frac{\Phi_0}{B_{\text{eff}}} = \frac{\Phi_0}{B - 2p\Phi_0} = \frac{1}{\frac{1}{\nu} - 2p} = 1$$

IQHE!

For filling $\nu > \frac{1}{2p+1}$ higher $\mathcal{L}$ -level become occupied. Without interactions these higher $\mathcal{L}$ levels might be energetically degenerate with the lowest $\mathcal{L}$ -level

With interactions the lowest $\mathcal{L}$ -level wavefunction screens all small pseudopotentials, while states with components in higher $\mathcal{L}$ -levels have interaction energy $\Rightarrow$ splitting



(C) Dressed composite fermions

Bare composite wavefunctions have singularities. To avoid them we slightly modify the phase operator. The modification is different for composite bosons and composite fermions. We here discuss CF's in the LLL with $m=2p$

$$\hat{\Phi}_{2p}(z) \rightarrow \hat{\Psi}_{2p}(z) = e^{-i2p\hat{\theta}(z) - \hat{Q}_{cs}(z)} \hat{\Psi}(z)$$

(403)

where we added a scalar field

$$(404) \quad \hat{Q}_{cs}(z) \equiv 2p \int d^2z' \ln\left(\frac{|z-z'|}{2\ell_0}\right) \hat{n}(z') - \frac{e\bar{B}_{cs}}{4t} |z|^2$$

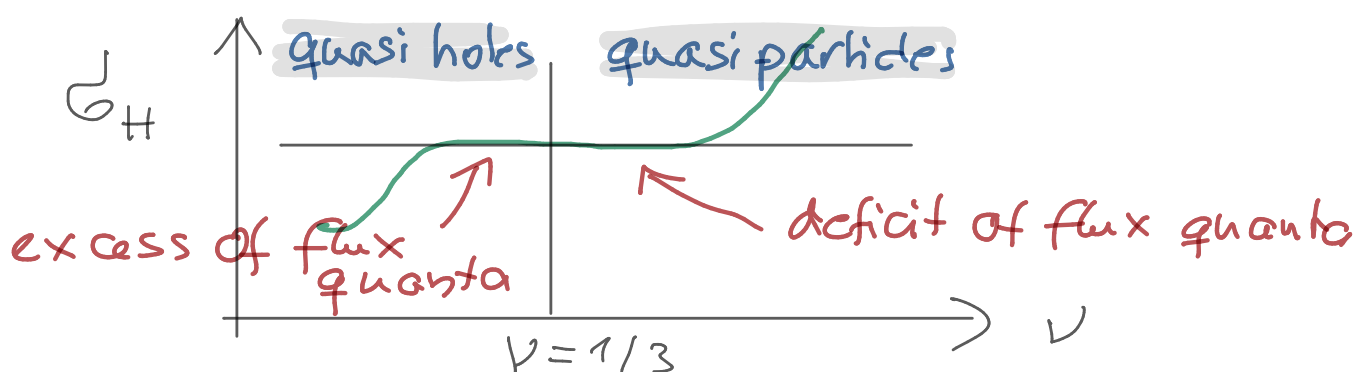
$\bar{B}_{cs}$ is the homogenized Chern-Simon field

$$(405) \quad \bar{B}_{cs} = 2p \Phi_0 B$$

The transformation properties (commutation / anticommutation) are not modified

9.5 Elementary excitations: quasi-holes and quasi-particle states

FQH states are incompressible. Moving away from strict fractional filling by changing the magnetic field leads to elementary excitations which we want to discuss now

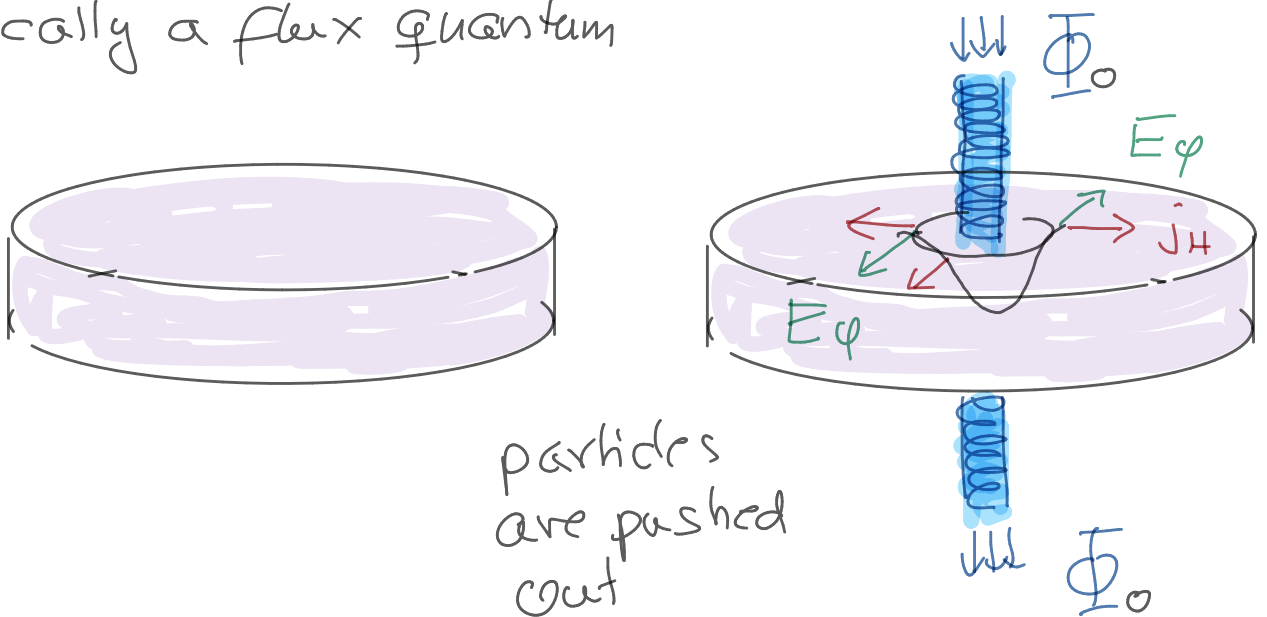


Both quasi-holes (qh) and quasi-particles (qp) are pinned to disorder / impurities and thus do not contribute to the transport which explains the presence of the QH plateaus

... so disorder is in fact important to see plateaus at all

(A) fundamentals of elementary excitations

let's start with a thought experiment by inserting locally a flux quantum

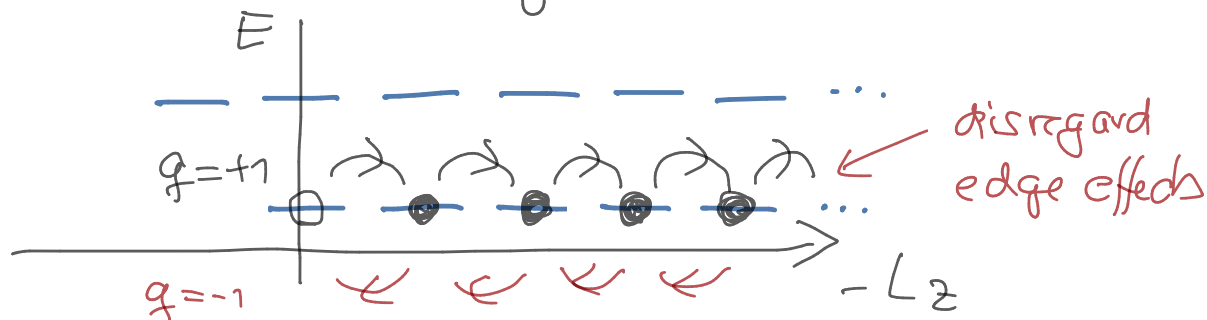


- after insertion of one flux quantum Φ_0 the angular momentum with respect to the origin is increased (reduced) by one unit

$$(406) \quad \lambda(z) = z^{\ell} \quad \rightarrow \quad \lambda'(z) = e^{i q \varphi} z^{\ell}$$

$q = \pm 1$

φ is the azimuthal angle around the z axis



- If flux insertion is adiabatic the wavefunction follows adiabatically

$$(407) \quad \psi(z_1 \dots z_n) \rightarrow \psi'(z_1 \dots z_n) = e^{\pm i\varphi} \psi$$

This is a (singular) gauge transformation that does not change the Hamiltonian (disregard edge effects!) except at origin where $e^{\pm i\varphi}$ is undefined. $\Rightarrow$ state after flux insertion should again be an eigenstate of H (see figure above)

$\Rightarrow$ elementary excitation

$$q = +1$$

quasi-hole

$$q = -1$$

quasi-particle

problem: $\psi' = e^{\pm i\varphi} \psi$ is no longer in LLL
need instead increase or reduce
power of z

$$q = +1 \quad \lambda(z) = z^l \longrightarrow \lambda_{qh}(z) = z \lambda(z) = r e^{+i\varphi} z^l$$

$$q = -1 \quad \lambda(z) = z^l \longrightarrow \lambda_{qp}(z) = \frac{\partial}{\partial z} \lambda(z) = \frac{1}{r} e^{-i\varphi} z^l$$

use angular-momentum ladder operators

$$(408) \quad b_j^+ = \frac{1}{\sqrt{2}} \left(z_j - \frac{\partial}{\partial z_j^*} \right) \quad b_j = \frac{1}{\sqrt{2}} \left(z_j^* + \frac{\partial}{\partial z_j} \right)$$

$$(409) \quad \boxed{\psi(z_1 \dots z_N) \longrightarrow \psi_{qh}(z_1 \dots z_N) = \prod_{j=1}^N b_j^+ \psi_L(z_1 \dots z_N)}$$

quasi-hole increases angular momentum of all particles by one unit

$$(410) \quad \boxed{\psi(z_1 \dots z_N) \longrightarrow \psi_{qp}(z_1 \dots z_N) = \prod_{j=1}^N b_j \psi_L(z_1 \dots z_N)}$$

quasi-particle decreases angular momentum of all particles by one unit

for

$$\psi_L(z) = \lambda(z) e^{-\sum_j |z_j|^2}$$

different length unit used for simplicity

$$(411) \quad b_e \psi_L(z) \sim \left(\frac{\partial}{\partial z_e} \lambda(z) \right) e^{-\sum_j |z_j|^2}$$

$$b_e^+ \psi_L(z) \sim z \lambda(z) e^{-\sum_j |z_j|^2}$$

quasi-hole wavefunction

(412)

$$\psi_{qh} = \prod_{e=1}^N z_e \psi_{LN}$$

quasi-particle wavefunction

more complicated

(413)

$$\psi_{qp} = \frac{\prod_{j=1}^N \frac{\partial}{\partial z_j} \prod_{j < s} (z_j - z_s)^m}{\prod_{j < s} (z_j - z_s)^m} \psi_{LN}$$

remark:

⚠ qp excitation for integer QH state as qp generation corresponds to removal of angular momentum. Fermions would then be pushed to the next higher Landau level.

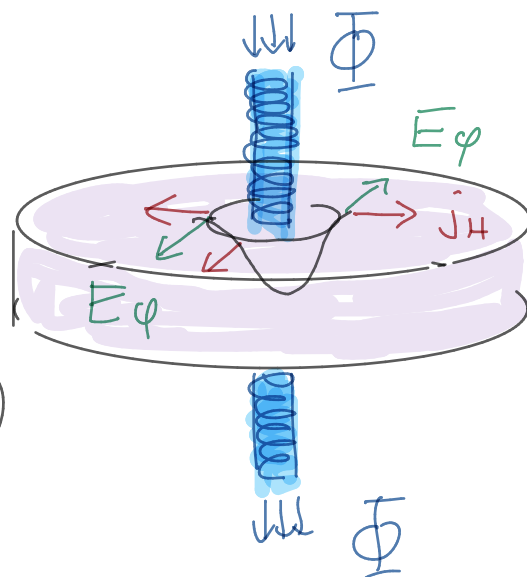
An interesting question is that of the effective charge of quasi-holes / quasiparticles

- flux insertion by $\pm \Phi_0 \Rightarrow$ electric field in time T (adiabatic)

$$(414) \quad E_{\varphi} = \frac{d\Phi}{dt} \frac{1}{2\pi r}$$

$$\Rightarrow (415) \quad j_H = \frac{eS}{B} E_{\varphi}$$

for large r ($j = cE$)



Total change of charge

$$q^* = 2\pi r \int_0^T dt j_H = \pm \frac{eS}{B} \int_0^T dt \frac{d\Phi}{dt}$$

$$(416) \quad q^* = \pm \frac{eS}{B} \Phi_0 = \pm \nu e$$

quasi-hole / quasi-particle excitations on top of an incompressible FQH fluid with filling ν have a fractional charge

- What happens if we add / subtract a bare fermion from the system?

$$F = \begin{array}{c} \uparrow\downarrow \\ \bigcirc \end{array} + \begin{array}{c} \downarrow\downarrow \end{array}$$

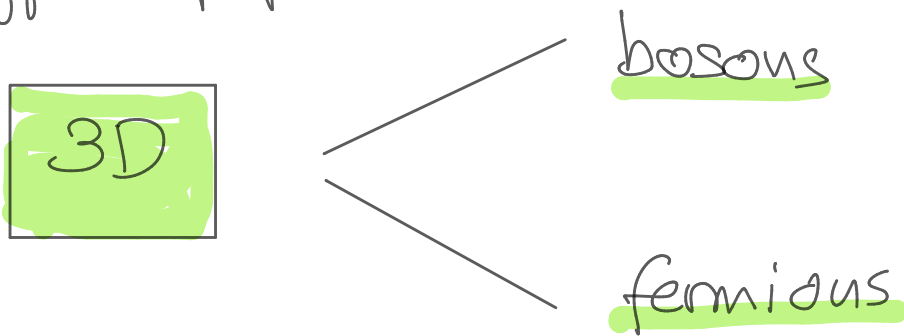
$F \qquad 2CF$

$\Rightarrow$ loss (addition) of composite fermion
and creation of $2p$ local flux quanta
i.e. creation of quasi-holes

- We will now argue that quasi-holes / particles have an unusual transmutation property: they are anyons

(B) Anyons

In the real 3D world there are only two types of particles



(i) spin

bosons: integer S

fermions: half-integer S

(ii) exchange statistics

$$\psi(x_2, x_1) = \begin{cases} \psi(x_1, x_2) & B \\ -\psi(x_1, x_2) & F \end{cases}$$

(iii) Pauli exclusion for fermions

The spin of bosons/fermions is enforced to be integer / half-integer by the non-Abelian spin algebra

$$(417) \quad [S_i, S_j] = i\hbar \varepsilon_{ijk} S_k$$

This changes if we are in 2D

2D

only S_z no algebra!

quantization of spin not enforced

$\Rightarrow$ (i) quasiparticles in 2D can have fractional spin

How about exchange statistics? Exchanging two identical particles can only affect the phase of the wavefunction

$$(418) \quad \psi(\vec{r}_2, \vec{r}_1) = e^{i\alpha\pi} \psi(\vec{r}_1, \vec{r}_2)$$

$\alpha : \alpha \bmod 2$ statistics parameter

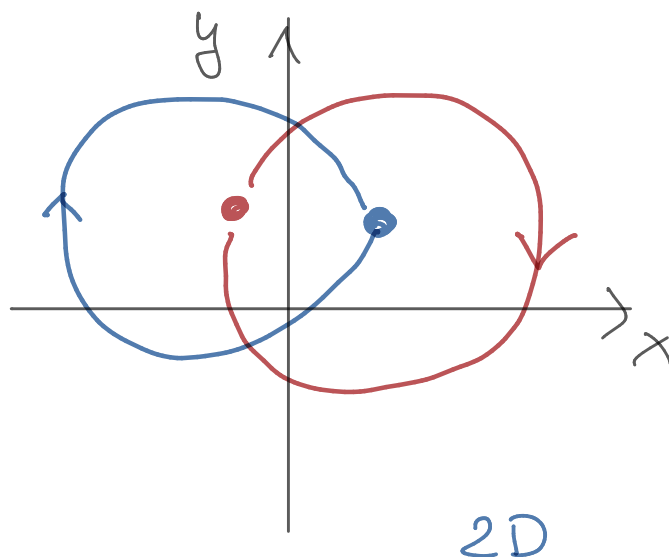
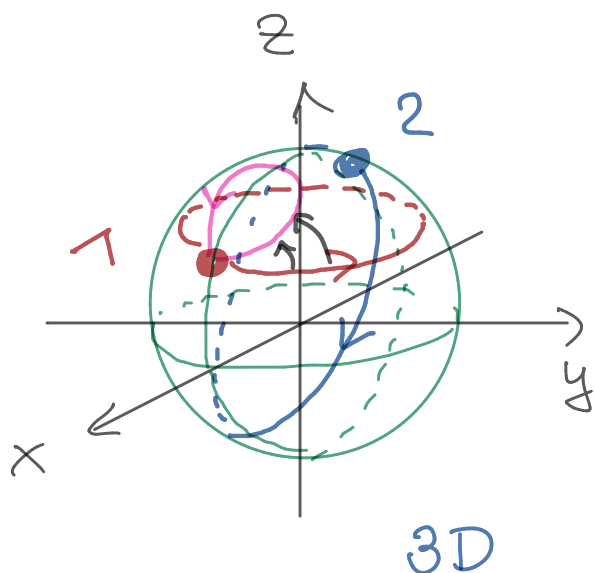
• $\alpha = ?$ Exchanging position twice leads to

$$\psi(\vec{r}_1, \vec{r}_2) = e^{i2\alpha\pi} \psi(\vec{r}_1, \vec{r}_2) \Rightarrow e^{2i\alpha\pi} = 1$$

$\alpha = 0$ bosons

$\alpha = 1$ fermions

But: argument for requiring $e^{2i\alpha\pi} = 1$ comes from ability of continuously deform exchange path which does not apply in 2D



quasi-particles in 2D can have
any exchange statistics (anyons)

(C) Quantum mechanics of anyons

consider composite-boson field $\hat{\Phi}(z)$

$$(419) \quad [\hat{\Phi}(z), \hat{\Phi}^\dagger(z')] = \delta^{(2)}(z-z')$$

define a new field

$$(420) \quad \boxed{\hat{\Psi}(z, t) = e^{i\alpha \hat{\Theta}(z, t)} \hat{\Phi}(z, t)} \quad \text{anyon field}$$

where $\hat{\Theta}(z)$ is again the angle operator

$$(421) \quad \hat{\Theta}(z) = \int d^2 z' \varphi(z - z') \underbrace{\hat{\Phi}^\dagger(z') \hat{\Phi}(z')}_{= \hat{n}(z')}$$

then following the same argument as in Sec. 9.4A $\Rightarrow$

$$(422) \quad \begin{aligned} \hat{\Psi}(y) \hat{\Psi}(x) &= e^{i\alpha\pi} \hat{\Psi}(x) \hat{\Psi}(y) \\ \hat{\Psi}^\dagger(y) \hat{\Psi}(x) &= e^{i\alpha\pi} \hat{\Psi}(x) \hat{\Psi}^\dagger(y) + \delta^{(2)}(x - y) \end{aligned}$$

Note that different from Sec. 9.4A α is not necessarily an integer!

9.6* Effective Hamiltonian of quasi-holes: Anyons and Chern-Simon field

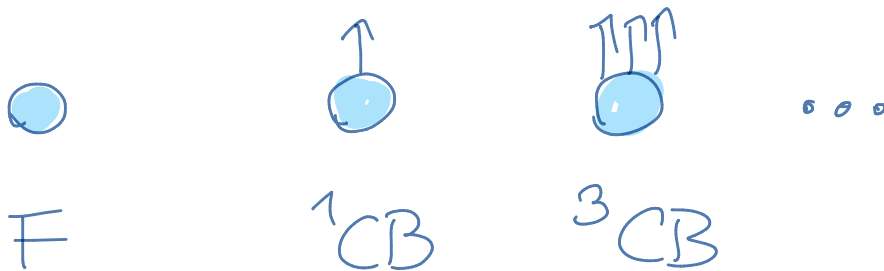
Now we want to sketch the derivation of an effective anyon Hamiltonian of quasiholes in the FQH system.

(A) FQHE as condensate of composite bosons

An alternative to the explanation of the FQHE as integer QHE of composite fermions in a reduced effective field is the interpretation in terms of composite bosons in a vanishing effective field.

composite bosons

^{2p+1}CB bound state of fermion and odd number $(2p+1)$ of flux quanta attached to them



effective magnetic field can be made to vanish

$$(423) \quad B^* = B - (2p+1) \frac{\Phi_0}{2\pi} = 0 \quad \hat{=} \quad \nu = \frac{1}{2p+1}$$

$$\text{F} = \text{3CB} + \text{chem-Simon flux quanta}$$

To construct the theory of quasi-holes it is more convenient to start from the composite-boson picture: $m=2p+1$

$$(424) \quad H = \frac{1}{2M} \int d^2z \, \phi_m^\dagger(z) (\tilde{P}_x + i\tilde{P}_y)(\tilde{P}_x - i\tilde{P}_y) \phi_m(z) + \frac{1}{2} N \hbar \omega_c$$

$$(425) \quad \text{where} \quad [\phi_m(z), \phi_m^\dagger(z')] = \delta^{(2)}(z-z')$$

describes the composite-boson fields

$$\tilde{P}_\mu = -i\hbar \frac{\partial}{\partial x_\mu} + e A_\mu^{\text{eff}}$$

$$A_\mu^{\text{eff}} = A_\mu + C_\mu \quad C_\mu \text{ chem-Simon field of } 2p+1 \text{ CB}$$

$$\text{For } \nu = \frac{1}{2p+1}$$

$$(426) \quad A_\mu^{\text{eff}} = 0$$

This is a convenient starting point to discuss quasiholes if $\nu = \frac{1}{2p+1} - \varepsilon$

(B) Quasiholes on top of a composite-boson fluid

To create N_{qh} quasiholes we start from a CB liquid with $A^{eff} = 0$ and perform a singular gauge trafo

$$(427) \quad \hat{\phi}_m \rightarrow e^{if} \hat{\phi}_m \quad (m=2p+1)$$

$$(428) \quad f = \sum_{r=1}^{N_{qh}} \hat{\Theta}(z - z_r)$$

where z_r are the positions of the quasiholes. (427) will change the Chern-Simons field

$$(429) \quad C_\mu \rightarrow C_\mu' = C_\mu - \frac{\hbar}{e} \partial_\mu f$$

Note that now $\nu = \frac{1}{2p+1} - \varepsilon$

$$(430) \quad A_\mu^{eff} = \underbrace{A_\mu + C_\mu}_{\neq 0} - \frac{\hbar}{e} \partial_\mu f \stackrel{!}{=} 0 \quad \text{on average}$$

$\neq 0$ since $\nu \neq \frac{1}{2p+1}$

The number N_{qh} of quasiholes created is such that the effective magnetic field including the

flux quanta associated to the quasiholes vanishes after averaging.

The new Chern Simons magnetic field is now

$$(431) \quad B_{CS} \rightarrow B_{CS} = m \Phi_0 n(z) + \Phi_0 n_{qh}(z)$$

instead of (399), where $(m = 2p + n)$

$$(432) \quad n_{qh}(z) = \sum_{r=1}^{N_{qh}} \delta^{(2)}(z - z_r(t))$$

is the density of quasi-holes which have positions $z_r(t)$.

Allowing for a kinetic energy of quasiholes with some (yet unknown but at the same time irrelevant) mass M_{qh} one can derive an effective Hamiltonian of quasiholes with

effective charge

$$q^* = -\frac{1}{m}e$$

exchange phase

$$\frac{\pi}{m}$$

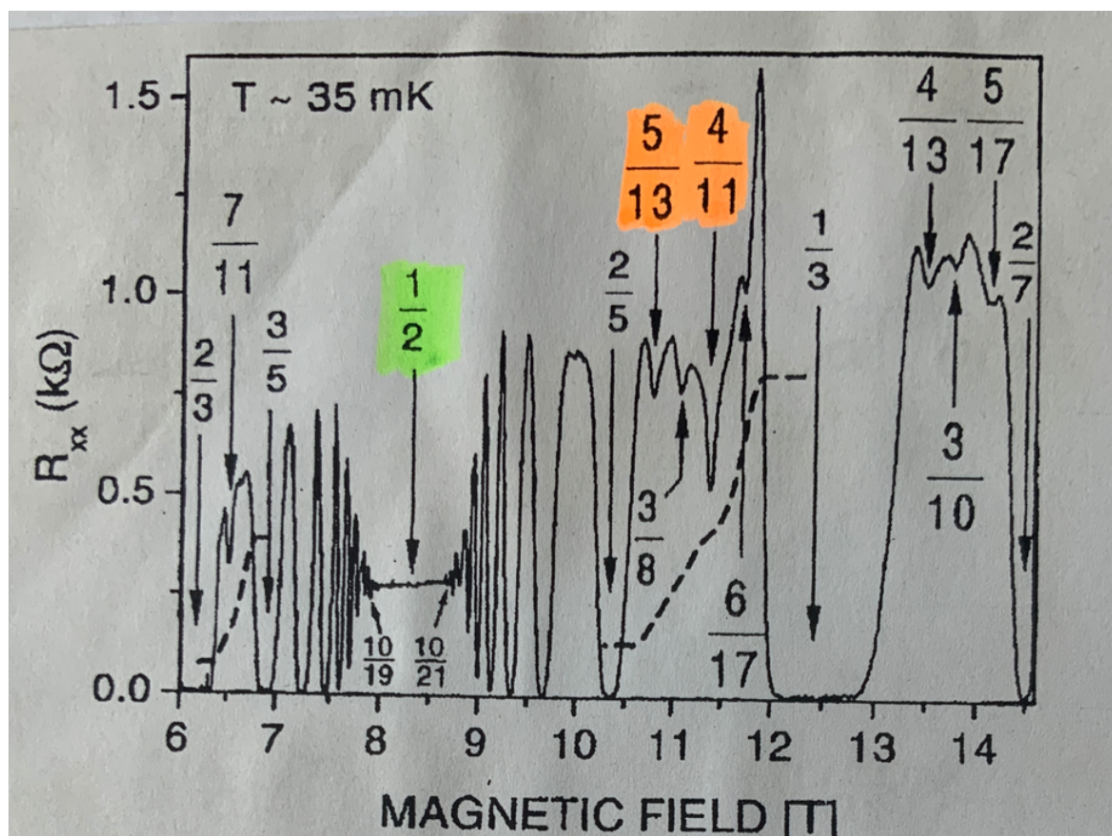
...

9.7 Interacting composite fermions

We have seen that the CF theory explains the existence of incompressible states at filling

$$(433) \quad \nu = \frac{n}{2pn \pm 1}$$

with Hall plateaus. But there are also other fillings where Hall plateaus can be observed



e.g. at filling $\nu = \frac{5}{13}$ or $\nu = \frac{4}{11}$

$\Rightarrow$ interactions between CF!

(A) higher FQH series

Pseudopotentials are not completely screened by the formation of composite fermions. So one can discuss the fractionalization of CF rather than of bare fermions

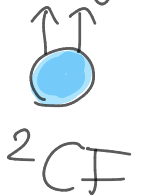
fermions at filling

$$\frac{1}{3} < \nu = \frac{\nu^*}{2\nu^* + 1} < \frac{2}{5}$$



can be mapped to ${}^2\text{CF}$ with effective filling

$$1 < \nu^* = 1 + \frac{f}{2f+1} < 2$$



full lowest
level

$\Rightarrow$ FQHE of ${}^2\text{CF}$ with fillings

$$\begin{array}{lll} f=1 & \nu^* = 1 + \frac{1}{3} & \Rightarrow \nu = \frac{4}{11} \\ f=2 & \nu^* = 1 + \frac{2}{3} & \Rightarrow \nu = \frac{5}{13} \end{array}$$

(B) Absence of FQH state at $\nu = \frac{1}{2}$

At $\nu = \frac{1}{2}$ there is no dip in R_{xx} but the state is still interesting

$$(434) \quad \nu = \frac{1}{2} = \lim_{n \rightarrow \infty} \frac{n}{2n \pm 1} \quad (\rho = 1)$$

i.e. ${}^2\text{CF}$ with filling

$$(435) \quad n = \frac{S}{B_{\text{eff}} \Phi_0} = \infty \quad \boxed{B_{\text{eff}} = 0}$$

These are free ${}^2\text{CF}$ without magnetic field which form a Fermi sea

(C) Exotic states at $\nu = 5/2$: CF-pairing

Willett et al. (PRL 59, 1776 (1987)) have seen a FQH plateau at $\nu = 5/2$. Here the Zeeman splitting is small and one has to consider both spin components

$$\nu = \frac{5}{2} = 2 + \frac{1}{2}$$

↑ ↑
lowest LL 2nd LL
for 2 spins 1 spin

$\Rightarrow$ Fermi sea of ^2CF in 2nd LL?
no!

Moore & Read (Nucl. Phys. B360, 362 (1991))
have shown that a good candidate for
the $\nu = \frac{5}{2}$ FQH state is a p-wave paired
state of spin-polarized electrons in the
2nd LL. "Pfaffian" state

Remark: • pairing of (composite) fermions
requires attractive interaction

in 2nd LL ^2CF do indeed
attract each other

• nature of $\nu = \frac{5}{2}$ state is still not
fully understood.