

3. Chern insulators

3.1 The Harper Hofstadter model

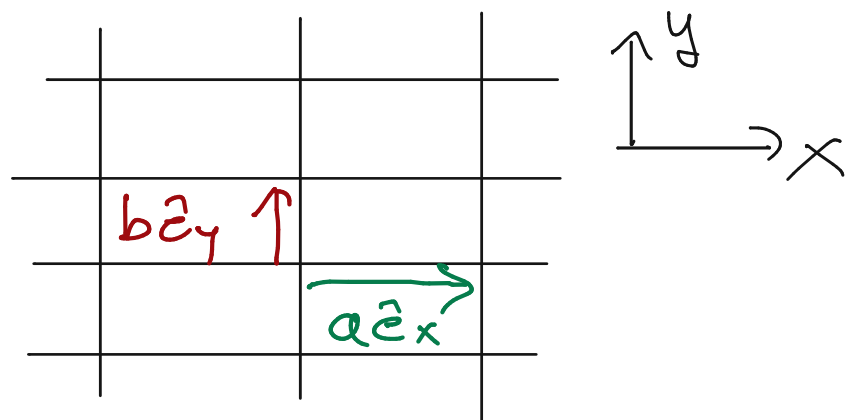
The Harper - Hofstadter model describes electrons on a rectangular lattice in a constant transverse magnetic field. So we first have to discuss lattice models in the presence of a magnetic field

(A) Bloch states in a magnetic field

• Without magnetic field

$$(135) \quad V(\vec{r}) = V(\vec{r} + \vec{R})$$

$$(136) \quad \vec{R} = n a \hat{e}_x + m b \hat{e}_y$$
$$n, m \in \mathbb{Z}$$



lattice constants a, b

the Hamiltonian is invariant under discrete lattice translations

$$(137) \quad T_{\vec{R}} = e^{\frac{i}{\hbar} \vec{R} \cdot \hat{p}} \quad [T_{\vec{R}}, H] = 0$$

$$T_{\vec{R}_1} T_{\vec{R}_2} = T_{\vec{R}_2} T_{\vec{R}_1} = T_{\vec{R}_1 + \vec{R}_2}$$

the eigenstates of H are Bloch functions,

$$(138) \quad \psi_{\vec{b}}(\vec{r}) = e^{i \vec{b} \cdot \vec{r}} u_{\vec{b}}(\vec{r}) \quad u_{\vec{b}}(\vec{r} + \vec{R}) = u_{\vec{b}}(\vec{r})$$

lattice momenta in 1. BZ

$$(139) \quad -\frac{\pi}{a} \leq k_x \leq \frac{\pi}{a} \quad -\frac{\pi}{b} \leq k_y \leq \frac{\pi}{b}$$

• now with magnetic field

$\vec{A}(\vec{r})$ is not translational invariant! 

eg. $\vec{A}(\vec{r}) = -By \vec{e}_x$

but

$$(140) \quad T_{\vec{R}} \vec{A}(\vec{r}) = \vec{A}(\vec{r} + \vec{R}) = \vec{A}(\vec{r}) + \underbrace{\delta \vec{A}(\vec{r})}_{\text{consider as gauge trans!}}$$
$$\delta \vec{A}(\vec{r}) = \vec{\nabla} \chi_{\vec{R}}(\vec{r}) \rightarrow \text{consider as gauge trans!}$$

Hamiltonian may be invariant under combined translation and gauge transformation

$$(141) \quad \boxed{T_{\vec{R}} = e^{i \frac{e}{\hbar c} \chi_{\vec{R}}(\vec{r})} T_{\vec{R}}} \quad \text{magnetic translation}$$

but magnetic translations are in general not Abelian

$$(142) \quad \text{in general} \quad [T_{\vec{R}_1}, T_{\vec{R}_2}] \neq 0 \quad \text{☹}$$

(B) Hofstadter Butterfly ; magnetic unit cell

Due to (142) there is in general no Bloch theorem, and the spectrum of the Harper-Hofstadter model

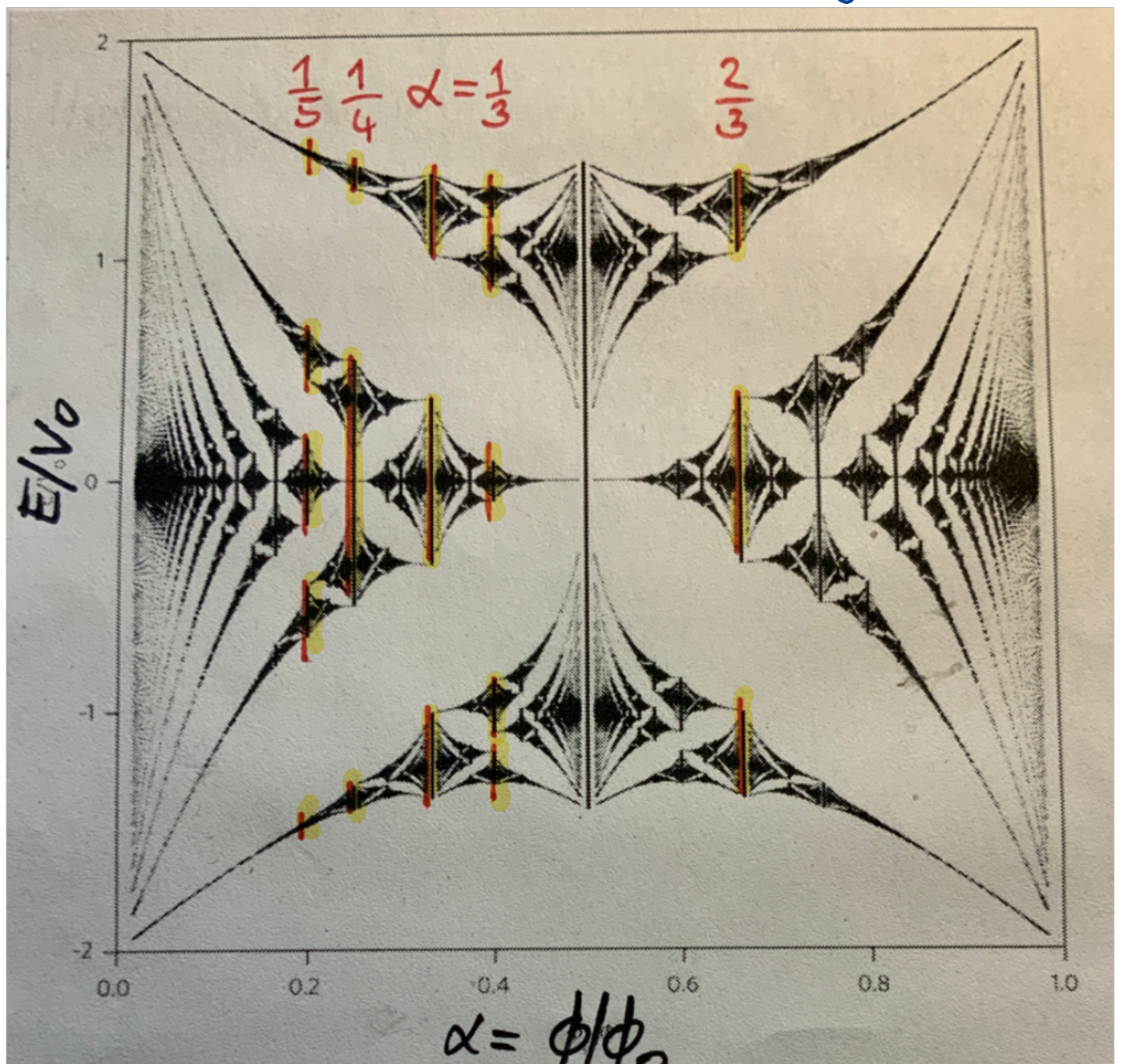
is in general
fractal

Hofstadter butterfly

Φ : flux per
plaquette

$$(143) \quad \alpha \equiv \frac{\Phi}{\Phi_0}$$

D.R. Hofstadter
Phys. Rev. B 14
2239 (1976)



However at rational values of α the spectrum seems
to have bands

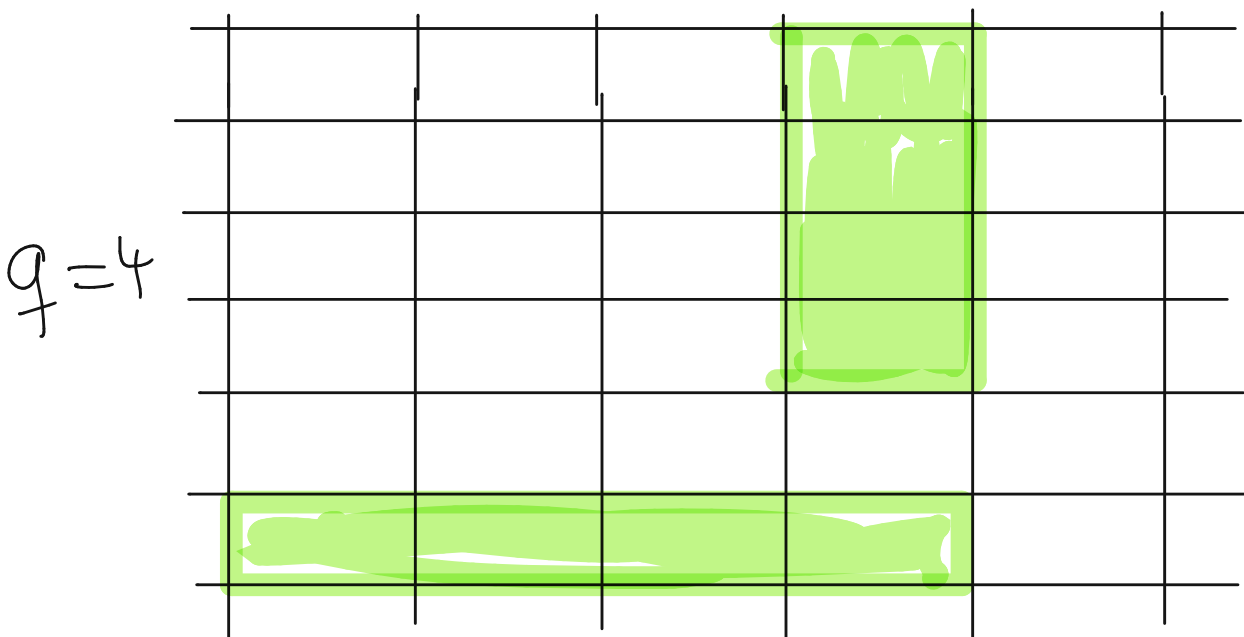
$\Rightarrow$

$$\alpha = \frac{\Phi}{\Phi_0} = \frac{p}{q}$$

(144)

p, q relative prime

introduce magnetic unit cell consisting of q normal
unit cells



magnetic unit cell
has flux

$$\Phi_{\text{tot}} = p \Phi_0$$

integer multiple of Φ_0

so let's define a magnetic lattice vector

$$(145) \quad \vec{R}_m = n \underline{q} a \vec{e}_x + m b \vec{e}_y \quad (\alpha = p/q)$$

The corresponding magnetic translation operators now form an Abelian group! 😊

Choose Landau gauge $\vec{A} = -By \vec{e}_x$

$$(146) \quad \overline{T}_{\vec{R}} = e^{-\frac{i}{e_0^2} \gamma \times T_{\vec{R}}} = \exp\left(-\frac{i}{e_0^2} \gamma \times\right) \exp\left(\frac{i}{\hbar} \vec{R}_m \cdot \vec{p}\right)$$

using Baker Hausdorff one easily shows

$$(147) \quad \overline{T}_{\vec{R}_1} \overline{T}_{\vec{R}_2} = \overline{T}_{\vec{R}_1 + \vec{R}_2} = \overline{T}_{\vec{R}_2} \overline{T}_{\vec{R}_1}$$

where $\vec{R}_1, \vec{R}_2$ are magnetic lattice vectors!

- Since the unit cell is now q times enlarged (here in x direction) the Brillouin zone (in k_x) is reduced

$\Rightarrow$ q magnetic subbands

$$(148) \quad -\frac{1}{q} \frac{\pi}{a} \leq k_x \leq \frac{1}{q} \frac{\pi}{a}$$

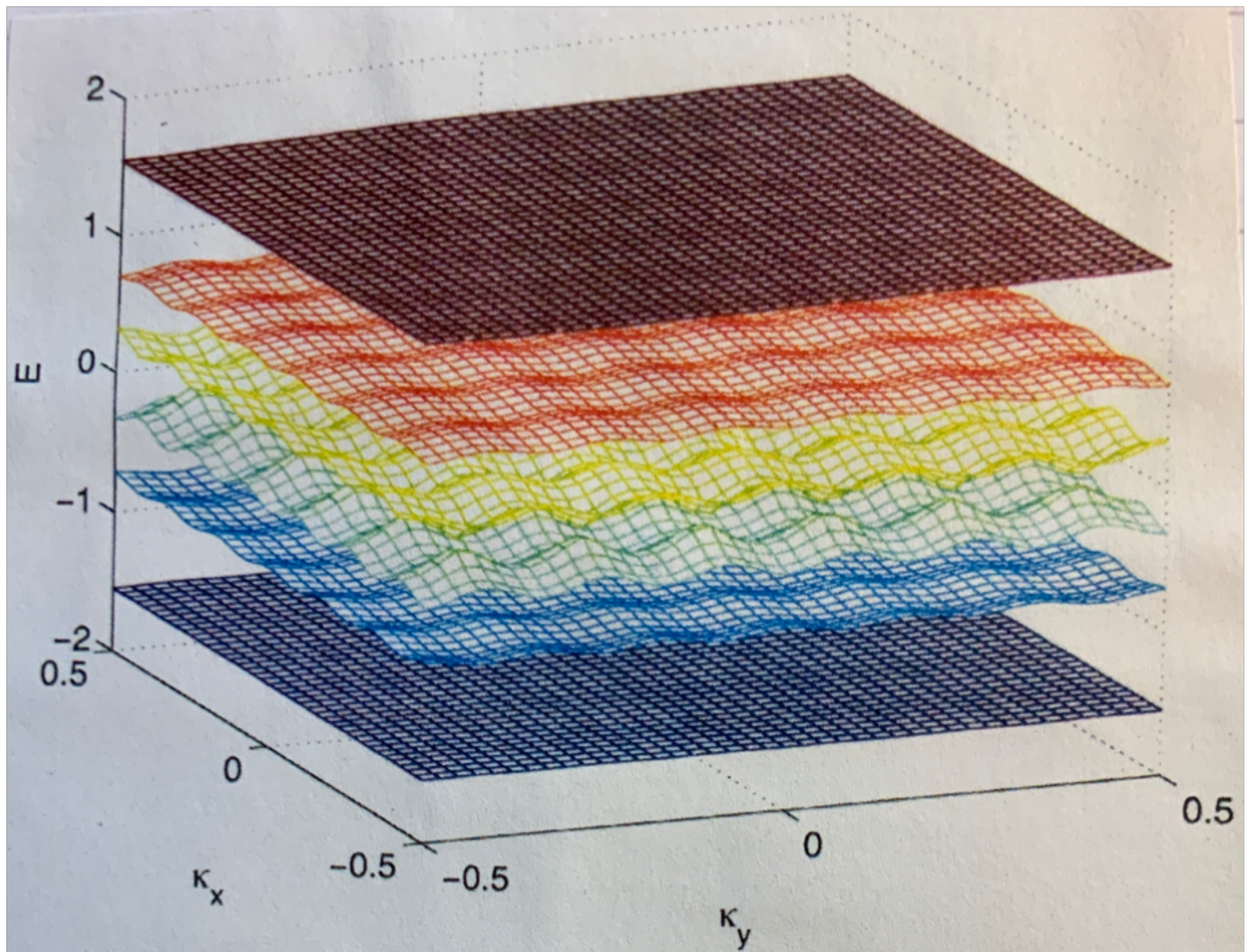
$$-\frac{\pi}{b} \leq k_y \leq \frac{\pi}{b}$$

Rem.: One could have chosen the y direction instead of x

E.g. if the lattice without a magnetic field has a unit cell of one site and we add a magnetic field with $q=6$ we get 6 magnetic subbands

6 subbands

note the
6-fold
periodicity
in both
 k_x and k_y



(C) Harper equation and calculation of the spectrum of the H-H model

We want to diagonalize the Harper-Hofstadter Hamiltonian with rational flux per plaquette

$$H = -t \sum_{m,n} \left(C_{m+1,n}^\dagger C_{m,n} + C_{m,n+1}^\dagger C_{m,n} e^{i \frac{2\pi p}{q} m} + \text{h.c.} \right) \quad (149)$$

Peierls phase

Fourier transformation with respect to original unit cell does not give diagonal Hamiltonian. We have to use the magnetic unit cell as proper unit cell. (now $a=b=1$)

$$(150) \quad C_{m,n} = \frac{1}{(2\pi)^2} \int_{-\pi/q}^{\pi/q} dk_x \int_{-\pi}^{\pi} dk_y e^{ik_x m} e^{ik_y n} \tilde{C}_{k_x k_y}$$

$$\Rightarrow (151) \quad H = \frac{1}{(2\pi)} \int_{-\pi/q}^{\pi/q} dk_x \int_{-\pi}^{\pi} dk_y \sum_{\nu=0}^{q-1} H_{k_x k_y}^{\nu}$$

magnetic sub-bands

where

$$(152) \quad H_{k_x k_y}^{\nu} = -t \left[2 \cos(k_x + 2\pi \frac{P}{q} \nu) \tilde{C}_{k_x + 2\pi \frac{P}{q} \nu, k_y}^+ \tilde{C}_{k_x + 2\pi \frac{P}{q} \nu, k_y} \right. \\ \left. + e^{-ik_y} \tilde{C}_{k_x + 2\pi \frac{P}{q} (\nu+1), k_y}^+ \tilde{C}_{k_x + 2\pi \frac{P}{q} \nu, k_y} \right. \\ \left. + e^{ik_y} \tilde{C}_{k_x + 2\pi \frac{P}{q} (\nu-1), k_y}^+ \tilde{C}_{k_x + 2\pi \frac{P}{q} \nu, k_y} \right]$$

Obviously $\sum_{\nu=0}^{q-1} H_{k_x k_y}^{\nu}$ is still not diagonal in k_x , but it corresponds to a 1D lattice Hamiltonian along " k_x " with q lattice sites and "hopping"

$$k_x + 2\pi \frac{P}{q} \nu \quad \begin{array}{c} \xrightarrow{te^{-ik_y}} \\ \xleftarrow{te^{ik_y}} \end{array} \quad k_x + 2\pi \frac{P}{q} (\nu+1)$$

and periodic boundary conditions. This can be solved like any 1D tight binding Hamiltonian,

$$(153) \quad H_{k_x k_y} |\varphi\rangle = \sum_{\nu=0}^{q-1} H_{k_x k_y}^{\nu} |\varphi\rangle = E_{k_x k_y} |\varphi\rangle$$

Ansatz: $|2\rangle = \sum_{\nu=0}^{q-1} \alpha_{\nu} \tilde{C}_{k_x + 2\pi \frac{P}{q}\nu, k_y}^+ |0\rangle$

$\Rightarrow \dots \Rightarrow$ Harper equations

$$(154) \quad \begin{aligned} -2 \cos(k_x + 2\pi \frac{P}{q}\nu) \alpha_{\nu} - e^{-ik_y} \alpha_{\nu-1} - e^{ik_y} \alpha_{\nu+1} \\ = \frac{1}{t} E_{k_x k_y} \alpha_{\nu} \end{aligned}$$

with boundary conditions

$$(155) \quad \alpha_{\nu+q} = \alpha_{\nu}$$

The exponential terms $e^{\pm ik_y}$ can be removed by $\alpha_j = \beta_j e^{-ik_y j}$. This gives

$$(156) \quad -\beta_{\nu-1} - \beta_{\nu+1} - 2 \cos(k_x + 2\pi \frac{P}{q}\nu) \beta_{\nu} = \frac{1}{t} E_{k_x k_y} \beta_{\nu}$$

with $\beta_{\nu+q} = \beta_{\nu} e^{ik_y q}$. Note k_y disappeared and is hidden in the boundary conditions $\Rightarrow$

$$(157) \quad E(k_x, k_y) = E(k_x, k_y + \frac{2\pi}{q} n) \quad n=1, \dots, q-1$$

$\Rightarrow$ q -fold periodicity of spectrum within the normal Brillouin zone with respect to k_y

Some general properties of solutions of Harper equations:

• for even q there are solutions with $E=0$

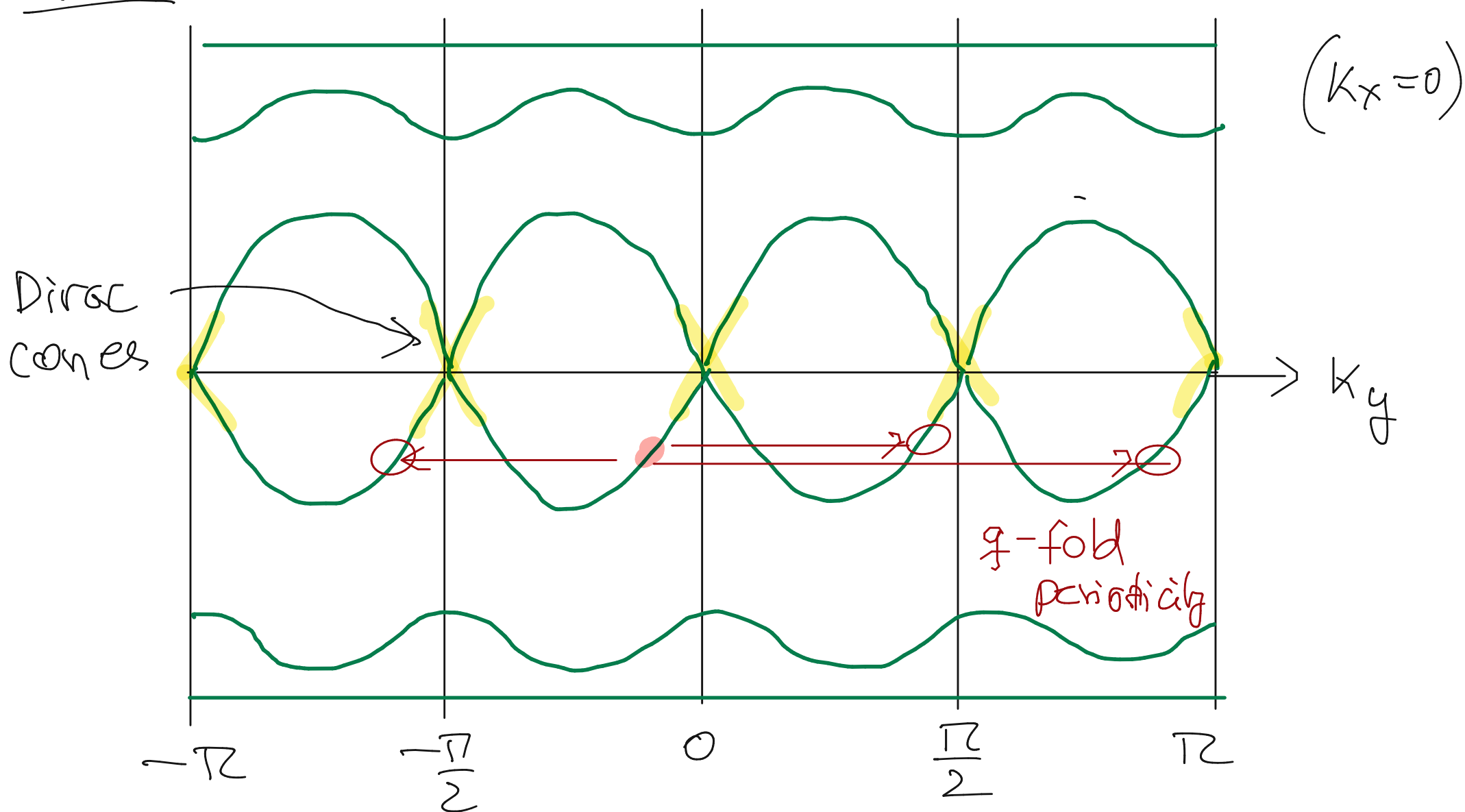
▶ $q=4n$ $E=0$ at $k_y = \frac{2m}{q}\pi$ $k_x=0$

▶ $q=4n+2$ $-||-$ $k_y = \frac{(2m+1)}{q}\pi$ $k_x = \pm \frac{\pi}{q}$

Close to these points the dispersion reads

(158) $E \approx \pm \gamma q t^{q/2} |\vec{k}| = \pm \gamma q t^{q/2} \sqrt{k_x^2 + k_y^2}$
 i.e. Dirac cones

$q=4$



In the chosen gauge ($\Rightarrow$ choice of magnetic unit cell)

$$-\frac{\pi}{q} \leq k_x \leq \frac{\pi}{q} \quad -\pi \leq k_y \leq \pi$$

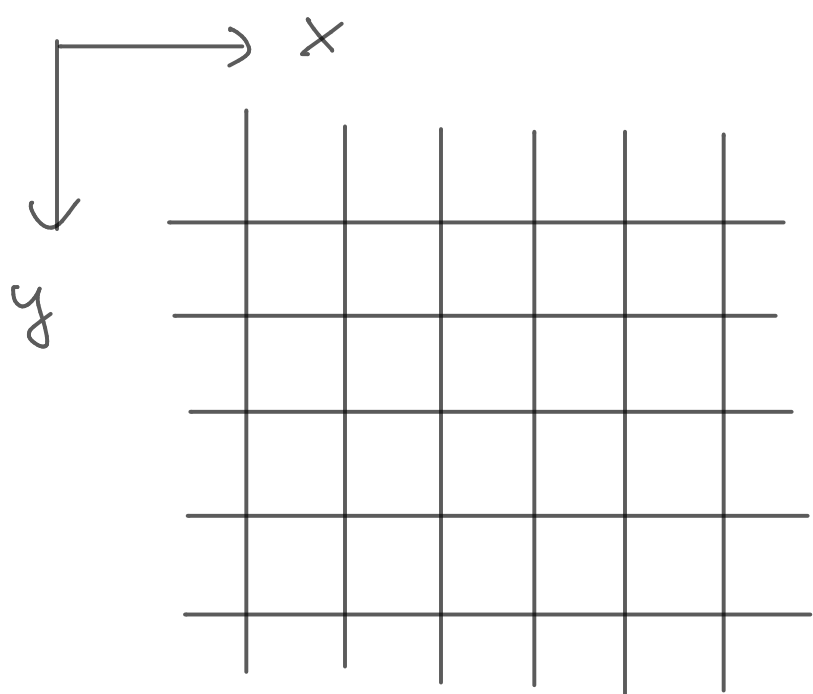
i.e. also periodicity in k_x !

3.2 Hall conductivity and Chern number

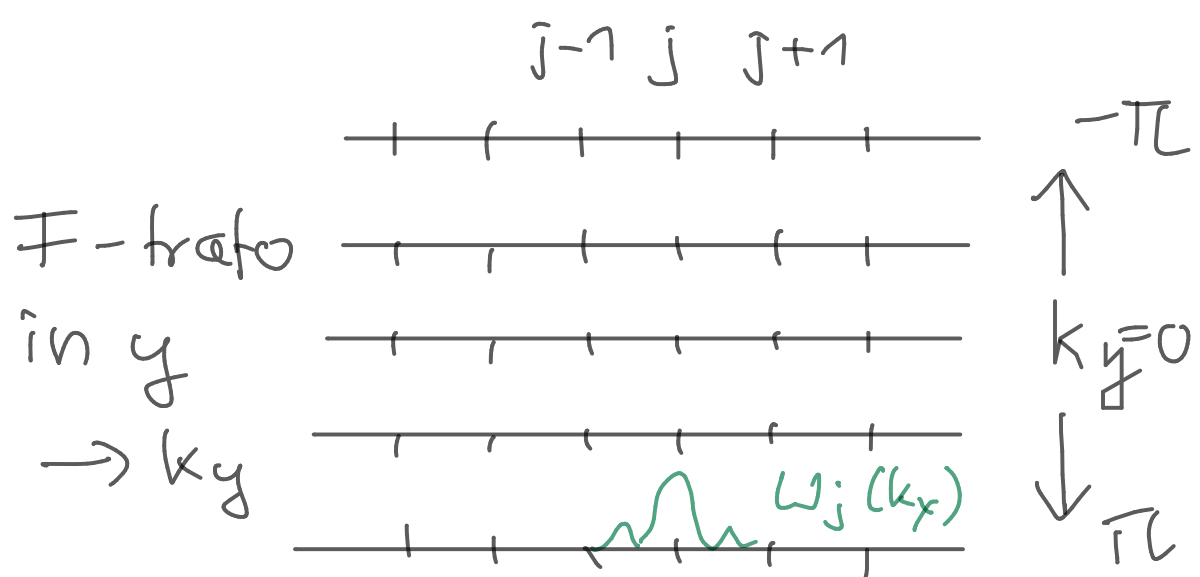
Now we want to argue that electrons hopping on a 2D lattice subject to an external magnetic field with commensurate flux per plaquette can have a quantized Hall conductivity determined by a topological invariant, the Chern number

(A) Hall conductivity

Harper-Hofstadter model $\alpha = p/q$



(magnetic) unit cells



k_y -periodic set of 1D chains

Separate Hamiltonian $h(k_y)$

Center of mass of Wannier center $\omega_j^v(k_y)$ in x direction for every value of k_y (polarization)

$$(159) \quad \begin{aligned} P_r(k_y) &= \langle \omega_j^v(k_y) | \hat{x} | \omega_j^v(k_y) \rangle \\ &= \frac{1}{2\pi} \phi_v^{2Ak}(k_y) \end{aligned}$$

Consider a filled magnetic subband (insulator) and apply a small electric field in the y-direction

$$(160) \quad \frac{dk_y}{dt} = E_y \quad \vec{E} = E_y \hat{e}_y$$

If $\vec{E}$ is small enough not to close insulating many-body gap $\rightarrow$ adiabatic following

$$|\omega_j^v(k_y)\rangle \rightarrow |\omega_j^v(k_y(t))\rangle$$

Hall conductivity

$$(161) \quad \sigma_{xy} = \frac{j_x}{E_y} \quad (\text{charge } e = 1)$$

$$(162) \quad j_x = \int_{BZ} dk_y \frac{dP_r(k_y(t))}{dt}$$

$\nearrow$
contribution from all electrons occupying the magnetic subband

(163)

$$\begin{aligned}
 \sigma_{xy} &= \int_{BZ} dk_y \frac{dP_\nu}{dt} \cdot \frac{dt}{dk_y} = \int_{BZ} dk_y \frac{\partial P_\nu(k_y)}{\partial k_y} \\
 &= \frac{1}{2\pi} \oint_{BZ} dk_y \frac{\partial \phi_\nu^{2A_u}(k_y)}{\partial k_y} \\
 &= \frac{1}{2\pi} \oint dk_x dk_y \underbrace{i \left[\langle \partial_{k_y} u^\nu | \partial_{k_x} u^\nu \rangle - \langle \partial_{k_x} u^\nu | \partial_{k_y} u^\nu \rangle \right]}_{\text{Berry curvature}} \\
 &= C \quad \text{Chem number}
 \end{aligned}$$

$\Rightarrow$ quantization of Hall conductivity in insulating bulk state

(B) Streda formula

To find the Chem number of a particular magnetic sub-band of the Harper-Hofstadter model one could just numerically integrate the Berry curvature. It turns out however that there is an alternative way to determine C_ν via the Streda formula that connects the Hall conductivity with the change of the electron density with magnetic field while keeping the Fermi energy constant

- current density in 2D electron gas

$$(164) \quad j_k = -G_{xy} \sum_k E_k E_e \quad \begin{array}{l} \text{2D} \\ \text{Levi-Civita} \end{array}$$

continuity equation

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= -\vec{\nabla} \cdot \vec{j} = -(\partial_x j_x + \partial_y j_y) \\ &= +G_{xy} (\partial_x E_y - \partial_y E_x) = G_{xy} (\vec{\nabla} \times \vec{E})_z \end{aligned}$$

With Maxwell eqs

$$\vec{\nabla} \times \vec{E} = \frac{1}{c} \frac{\partial \vec{B}}{\partial t} \Rightarrow \frac{\partial \rho}{\partial t} = G_{xy} \frac{1}{c} \frac{\partial B}{\partial t}$$

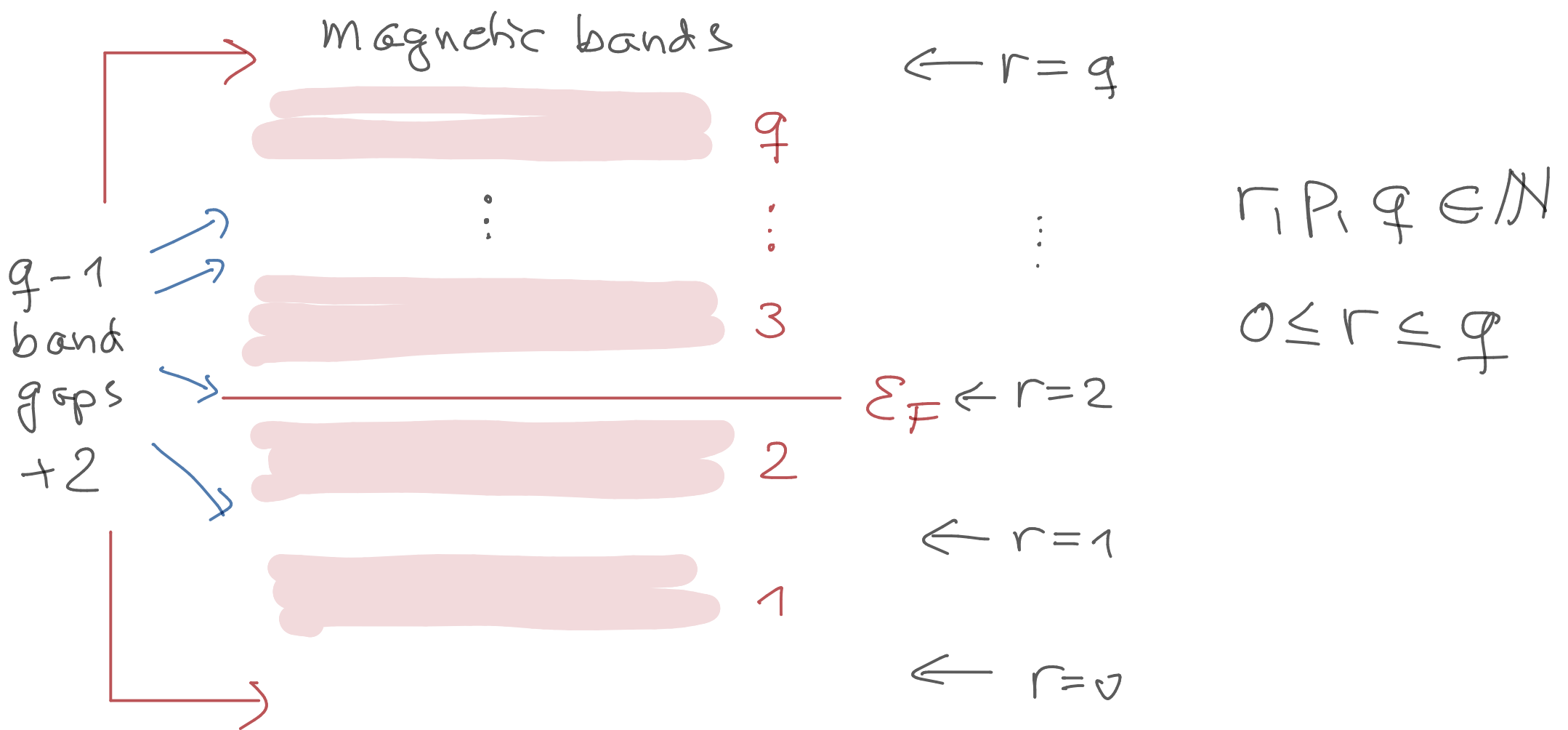
$$(165) \quad \boxed{G_{xy} = c \frac{\partial \rho}{\partial B}} \quad \begin{array}{l} \text{Streda formula} \\ (c : \text{speed of light}) \end{array}$$

Changing B changes the number of available states below the Fermi energy. So if we keep the Fermi energy constant additional electrons must flow into the sample

- Now assume flux per plaquette

$$(166) \quad \Phi = \frac{P}{g} \Phi_0 \Rightarrow g \text{ subbands}$$

E_F in the r 'th gap between subbands



One can easily see that there are integer numbers
 s_r and t_r such that $(t_r \text{ can be negative})$

$$(167) \quad r = \frac{q}{f} s_r + p t_r \quad |t_r| \leq \frac{q}{2}$$

Now consider a finite lattice with M lattice sites. Then the number of occupied lattice sites is $N = r/g M$ (each magn. subband has the same number of states)

(168) $S = e \frac{N}{A} = e \frac{r}{q} \frac{M}{A} = \frac{er}{qA_0}$

Where A_0 is area of unit cell. With the Shredg formula

$$G_{xy} = c \frac{\partial g}{\partial B} = ec \frac{1}{A_0} \frac{\partial}{\partial B} \left(s_r + \frac{P}{q} t_r \right)$$

Since $\frac{P}{q} = \frac{\Phi}{\Phi_0} = \frac{BA_0}{\Phi_0}$ only $\frac{\partial}{\partial B} \frac{P}{q} = \frac{A_0}{\Phi_0} \neq 0$

This finally gives the quantization of σ_{xy}

$$(169) \quad \sigma_{xy} = \frac{ec}{\Phi_0} t_r = \frac{e^2}{h} t_r \quad \begin{array}{l} t_r \text{ integer} \\ |t_r| \leq \frac{q}{2} \end{array}$$

Darboux Theorem

(i) q even

let assume $p=1$ for simplicity
i.e.

$$0 \leq r = qS_r + t_r \leq q$$

- $r < \frac{q}{2} \Rightarrow S_r = 0 \quad t_r = r$

contribution of each band to total Hall conductivity
(in unit of e^2/h)

$$(170) \quad \Delta t_r = t_{r+1} - t_r = 1 \quad \text{increase by 1}$$

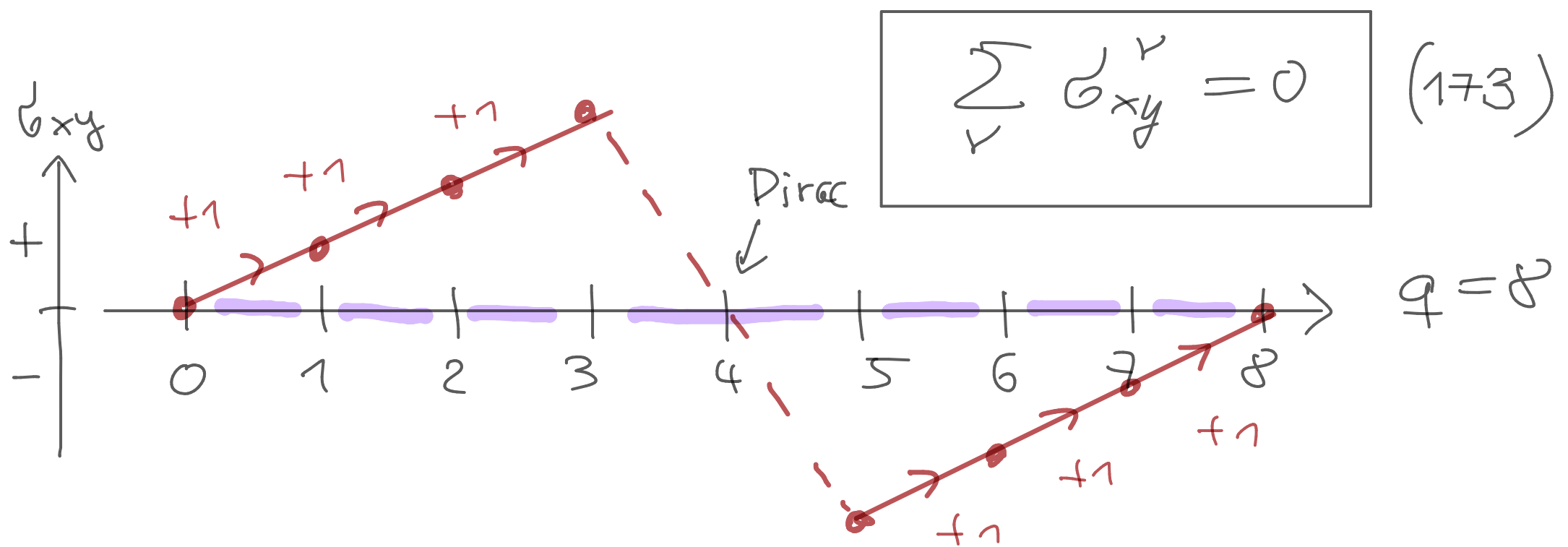
- $r > \frac{q}{2} \Rightarrow S_r = 1 \quad t_r = r - q$

$$(171) \quad \Delta t_r = t_{r+1} - t_r = 1 \quad \text{increase by 1}$$

- $r = \frac{q}{2}$ not a true gap (Dirac points)

$$\Delta t_{q/2} = t_{\frac{q}{2}+1} - t_{\frac{q}{2}-1} = -\frac{q}{2} + 1 - \frac{q}{2} + 1$$

$$(172) \quad \Delta t_{q/2} = 2 - q \quad \text{jump!}$$



(ii) q odd

- $r \leq \frac{q-1}{2} \Rightarrow S_r = 0 \quad t_r = r$

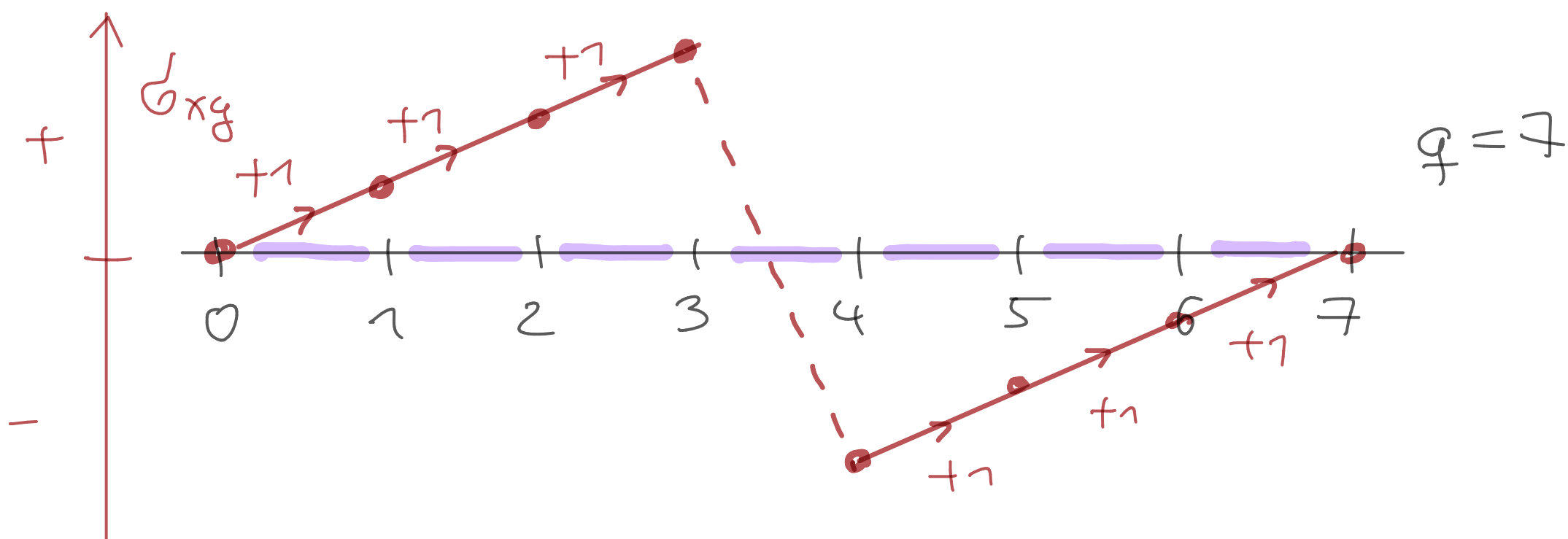
(174) $\Delta t_r = t_{r+n} - t_r = 1$ increase by 1

- $r \geq \frac{q+1}{2} \Rightarrow S_r = 1 \quad t_r = r - q$

(175) $\Delta t_r = t_{r+n} - t_r = 1$ increase by 1

- central energy band

(176) $\Delta t_{q/2} = t_{\frac{q+1}{2}} - t_{\frac{q-1}{2}} = 1 - q$ jump!



(C) Continuum limit: Landau levels

The Harper - Hofstadter model smoothly connects to the continuum model of a 2D electron gas in a magnetic field by considering the limit

$$(177) \quad \alpha = \frac{P}{g} \longrightarrow 0 \quad \text{i.e. } g \longrightarrow \infty$$

and having only a finite number of magnetic sub-bands occupied.

In the limit (177) all bands become flat (Landau levels)

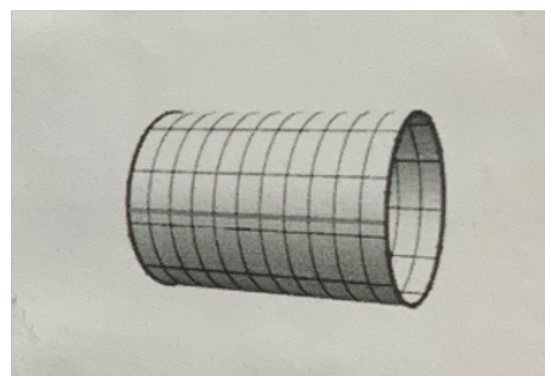
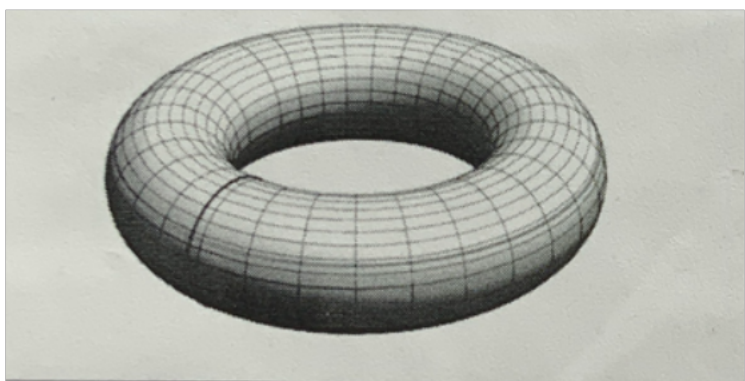
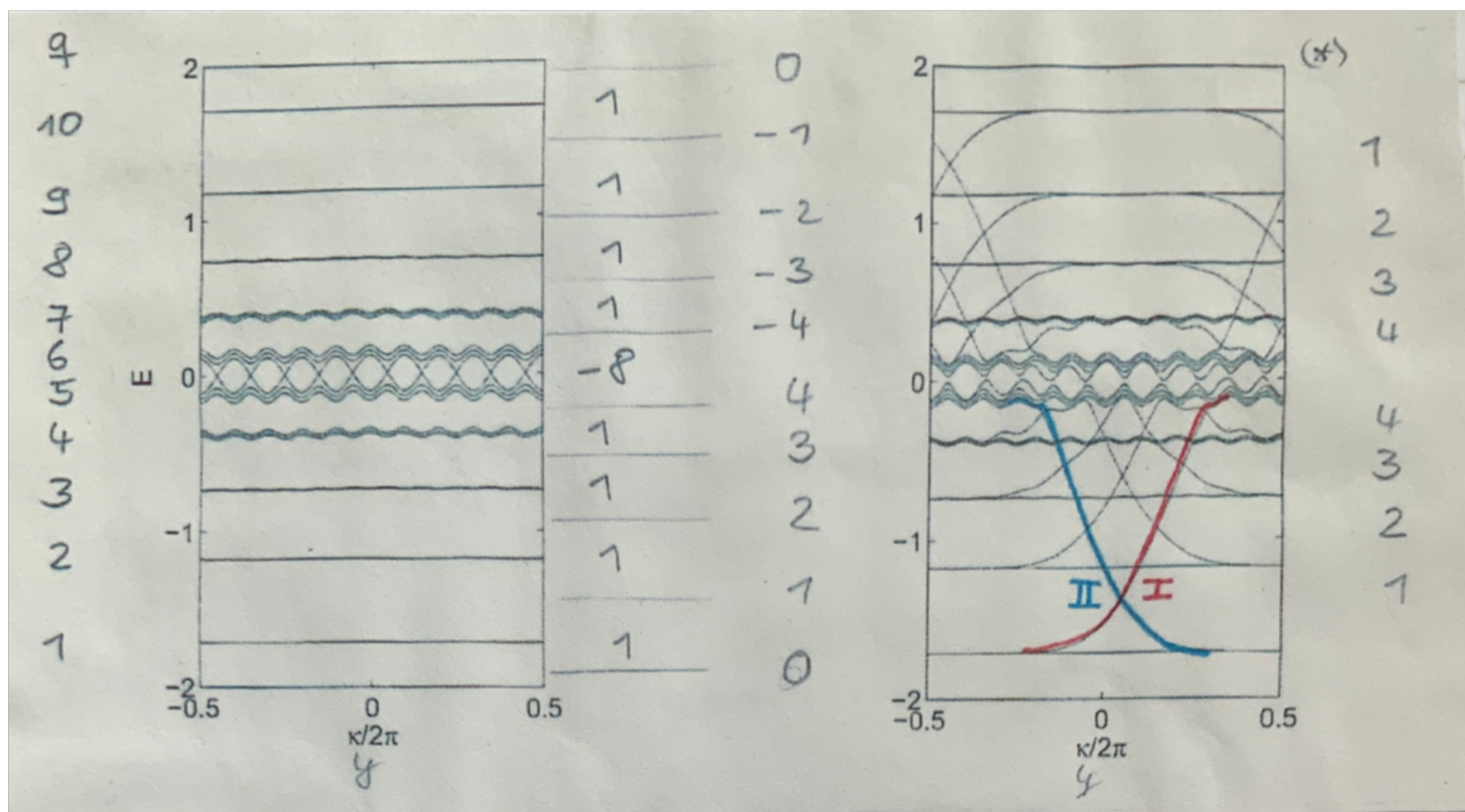
3.3 Edge modes and bulk-boundary correspondence

We have established the connection between the quantized Hall conductivity σ_{xy} and the Chern number in a QH system or Chern insulator. But what is the origin of the (almost) vanishing longitudinal resistivity? The system is an insulator after all!

(A) some numerical findings

periodic BC

open BC



- isolated modes in the case of open boundary conditions. These are edge modes similar to those we have encountered in the Qi-Wu-Zhang model.

- There are two types of modes

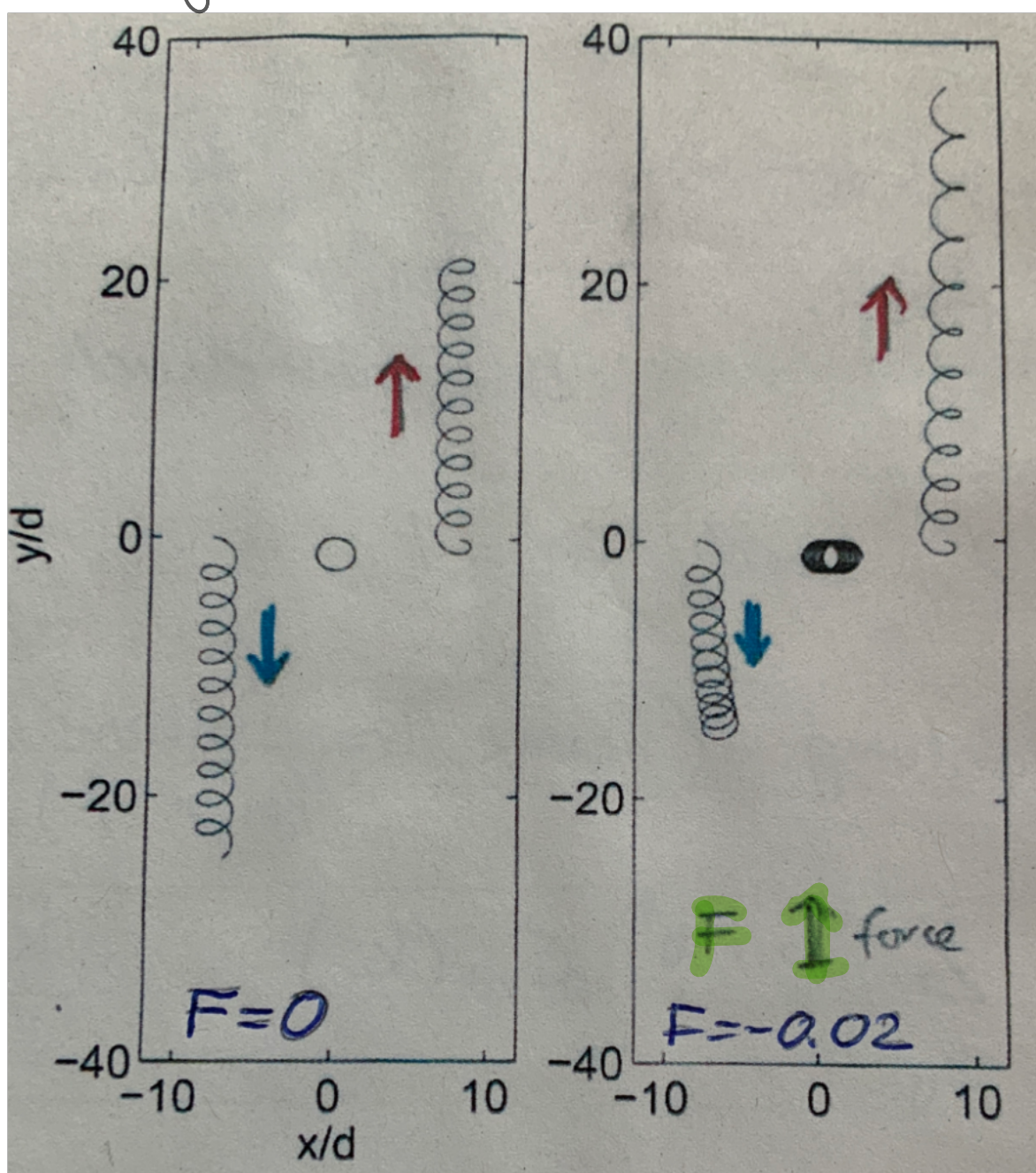
type I	●	$\frac{dw}{dk} > 0$	transport in "+" direction
type II	●	$\frac{dw}{dk} < 0$	transport in "-" direction

Chiral edge modes

- Level crossings are true level crossings!

In finite systems they are avoided but with exponentially small gap

single particle trajectories under a force $\vec{F}$ parallel to the strip



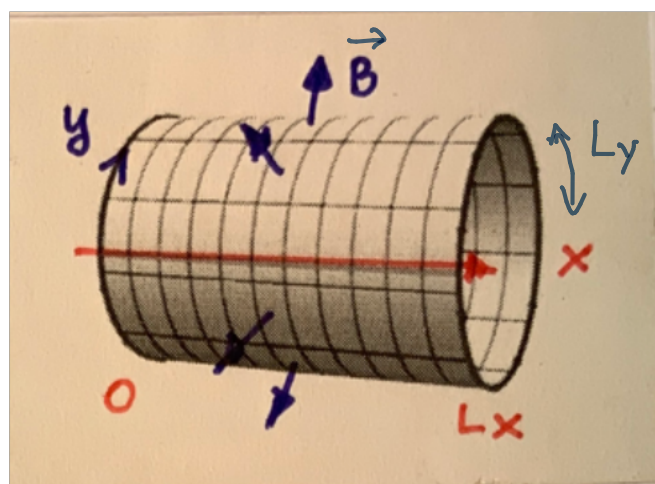
Without a proof:

The total number of edge modes (per direction) is equal to the absolute value of the sum of Chern numbers of all bands below the considered gap

bulk-boundary correspondence

(B) How to calculate the spectrum with open boundaries: transfer matrix method

Let us consider a cylinder geometry, i.e. periodic in y but with open boundaries in x direction in a magnetic field with flux per plaquette α



$$(178) \quad H = -t_x \sum_{m,n} C_{m+1,n}^\dagger C_{mn} - t_y \sum_{m,n} C_{m,n+1}^\dagger C_{mn} e^{2\pi i \alpha m} + \text{h.c.}$$

edges are at $m=0$ and $m=L_x$

F-transform with respect to y (periodicity)

$$n_y \in \mathbb{Z}$$

$$(179) \quad C_{mn} = \frac{1}{L_y} \sum_{k_y} e^{ik_y n} \tilde{C}_m(k_y)$$

$$k_y = \frac{2\pi}{L_y} n_y$$

$\Rightarrow$

$$(180) \quad H = -t_x \sum_{m,k_y} \tilde{C}_{m+1}^\dagger(k_y) \tilde{C}_m(k_y) - t_y \sum_{m,k_y} \tilde{C}_m^\dagger(k_y) \tilde{C}_m(k_y) e^{-ik_y + 2\pi i \alpha m}$$

Now we want to find the solution of the stationary Schrödinger eq.

$$H|\psi\rangle = E|\psi\rangle$$

for a single-particle state

$$(181) \quad |\psi\rangle = \sum_{m, k_y} \psi_m(k_y) \tilde{C}_m^\dagger(k_y) |0\rangle$$

inserting into the SE yields for each k_y

$$(182) \quad -t_x(\psi_{m+1} + \psi_{m-1}) - 2t_y \cos(k_y - 2\pi\alpha m) \psi_m = E \psi_m$$

This can be written in the form of a matrix recursion

$$(183) \quad \begin{pmatrix} \psi_{m+1}(k_y, E) \\ \psi_m(k_y, E) \end{pmatrix} = \underline{\underline{M}}_m(k_y, E) \begin{pmatrix} \psi_m(k_y, E) \\ \psi_{m-1}(k_y, E) \end{pmatrix}$$

where $\Sigma \equiv E/t_x$ and

$$(184) \quad \underline{\underline{M}}_m = \begin{pmatrix} -\Sigma - \frac{2t_y}{t_x} \cos(k_y - 2\pi\alpha m) & -1 \\ \textcircled{-1} ? & 0 \end{pmatrix}$$

with transfer matrix

- assume rational $\alpha = P/q$ and L_x commensurate with magnetic unit cell, i.e.

$$L_x = q \ell \quad \ell \in \mathbb{N}$$

$\Rightarrow$ transfer matrices periodic in m with period q

$\Rightarrow$ total transfer matrix over q sites

$$(185) \quad \underline{\underline{M}}(k_y, \varepsilon) = \underline{\underline{M}}_q(\varepsilon) \underline{\underline{M}}_{q-1}(\varepsilon) \cdots \underline{\underline{M}}_1(\varepsilon)$$

(we dropped k_y for simplicity) $\underline{\underline{M}}(\varepsilon)$ is 2×2

$$\underline{\underline{M}}(\varepsilon) = \begin{pmatrix} M_{11}(\varepsilon) & M_{12}(\varepsilon) \\ M_{21}(\varepsilon) & M_{22}(\varepsilon) \end{pmatrix}$$

• From eq. (184) we see:

- $M_{11}(\varepsilon)$ polynomial in ε of degree q
- $M_{12}(\varepsilon), M_{21}(\varepsilon)$ polynomial in ε of degree $q-1$
- $M_{22}(\varepsilon)$ polynomial in ε of degree $q-2$

Since $L_x = q\ell \Rightarrow$ from (183)

$$(186) \quad \begin{pmatrix} \psi_{L_x+1} \\ \psi_{L_x} \end{pmatrix} = \left(\underline{\underline{M}}(\varepsilon) \right)^{\ell} \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix}$$

bulk modes

open boundary conditions imply (ψ_{L_x+1} is irrelevant)

$$(187) \quad \psi_{L_x} = \psi_0 \stackrel{!}{=} 0 \quad \begin{pmatrix} \psi_{L_x+1} \\ 0 \end{pmatrix} \stackrel{!}{=} \underline{\underline{M}}^{\ell} \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix}$$

$\Rightarrow$

(188)

$$\boxed{\left[\left(\underline{\underline{M}}(\varepsilon) \right)^{\ell} \right]_{21} \stackrel{!}{=} 0}$$

for every k_y

$\left(\underline{\underline{M}}^e\right)_{2,1}$ is a polynomial of degree $q\ell-1 = L_x-1$

$\Rightarrow L_x-1$ real roots $\varepsilon_n = \varepsilon_n(k_y)$ which are the eigenvalues of the problem with open BC

(C) Edge modes

Let us construct the edge modes. ε_q . (188) can be fulfilled if we choose

$$(189) \quad \underline{\underline{M}}_{2,1}(\varepsilon) = 0 \quad (\text{note } \underline{\underline{M}} \text{ not } \underline{\underline{M}}^e)$$

Since a product of triangular matrices is again a triangular matrix. $\underline{\underline{M}}_{2,1}(\varepsilon)$ is a polynomial in ε of degree $q-1$ i.e. (189) has $q-1$ solutions

$$(190) \quad \varepsilon_j \quad j=1,2,\dots,q-1 \quad \text{set } \varepsilon_i < \varepsilon_j \quad \text{for } i < j$$

One can show that (189) does not have degenerate solutions i.e. all ε_j are different for a fixed value of k_y .

• Classification of the $q-1$ solutions

from (186) ($L_x = q\ell$) $e' < \ell$

$$\begin{pmatrix} \psi_{qe'+1}(\varepsilon_j) \\ \psi_{qe'}(\varepsilon_j) \end{pmatrix} = \begin{pmatrix} M_{11}(\varepsilon_j)^{e'} & B \\ 0 & C \end{pmatrix} \begin{pmatrix} \psi_1(\varepsilon_j) \\ 0 \end{pmatrix}$$

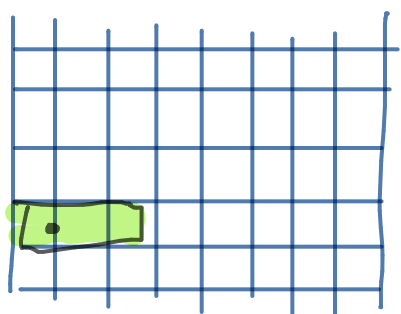
$\Rightarrow$

$$\frac{2^q q e^{i+1}}{2^q 1} = M_{11}(\varepsilon_j) e^i$$

$$|M_{11}(\varepsilon_j)| < 1$$

mode localized
at magnetic unit
cell containing
 $m=1$

edge mode
on the left

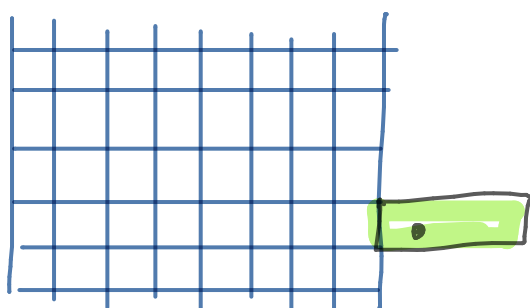


$m=0$ L_x

$$|M_{11}(\varepsilon_j)| > 1$$

mode localized
at magnetic unit
cell containing
 $m = q\ell + 1 = L_x + 1$

edge mode
on the right

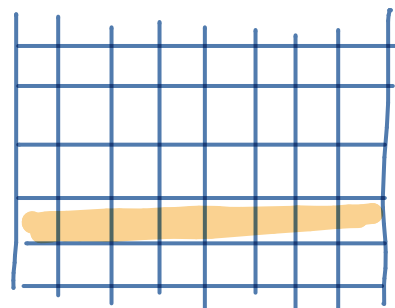


0 L_x

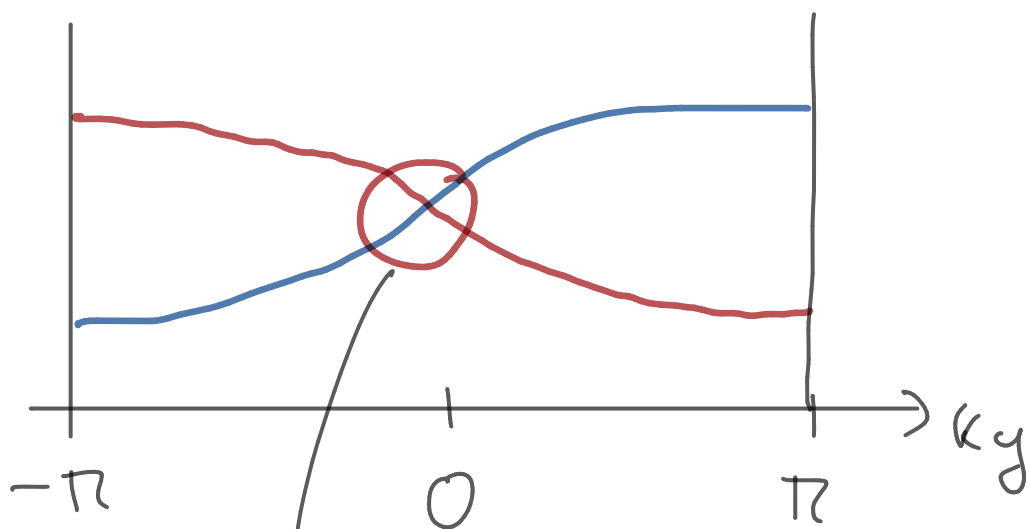
$$|M_{11}(\varepsilon_j)| = 1$$

delocalized
mode

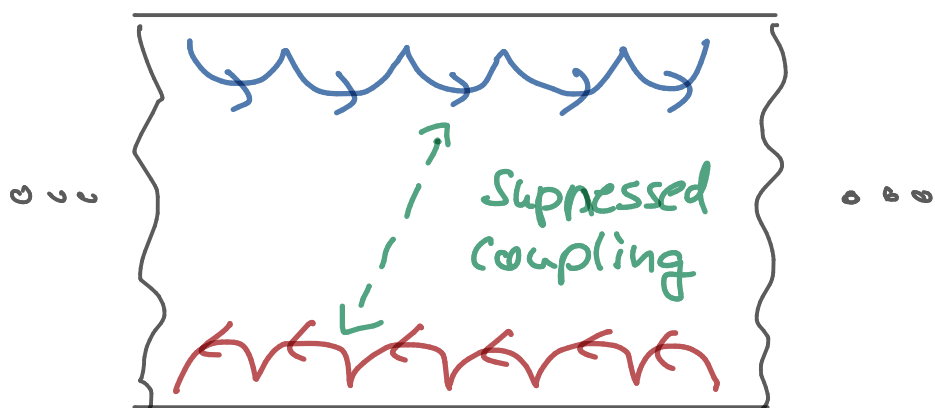
bulk mode



0 L_x



exponentially suppressed
coupling



$$\Rightarrow \sigma_{xx} = 0 \quad (197)$$