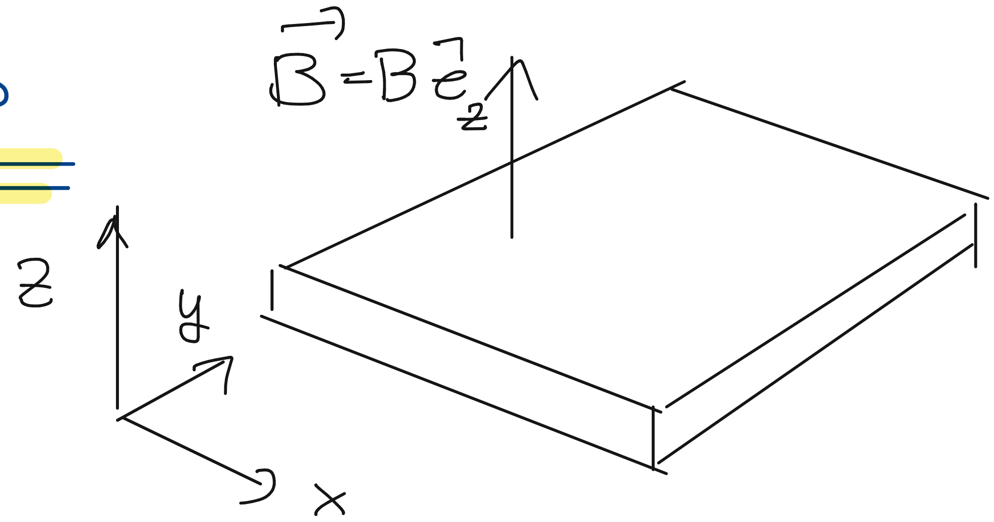
2. Integer quantum Hall system

2.1 Landau levels



Free electrons in magnetic field
+ 2D confinement

(100) $$H = \frac{1}{2m}\left(\vec{p} + \frac{e}{c}\vec{A}\right)^2 \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

(A) Landau gauge

(101) $$\vec{A} = B(-y, 0, 0)$$

gauge breaks rotational symmetry

(102) $$H = \frac{1}{2m}\left[\left(\hat{p}_x - \frac{eB}{c}\hat{y}\right)^2 + \hat{p}_y^2\right]$$

$$[H, \hat{p}_x] = 0 \implies \hat{p}_x = p_x = \hbar k_x$$

conserved

characteristic length

(104) $$\boxed{l_0 = \sqrt{\frac{\hbar c}{eB}}}$$

magnetic length

introduce dimensionless variables

$$y \to y' = \frac{y}{l_0} - l_0 k_x \qquad p_y \to p_y' = \frac{l_0}{\hbar} p_y$$

- now let's fix k_x (constant of motion)

(105) $$H[k_x] = \hbar\omega_c\left(\frac{1}{2}y'^2 + \frac{1}{2}p_{y'}^2\right) + \text{const.}$$

this is an harmonic oscillator

(106) $$\boxed{\omega_c = \frac{eB}{mc}}$$ cyclotron frequency

energies

(107) $$\boxed{E_n = \left(n+\frac{1}{2}\right)\omega_c}$$ Landau levels

- does not depend on k_x $\Rightarrow$ degeneracy
- y-coordinate of oscillators depends on k_x

(B) symmetric gauge

(108) $$\vec{A} = \frac{1}{2}(\vec{B}\times\vec{r}) = \frac{B}{2}(-y, x, 0)$$

gauge respects rotational symmetry

in dimensionless units: (l_0, ω_c)

(109) $$H = \frac{\hbar\omega_c}{2}\left[\left(-i\frac{\partial}{\partial x} - \frac{y}{2}\right)^2 + \left(-i\frac{\partial}{\partial y} + \frac{x}{2}\right)^2\right]$$

$\Rightarrow$ introduce complex coordinates in (x,y) plane

(110) $$\boxed{z = x - iy \qquad z^* = x + iy = re^{i\theta}}$$

(111) $\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial z^*}$ $\qquad$ $\frac{\partial}{\partial y} = -i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*}\right)$

where z and z^* are treated as independent variables

$\Rightarrow$ define two types of ladder operators

(112)
$$\boxed{\begin{aligned} \hat{a} &= \frac{1}{\sqrt{2}}\left(\frac{z}{2} + 2\frac{\partial}{\partial z^*}\right) \\ \hat{a}^{\dagger} &= \frac{1}{\sqrt{2}}\left(\frac{z^*}{2} - 2\frac{\partial}{\partial z}\right) \end{aligned}}$$

$\frac{\partial}{\partial z^*} = \frac{1}{2}(\partial_x - i\partial_y)$

$\frac{\partial}{\partial z} = \frac{1}{2}(\partial_x + i\partial_y)$

?

(113)
$$\boxed{\begin{aligned} \hat{b} &= \frac{1}{\sqrt{2}}\left(\frac{z^*}{2} + 2\frac{\partial}{\partial z}\right) \\ \hat{b}^{\dagger} &= \frac{1}{\sqrt{2}}\left(\frac{z}{2} - 2\frac{\partial}{\partial z^*}\right) \end{aligned}}$$

(114) $[a, a^{\dagger}] = [b, b^{\dagger}] = 1 \qquad [a, b^{(\dagger)}] = 0$

one finds that the Hamiltonian depends only on $a, a^{\dagger}$

(115)
$$\boxed{H = \hbar\omega_c\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right)}$$

again Landau levels

$\Rightarrow$ degeneracy of Landau levels with respect to $b^{\dagger}b$

angular momentum in z direction

$$[H, \hat{L}_z] = 0 \qquad \text{by symmetry}$$

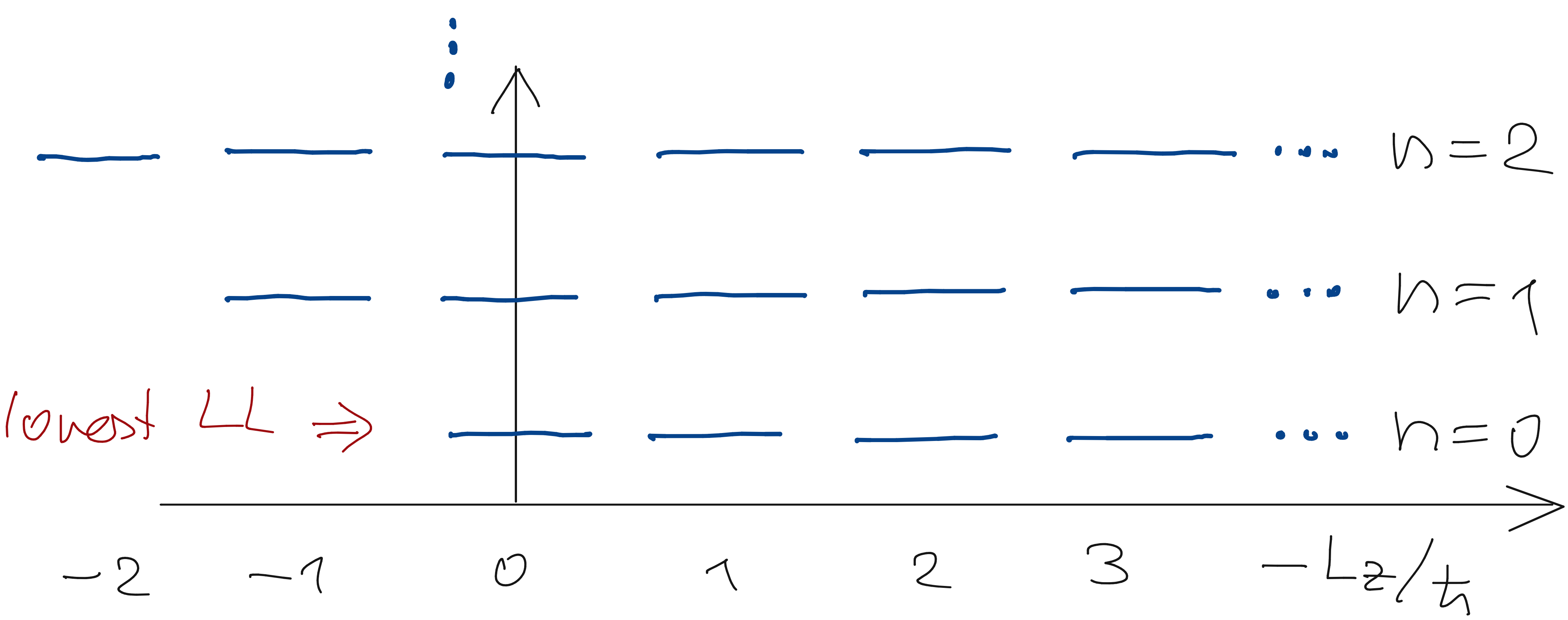
$$L_z = -i\hbar \frac{\partial}{\partial \theta} = -\hbar \left(z \frac{\partial}{\partial z} - z^* \frac{\partial}{\partial z^*} \right)$$

(116) $$\boxed{\hat{L}_z = \hbar \left(a^\dagger a - b^\dagger b \right)}$$

eigenvalues of $\hat{L}_z$: $\hbar m$

$$m = n, n-1, n-2, \ldots$$

spectrum of Landau levels (LL)



wavefunctions

(117) $$|n,m\rangle = \frac{b^{\dagger\, n+m} \, a^{\dagger\, n} \, |0\rangle}{\sqrt{(n+m)!\, n!}} \qquad L_z = -\hbar m, \quad E = \hbar \omega_c n$$

$$\psi_{nm}(z) = \mathcal{N}\left(\frac{z^*}{2} - 2\frac{\partial}{\partial z}\right)^n \left(\frac{z}{2} - 2\frac{\partial}{\partial z^*}\right)^{m+n} e^{-|z|^2/4} \tag{118}$$

(in normalized units)

2.2 Lowest Landau level (LLL)

For the physics at low temperature the lowest Landau level is of particular impordance

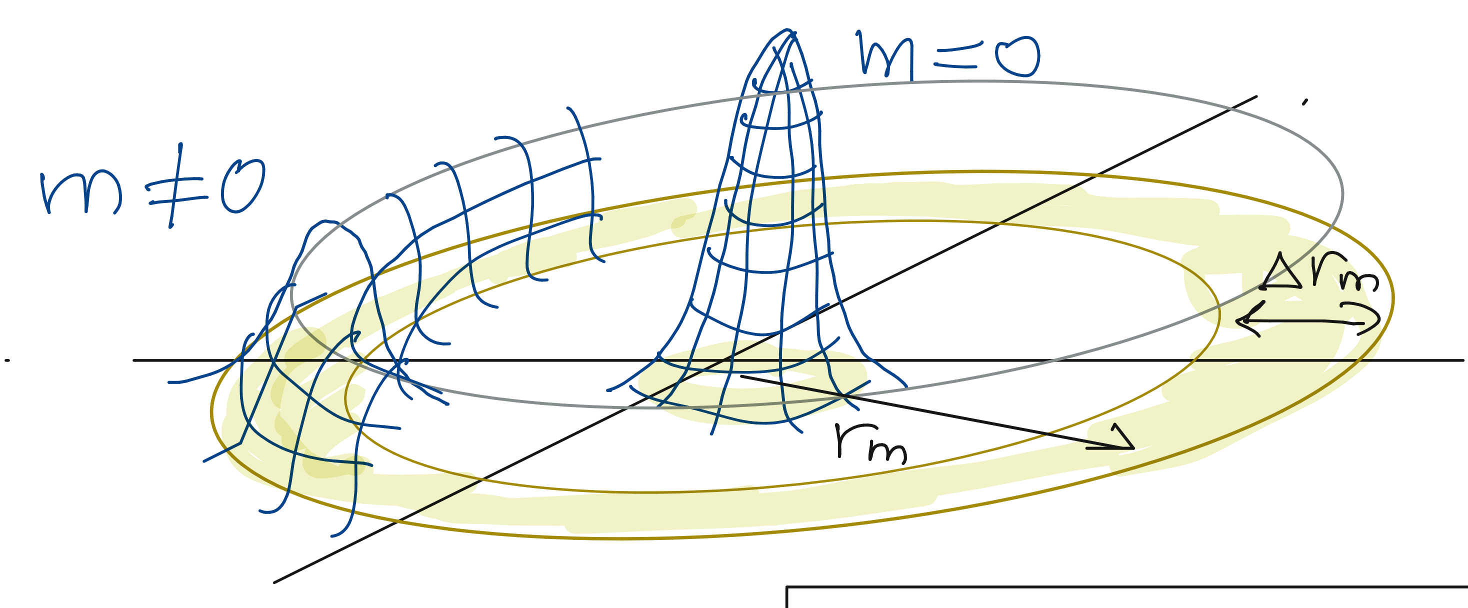
$$n = 0$$

$$\psi_{0m}(z) = \mathcal{N}\left(\frac{z}{2} - 2\frac{\partial}{\partial z^*}\right)^m e^{-|z|^2/4} \tag{119}$$

so

$$\boxed{\psi_{0m}(z) = \mathcal{N}\, z^m e^{-|z|^2/4}} \tag{120}$$

analytic wavefunction (apart from Gaussian)



$$\Delta r_m = \frac{l_0}{\sqrt{2m}}$$

$$r_m = l_0\sqrt{2m}$$

(121) area: $\boxed{A_m = 2\pi r_m \Delta r_m = 2\pi l_0^2}$

Number of states in an area A ($\hat{=}$ degeneracy)

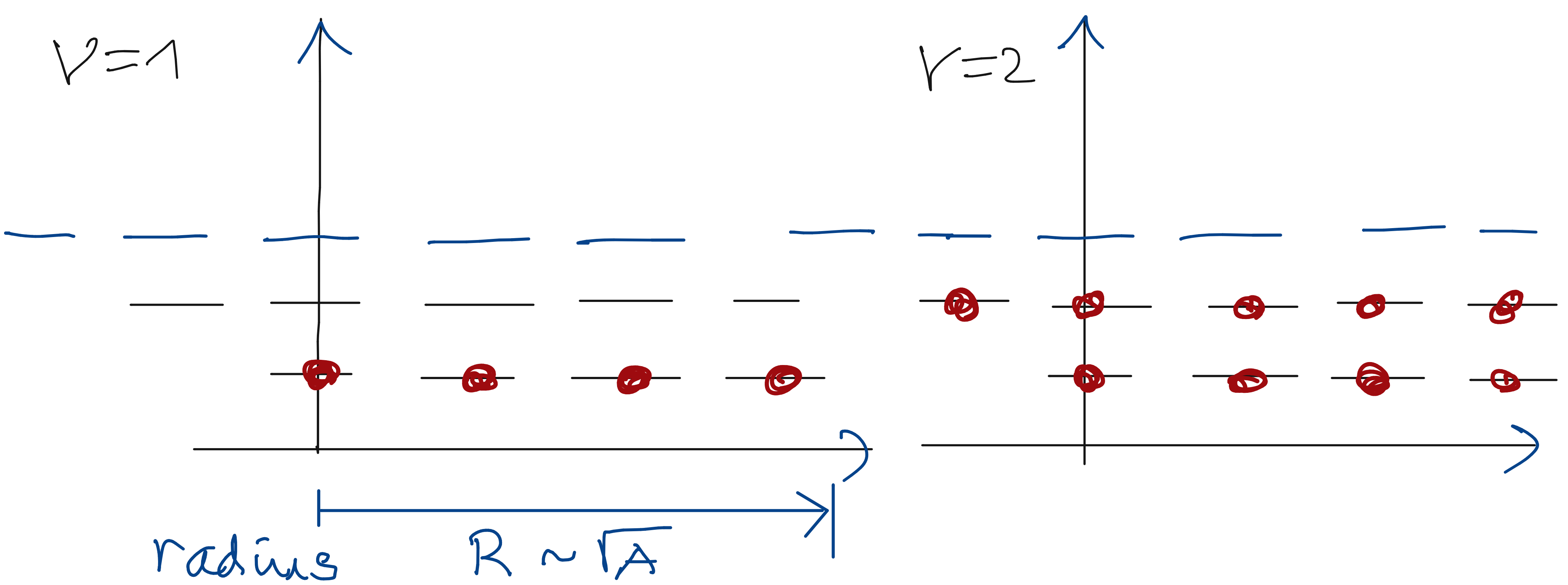
$$N = \frac{A}{2\pi l_0^2} = \frac{AB}{\Phi_0} = \frac{\Phi}{\Phi_0} \tag{122}$$

where $\boxed{\Phi_0 = \frac{hc}{e}}$ Dirac flux quantum

Since we consider (spin-polarized) electrons, every state can be occupied by one electron (Pauli principle) there can be at most one particle in $2\pi l_0^2$ if we stay in the LLL

in general: filling fraction

$$\nu = \frac{N_e}{N} = \frac{N_e\, 2\pi l_0^2}{A} = 2\pi \rho_e l_0^2 = \frac{\rho_e}{B/\Phi_0} \tag{123}$$



- many-body wavefunction of filled LLL ($\nu=1$)

Slater determinant

(124)

$$\psi_{\nu=1}(z_1,\dots,z_N) \sim \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & & z_N \\ z_1^2 & z_2^2 & \ddots & z_N^2 \\ \vdots & \vdots & & \vdots \\ z_1^N & z_2^N & \cdots & z_N^N \end{vmatrix} e^{-\frac{1}{4}\sum_{j=1}^{N}|z_j|^2}$$

Every electron can have all angular momenta, i.e. all wavefunctions $z_j^m e^{-\frac{1}{4}|z_j|^2}$ with $m=0,1,..N$ occur in many-body state

$$\boxed{\psi_{\nu=1}(z_1,..,z_N) \sim \prod_{j<k} (z_j - z_k) \exp\left\{-\frac{1}{4}\sum_j |z_j|^2\right\}}$$

(125)

Pauli exclusion + antisymmetry

normalization for $|z_j| \to \infty$

2.3 Discretization and lattice model

We will see now that the integer quantum Hall effect can be considered as limiting case of a Bloch Hamiltonian with topological properties

discretization in real space (square lattice)

(126)
$$x \rightarrow x_j = j\Delta x$$
$$y \rightarrow y_\ell = \ell \Delta x$$
$$\psi(\vec{r}) = \psi(x,y) \rightarrow \psi_{j\ell}$$

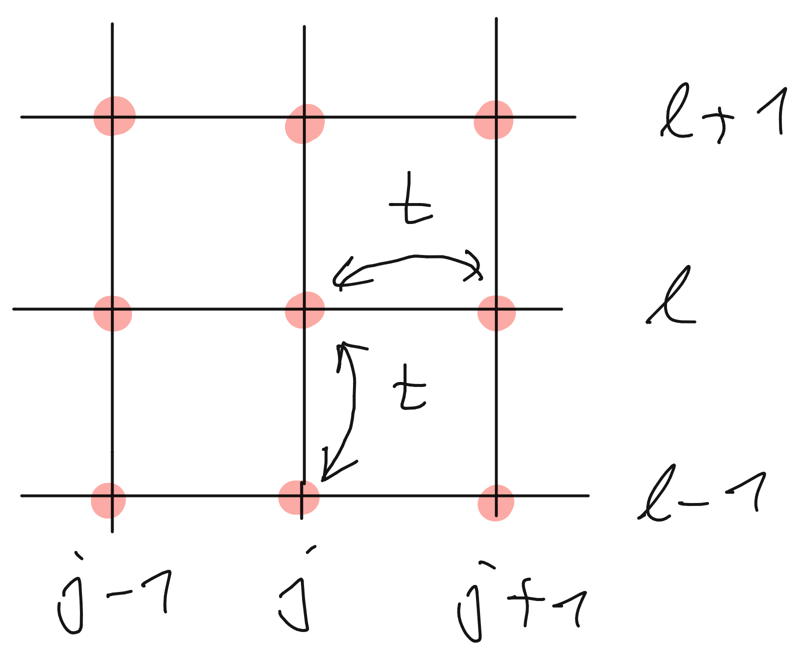
kinetic energy

(127)
$$\Delta\psi = \frac{1}{\Delta x^2}\left(\psi_{j+1,\ell} + \psi_{j-1,\ell} + \psi_{j,\ell+1} + \psi_{j,\ell-1} - 4\psi_{j\ell}\right)$$

$$i\hbar\frac{\partial}{\partial t}\psi = H\psi = -\frac{\hbar^2}{2m}\Delta\psi$$

(128)
$$H\psi_{j\ell} = -t\left(\psi_{j+1,\ell} + \psi_{j-1,\ell} + \psi_{j,\ell+1} + \psi_{j,\ell-1} - 4\psi_{j\ell}\right)$$

$$t = \frac{\hbar^2}{2m(\Delta x)^2}$$



in second quantization

$$\boxed{H = -t \sum_{je} \left(C^{\dagger}_{j+1,e} C_{je} + C^{\dagger}_{j,e+1} C_{je} + h.a. \right)} \qquad (129)$$

Now lets add the interaction with the magnetic field in Landau gauge.

$$\vec{A} = -By\,\vec{e}_x$$

$$H\psi = -\frac{\hbar^2}{2m}\left[\left(\frac{\partial}{\partial x} - \frac{i}{\hbar}\frac{eBy}{c}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2\right]\psi \qquad (130)$$

$$\left(\frac{\partial}{\partial x} - \frac{ieBy}{\hbar c}\right)\psi(x,y) = \frac{1}{\Delta x}\left(\psi_{j+1,e} - \psi_{j,e} - \frac{ieB\Delta x^2}{\hbar c} e\, \psi_{je}\right)$$

$$= \frac{1}{\Delta x}\left(\psi_{j+1,e} - \psi_{j,e} - \frac{ieB\Delta x^2}{\hbar c} e\, \psi_{j+1,e}\right)$$

$$= \frac{1}{\Delta x}\left(\psi_{j+1,e}\underbrace{\left(1 - \frac{ieB\Delta x^2}{\hbar c} e\right)}_{\approx \exp\left(-\frac{ieB\Delta x^2}{\hbar c} e\right)} - \psi_{je}\right) \qquad (131)$$

Peierls phase

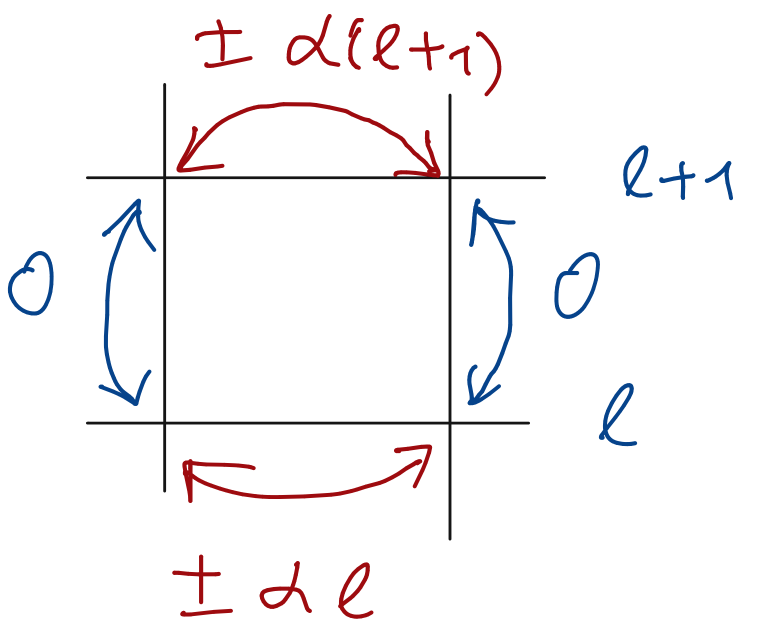
$$H\psi_{je} = -t\left(e^{-i\alpha e}\psi_{j+1,e} + e^{i\alpha e}\psi_{j-1,e} + \psi_{j,e+1} + \psi_{j,e-1} - 4\psi_{j,e}\right)$$

$$\boxed{\alpha = \frac{eB}{\hbar c}(\Delta x)^2 = \left(\frac{\Delta x}{l_0}\right)^2} \qquad (132)$$

Peierls phase

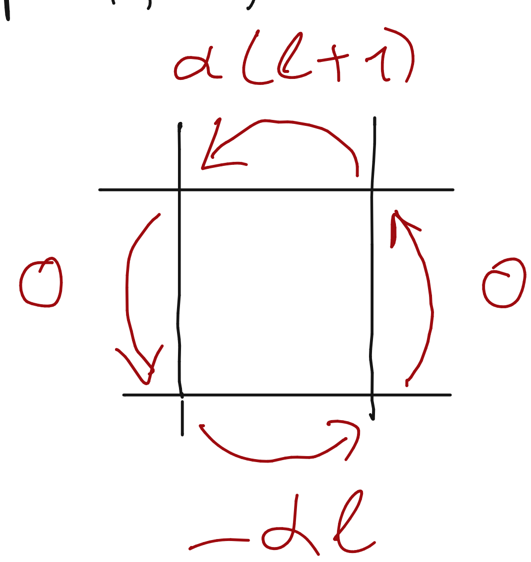
so in second quantization we obtain complex hopping amplitudes

$$\boxed{\begin{aligned} H = & -t \sum_{jl} \left(e^{-i\alpha l} c^\dagger_{j+1\,l} c_{jl} + h.a. \right) \\ & -t \sum_{jl} \left(c^\dagger_{j\,l+1} c_{jl} + h.a. \right) + 4t \sum_{jl} c^\dagger_{jl} c_{jl} \end{aligned}} \qquad (133)$$



in chosen gauge: phases when hopping in x-direction

The phase picked up upon a closed loop in any plaquette



$$\boxed{\begin{aligned} \Delta\phi = \alpha = \left(\frac{\Delta x}{l_0}\right)^2 \\ = \frac{B\Delta x^2}{\Phi_0} = \frac{\text{magn. flux per plaquette}}{\text{Dirac flux quantum}} \end{aligned}} \qquad (134)$$