

10. Topological lattice models with interactions and fractional Chern insulators

10.1 Rice-Mele model with interactions

Superlattice Bose-Hubbard model

We first consider the simplest topological models in 1D, the SSH and Rice-Mele model with onsite interactions for bosonic particles

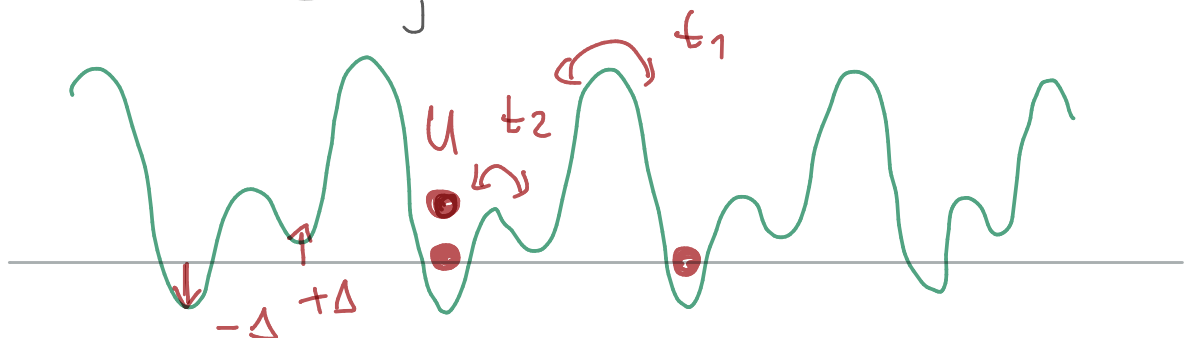
Superlattice Bose-Hubbard model (SLBHM)

RM

$$H = -t_1 \sum_{j \text{ even}} (a_{j+1}^\dagger a_j + \text{h.c.}) - t_2 \sum_{j \text{ odd}} (a_{j+1}^\dagger a_j + \text{h.c.}) + \Delta \sum_j (-1)^j a_j^\dagger a_j + \frac{U}{2} \sum_j a_j^{\dagger 2} a_j^2$$

(436)

repulsive interaction



The single particle Hamiltonian has chiral symmetry for $\Delta=0$ and non-trivial winding of 2π phase when encircling the origin. The SSH model ($\Delta=0$) has edge states. What survives with interactions?

let us consider the grand canonical setting

$$(437) \quad K = H - \mu N \quad N = \sum_j a_j^\dagger a_j$$

(A) Hardcore limit and mapping to fermionic RMM

hardcore limit $U \rightarrow \infty \Rightarrow$ only $n=0$ and $n=1$ particles per lattice site

$$\frac{U}{2} a_i^{\dagger 2} a_i^2 = \frac{U}{2} a_i^\dagger a_i (a_i^\dagger a_i - 1)$$

hard core bosons = spin 1/2 particle

$$(438) \quad \boxed{\begin{aligned} [a_i, a_j^\dagger] &= \delta_{ij} & a_i^{\dagger 2} &= a_j^2 = 0 \\ a_i &= c_i^- & a_i^\dagger &= c_i^+ \end{aligned}}$$

Wigner-Jordan mapping to fermions

$$(439) \quad \boxed{b_e^- = \exp \left\{ -i\pi \sum_{j < e} c_j^\dagger c_j \right\} c_e}$$

non-local transformation

$$(440) \quad \{c_j, c_e^\dagger\} = \delta_{ej} \quad \{c_j, c_e\} = 0$$

check: $e' < e$

$$\begin{aligned} b_e^- b_{e'}^- &= e^{-i\pi \sum_{j < e} c_j^\dagger c_j} c_e e^{-i\pi \sum_{j < e'} c_j^\dagger c_j} c_{e'} \\ &= e^{-i\pi \sum_{j=e'}^{e-1} c_j^\dagger c_j} c_e c_{e'} = -e^{-i\pi \sum_{j=e'}^{e-1} c_j^\dagger c_j} c_{e'} c_e \end{aligned}$$

$$\begin{aligned} b_{e'}^- b_e^- &= e^{-i\pi \sum_{j < e'} c_j^\dagger c_j} c_{e'} e^{-i\pi \sum_{j < e} c_j^\dagger c_j} c_e \\ &= e^{-i\pi \sum_{j=e'+1}^e c_j^\dagger c_j} \underbrace{c_{e'} e^{-i\pi c_{e'}^\dagger c_{e'}}}_{= e^{-i\pi(c_{e'}^\dagger c_{e'} + 1)}} c_e \\ &= e^{-i\pi} e^{-i\pi \sum_{j=e'}^e c_j^\dagger c_j} c_{e'} c_e = b_e^- b_{e'}^- \end{aligned}$$

$$\Rightarrow [b_e^-, b_{e'}^-] = 0 \quad \square$$

• Wigner-Jordan transform of Hamiltonian in hard-core limit

$$a_e^\dagger a_e = b_e^\dagger b_e = c_e^\dagger e^{+i\pi \sum_{j < e} c_j^\dagger c_j} e^{-i\pi \sum_{j < e} c_j^\dagger c_j} c_e$$

$$(440) \quad = c_e^\dagger c_e$$

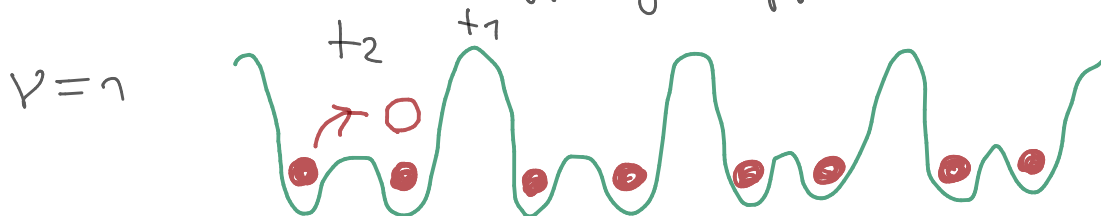
$$a_{e+1}^\dagger a_e = c_{e+1}^\dagger e^{\pi i \sum_{j < e+1} c_j^\dagger c_j} e^{-i \pi \sum_{j < e} c_j^\dagger c_j} c_e$$

$$(441) \quad = \underbrace{c_{e+1}^\dagger e^{i \pi c_e^\dagger c_e}}_{c_e^\dagger n_{e+1} = 0} c_e = c_{e+1}^\dagger c_e$$

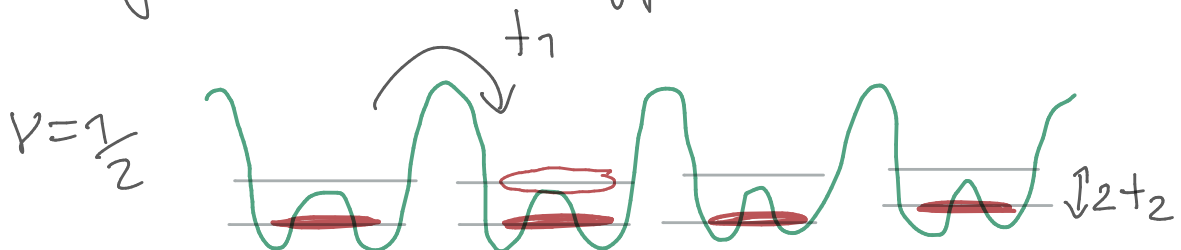
SLBHM in hard core limit is isomorph to the fermionic RMM $\Rightarrow$ identical topological features

(B) Going away from the hard-core limit

- small t/u Mott insulator (MI)
hopping suppressed



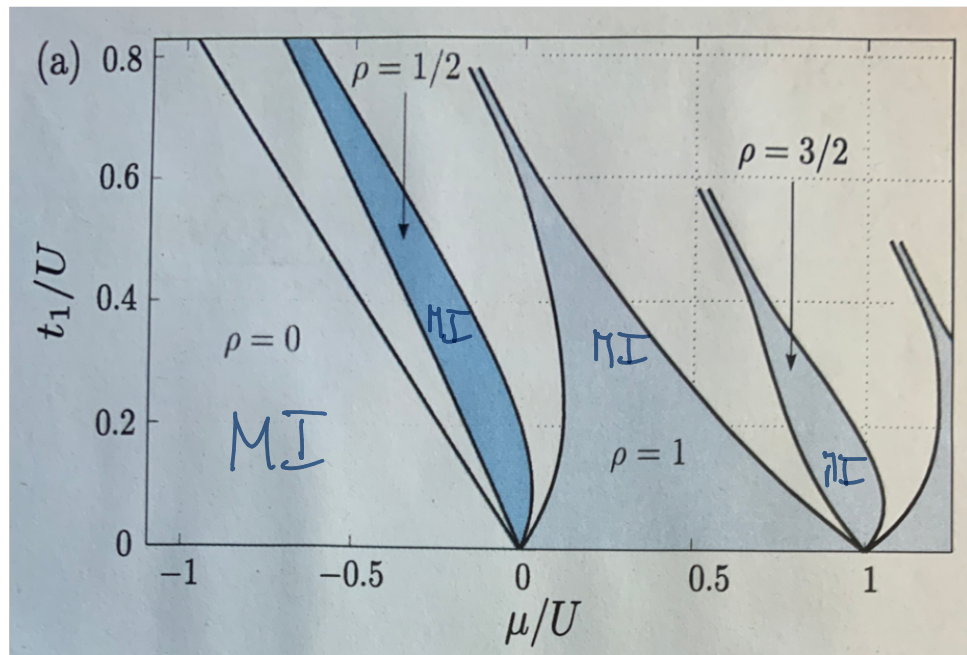
hopping costs interaction energy U
gains kinetic energy t_2



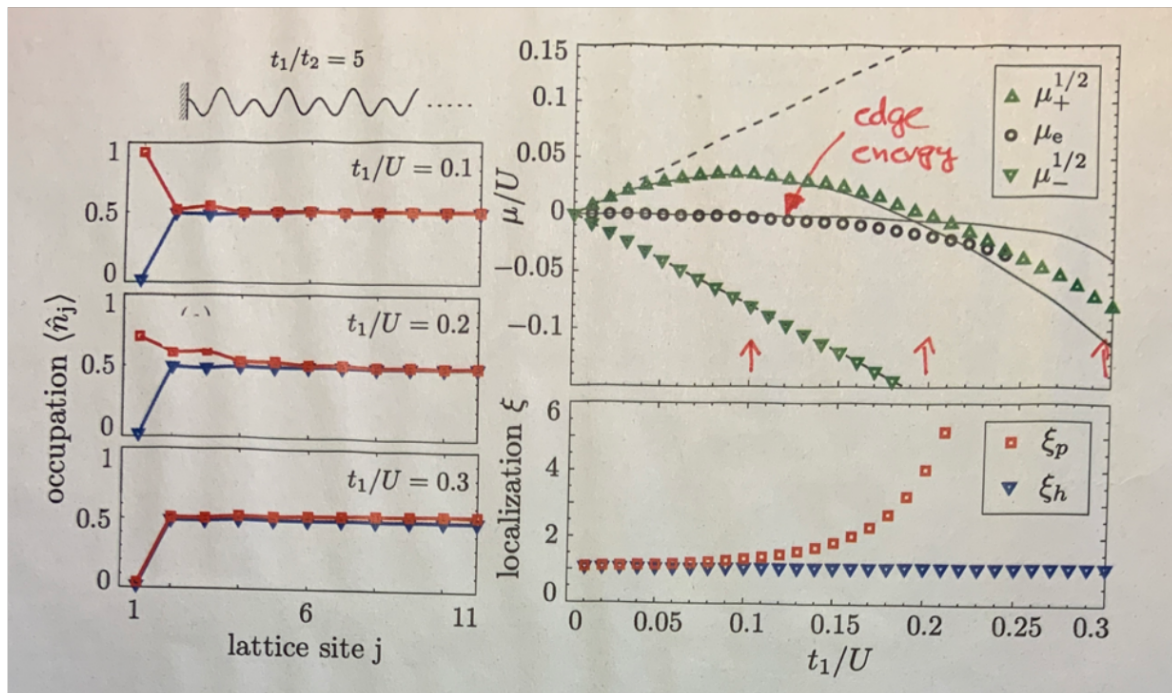
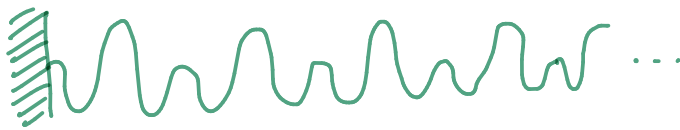
hopping costs energy $2t_2$ gains energy $\sim t_1 < t_2$

- large t/u motion allowed superfluid phase (SF)

phase diagram for $\Delta=0$



edge state in topologically nontrivial phase of SSH



→ edge state no longer at zero energy $\Rightarrow$ absence of PHS !

$$(442) \quad \mu_e \simeq -2t_z^2 \frac{U-2t_1}{(U+t_1)(U-3t_1)}$$

chemical potential of edge

$\mu < \mu_e$ edge empty

$\mu > \mu_e$ edge occupied

10.2 Many-body topological invariant

For finite interactions the single particle lattice momentum is no longer a good quantum number

$\Rightarrow$ Bloch function Zak phase
not meaningful

Q. Niu, D. Thouless J. Phys A 17, 2453 (1984)

For noninteracting gapped fermion systems a topological invariant is the winding of the Zak phase related to particle transport and a winding of the polarization

$$(443) \quad \Delta n = \nu = \frac{1}{2\pi} \oint d\lambda \frac{\partial \phi_{ZAK}}{\partial \lambda} = \Delta P$$

$$(444) \quad P = \frac{1}{2\pi} \text{Im} \ln \langle \exp(\frac{2\pi i}{L} \hat{X}) \rangle$$

Δn and ΔP remain well defined also for

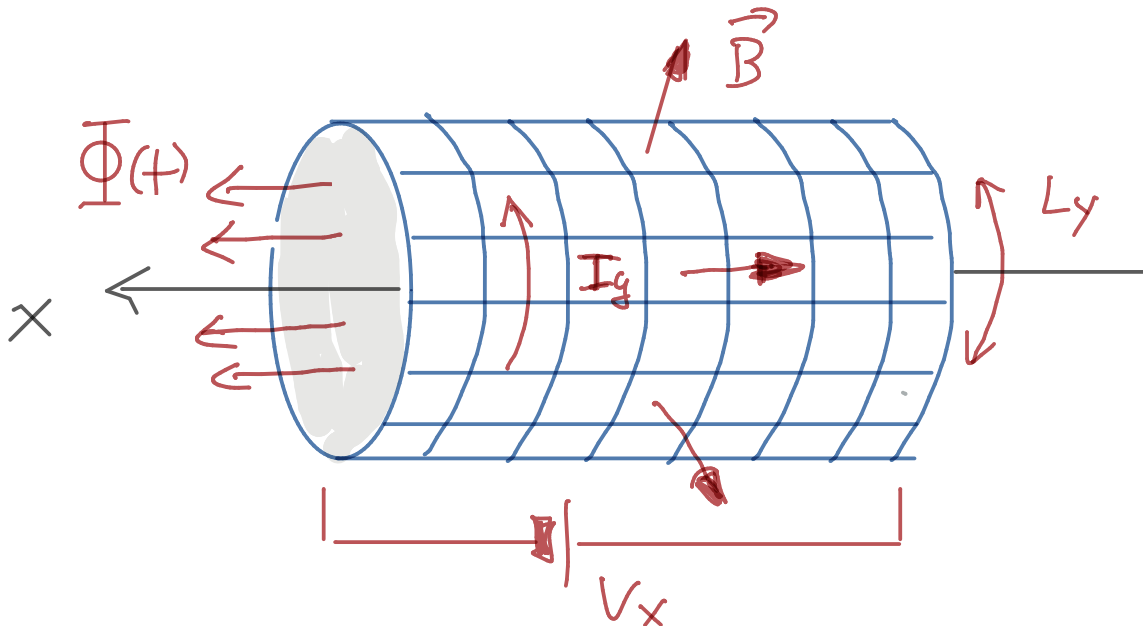
interacting particles. Following Niu and Thouless one can define a

mang-body Zak phase

To understand this mang-body generalization let us first look at non-interacting systems again and discuss an argument by Laughlin

(A) Laughlin's flux insertion

Consider a lattice wrapped to a cylinder i.e. with periodic boundary conditions



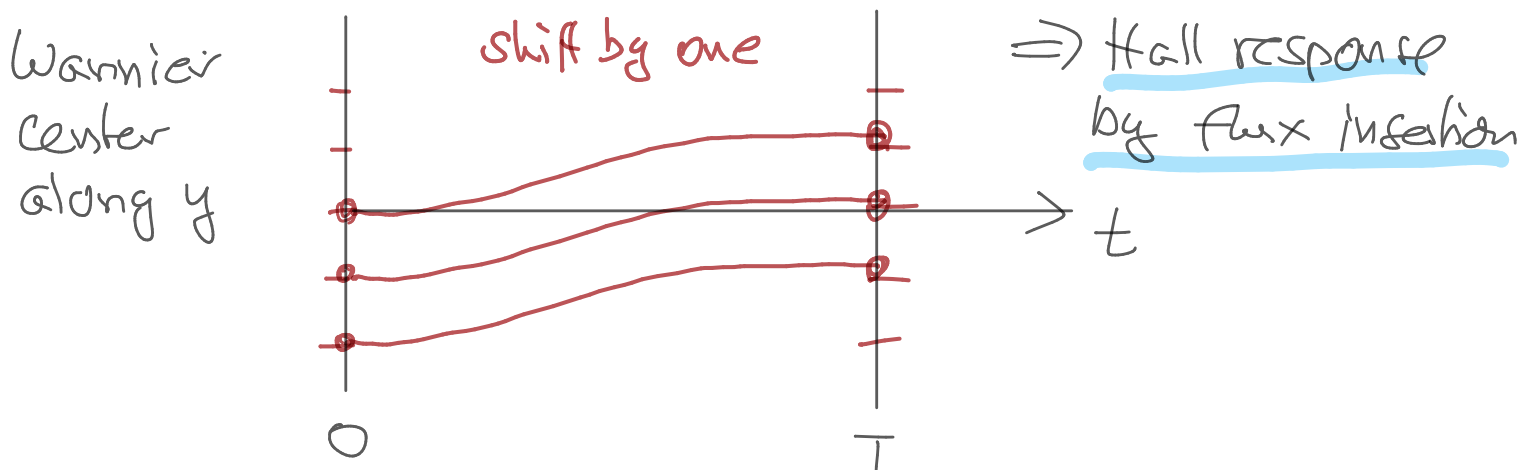
- periodic in y k_y quantized $k_y = m \frac{2\pi}{L_y}$
- adiabatic flux insertion through the tube

$$(445) \quad \vec{A} \rightarrow \vec{A} + \frac{\Phi(t)}{L_y} \vec{e}_y$$

so $k_y \rightarrow$

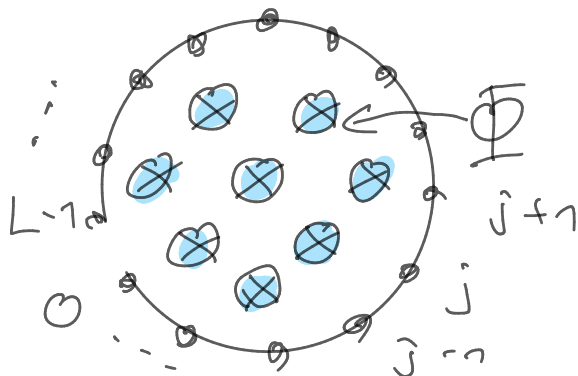
$$(446) \quad k_y(t) = k_y + \frac{\Phi(t) e}{L_y \hbar c} = k_y + \frac{2\pi}{L_y} \frac{\Phi(t)}{\Phi_0}$$

i.e. if we insert one flux quantum at lattice momenta increase by one unit



Flux insertion and boundary twist along y

The gauge potential (445) corresponds to a Peierls phase along the y -direction



$$(447) \quad C_{j+n}^\dagger C_j \rightarrow C_{j+n}^\dagger C_j e^{i\alpha}$$

$$\alpha = \frac{2\pi}{L} \frac{\Phi}{\Phi_0}$$

Hamiltonian with periodic boundary conditions

$$(448) \quad H_g = -t \sum_{j=0}^{L-1} C_{j+1}^\dagger C_j e^{i \frac{2\pi}{L} \frac{\Phi}{\Phi_0}} + \text{h.c.}$$

$$(449) \quad C_L = C_0 \quad \text{PBC}$$

With a unitary transformation

$$(450) \quad \tilde{C}_j = C_j e^{i \alpha_j}$$

the Peierls phase in (448) can be removed

$$(451) \quad \tilde{H}_g = -t \sum_{j=0}^{L-1} \tilde{C}_{j+1}^\dagger \tilde{C}_j$$

but the boundary conditions become twisted

$$(452) \quad \boxed{\tilde{C}_L = \tilde{C}_0 e^{i 2\pi \frac{\Phi}{\Phi_0}}} \quad \text{TBC}$$

So Laughlin's flux insertion is equivalent to adiabatically changing the twist angle in the twisted BC from 0 to 2π !

A twist of boundary conditions can be transferred to interacting system where the lattice momentum k_y is no good quantum number.

(B) The Niu-Thouless Wu invariant

consider interacting N -particle Hamiltonian

$$(453) \quad H(t) = \underbrace{\sum_{j=1}^N \left(-\frac{1}{2} \frac{\partial^2}{\partial x_j^2} + V(x_j, t) \right)}_{\text{free part}} + \underbrace{\sum_{i>j}^N U(x_i - x_j)}_{\text{interactions}}$$

where $V(x_j, t)$ is a time periodic potential.

Twisted boundary conditions for the N -particle wavefunction would read

$$(454) \quad \psi(x_1, x_2, \dots, x_j + L, \dots, x_N) = e^{i\alpha L} \psi(x_1, \dots, x_j, \dots, x_N) \quad \forall j$$

- instantaneous eigenstate of $H(t)$ with TBC

$$|E_n(\alpha; t)\rangle$$

- assume that there is finite many body gap in the period of $V(t)$ in $t=0 \dots T$

$\Rightarrow$ many body Thouless pump

Density matrix of many-body system

$$(455) \quad \rho(t) = \rho_{\text{stat}}(t) + \delta \rho(t)$$

$$(456) \quad S_{ad}(t) = |E_0(\alpha; t)\rangle \langle E_0(\alpha; t)|$$

start in ground state at $t=0$ $|E_0(\alpha, t=0)\rangle$

$$i\dot{S} = [H(H), S(t)] = [H(t), \delta S(t)]$$

Now $n \neq 0$

$$\langle E_0 | [H, \delta S] | E_n \rangle = (E_0 - E_n) (\delta S)_{0n}$$

$$(\delta S)_{0n} = \frac{\langle E_0 | [H, \delta S] | E_n \rangle}{E_0 - E_n} = i \frac{\langle E_0 | \dot{S} | E_n \rangle}{E_0 - E_n}$$

$$\approx \frac{i \langle E_0 | \dot{S}_{ad} | E_n \rangle}{E_0 - E_n} = i \frac{\langle \partial_t E_0 | E_n \rangle}{E_0 - E_n}$$

Transported charge ($\hat{p}$ - particle momentum)

$$\Delta n = \frac{1}{L} \int_0^T dt \langle \hat{p} \rangle = \frac{1}{L} \int_0^T dt \text{Tr} \{ S(t) \hat{p} \}$$

(457)

$$= \frac{1}{L} \int_0^T dt (1 + \delta S_{00}(t)) \underbrace{\langle E_0 | \hat{p} | E_0 \rangle}_{=0}$$

$$+ \frac{1}{2L} \int_0^T dt \sum_{n \neq 0} \left(\delta S_{0n} \langle E_n | \hat{p} | E_0 \rangle + \text{c.c.} \right)$$

+ other contributions
negligible

In the instantaneous eigenstate $\hat{p}$ vanishes, so

$$(458) \Delta n = \frac{1}{2L} \int_0^T dt \sum_{n \neq 0} \left(i \frac{\langle \partial_t E_0 | E_n \rangle}{E_0 - E_n} p_{n0} + \text{c.c.} \right)_\alpha$$

The index " α " denotes the chosen boundary conditions.

For the following discussion we remove the TBC by a unitary trafo

$$(459) \quad H(t) \rightarrow H(t, \alpha) = T(\alpha) H(t) T^{-1}(\alpha) \\ |E_n\rangle \rightarrow |\hat{E}_n(t; \alpha)\rangle = T(\alpha) |E_n(t; \alpha)\rangle$$

For the new real-space wavefunctions ϕ we find

$$(460) \quad \phi(x_1, \dots, x_N) = e^{-i\alpha(x_1 + x_2 + \dots + x_N)} \psi(x_1, \dots, x_N)$$

which now fulfill periodic BC

$$(461) \quad \phi(x_1, \dots, x_j + L, \dots, x_N) = \phi(x_1, \dots, x_j, \dots, x_N)$$

The momentum operator gets a gauge term

$$\begin{aligned} \hat{\tilde{p}} &= T(\alpha) p T(\alpha)^{-1} = T(\alpha) \sum_{j=1}^N \left(-i \frac{\partial}{\partial x_j} \right) T(\alpha)^{-1} \\ &= \sum_{j=1}^N -i \frac{\partial}{\partial x_j} + \alpha \end{aligned}$$

so

$$(462) \quad \hat{\tilde{p}} = \frac{\partial H(\alpha)}{\partial \alpha}$$

Then

$$(463) \quad \langle \tilde{E}_n | \hat{p} | \tilde{E}_0 \rangle = \langle \tilde{E}_n | \frac{\partial H(t, \alpha)}{\partial \alpha} | \tilde{E}_0 \rangle$$

Using

$$\frac{\partial}{\partial \alpha} | \hat{E}_0 \rangle = E_0 | \tilde{E}_0 \rangle$$

← eigenvalues are invariant!

$$\left(\frac{\partial}{\partial \alpha} H(\alpha) \right) | \hat{E}_0 \rangle + H(\alpha) \partial_\alpha | \hat{E}_0 \rangle = E_0 \partial_\alpha | \hat{E}_0 \rangle$$

$$(464) \Rightarrow \langle \tilde{E}_n | \frac{\partial H(\alpha)}{\partial \alpha} | \hat{E}_0 \rangle = (E_0 - E_n) \langle \tilde{E}_n | \partial_\alpha \hat{E}_0 \rangle$$

Thus

$$\Delta n(\alpha) = \frac{1}{2L} \int_0^T dt \sum_{n \neq 0} i \left(\underbrace{\langle \partial_t \hat{E}_0 | \tilde{E}_n \rangle \langle \tilde{E}_n | \partial_\alpha \hat{E}_0 \rangle}_{=1} + \text{c.c.} \right)$$

so

$$(465) \quad \Delta n(\alpha) = \frac{1}{2L} \int_0^T dt i \left(\langle \partial_t \hat{E}_0 | \partial_\alpha \hat{E}_0 \rangle - \langle \partial_\alpha \hat{E}_0 | \partial_t \hat{E}_0 \rangle \right)$$

Now in a large system it is reasonable to assume that Δn should not depend of α in the

small interval

$$(466) \quad -\frac{\pi}{L} \leq \alpha \leq \frac{\pi}{L}$$

so we can average eq. (465) over this interval

$$(467) \quad \overline{\Delta n} = \frac{i}{4\pi L} \int_{-\pi/L}^{\pi/L} d\alpha \int_0^T dt \left[\left\langle \frac{\partial \hat{E}_0}{\partial t} \middle| \frac{\partial \hat{E}_0}{\partial \alpha} \right\rangle - \left\langle \frac{\partial \hat{E}_0}{\partial \alpha} \middle| \frac{\partial \hat{E}_0}{\partial t} \right\rangle \right]$$

Rescaling $\beta = \alpha L$ finally leads to an expression which looks similar to the non-interacting invariant

$$\overline{\Delta n} = \frac{1}{4\pi} \int_{-\pi}^{\pi} d\beta \int_0^T dt \, i \left[\left\langle \frac{\partial \hat{E}_0}{\partial t} \middle| \frac{\partial \hat{E}_0}{\partial \beta} \right\rangle - \left\langle \frac{\partial \hat{E}_0}{\partial \beta} \middle| \frac{\partial \hat{E}_0}{\partial t} \right\rangle \right]$$

$$(468) \quad \underbrace{\text{NTW invariant}}_{\text{Berry curvature}}$$

$$(469) \quad \overline{\Delta n} = \nu = \frac{1}{2\pi} \oint dt \, \frac{\partial \phi_{MB}(t)}{\partial t}$$

winding of a many-body Zak phase

$$(470) \quad \phi_{MB} = \text{Im} \int_{-\pi}^{\pi} d\beta \left\langle E_0 \middle| \frac{\partial E_0}{\partial \beta} \right\rangle$$

(C) Symmetry protected topology of SLBHM

We have seen that for finite U there is no longer particle-hole symmetry (charge conjugation) in SLBHM.

So what protects the topology in this 1D system?

$$H = -t_1 \sum_{j \text{ even}} a_{j+1}^\dagger a_j - t_2 \sum_{j \text{ odd}} a_{j+1}^\dagger a_j + h.c. + \frac{U}{2} \sum_j a_j^\dagger a_j^2 \quad (\Delta=0)$$

consider twisted Hamiltonian $a_j \rightarrow a_j e^{i\beta j}$

$$(471) \quad H \rightarrow \tilde{H} = -t_1 e^{i\beta} \sum_{j \text{ even}} a_{j+1}^\dagger a_j - t_2 e^{i\beta} \sum_{j \text{ odd}} a_{j+1}^\dagger a_j + \frac{U}{2} \sum_j a_j^\dagger a_j^2$$

Symmetry $[\tilde{H}, \hat{S}] = 0$

$$(472) \quad \boxed{\hat{S} = I K}$$

I inversion about $j=0$, K - complex conjugation

$$(473) \quad |E_0(\beta)\rangle = e^{i\chi_\beta} \hat{S} |E_0(\beta)\rangle$$

$$\begin{aligned} A(\beta) &= i \langle E_0 | \partial_\beta E_0 \rangle = i \langle S E_0 | \partial_\beta S E_0 \rangle - \partial_\beta \chi_\beta \\ &= i \langle K E_0 | \partial_\beta K E_0 \rangle - \partial_\beta \chi_\beta = i \langle E_0^* | \partial_\beta E_0^* \rangle - \partial_\beta \chi_\beta \end{aligned}$$

$$(474)$$

$$= i \langle \partial_\beta E_0 | E_0 \rangle - \partial_\beta \chi_\beta$$

$$\begin{aligned}
 \phi_{MB} &= \frac{1}{2i} \int_{-\pi}^{\pi} d\beta \left(\langle E_0 | \partial_\beta E_0 \rangle - \langle \partial_\beta E_0 | E_0 \rangle \right. \\
 &\quad \left. \langle \partial_\beta E_0 | E_0 \rangle + i \partial_\beta \chi_\beta \right) \\
 (475) \quad &= \frac{1}{2} \int_{-\pi}^{\pi} d\beta \partial_\beta \chi_\beta = \frac{1}{2} (\chi_\pi - \chi_{-\pi})
 \end{aligned}$$

$$\text{Since } \chi_\pi = \chi_{-\pi} \text{ mod } 2\pi$$

$$(476) \quad \boxed{\phi_{MB} = 0, \pi}$$

As in the fermionic SSH model the many-body Zak phase is either 0 or π

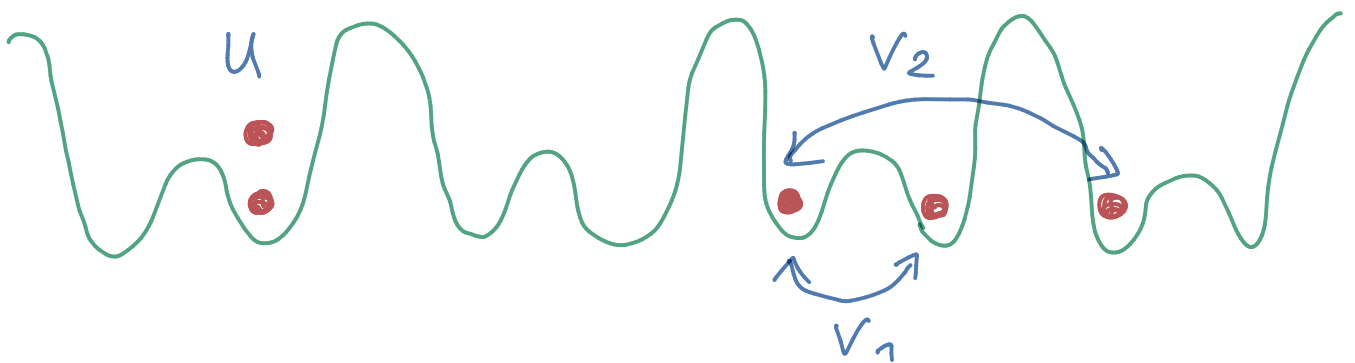
10.3 The extended SLBHM: fractional topological charges

As in the FQHE gapped phases with fractional topological charges can also appear in 1D lattice models:

For this we extend the SLBHM by non-local interactions

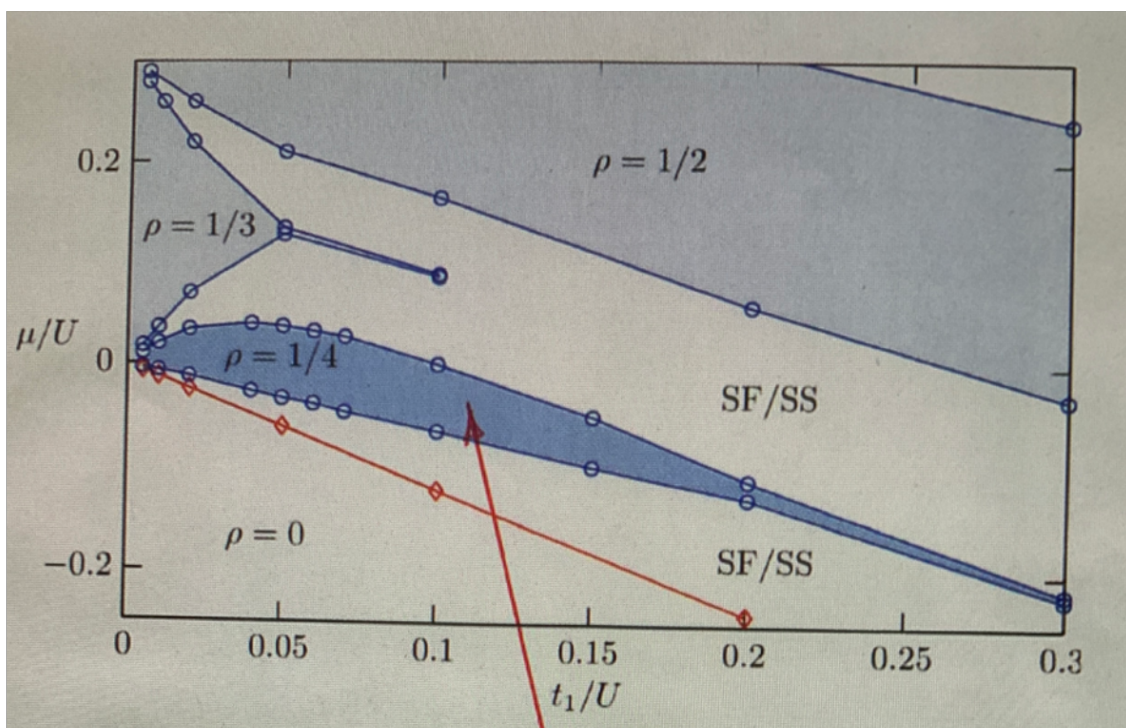
$$(477) \quad H = H_{\text{SLBHM}} + V_1 \sum_{\langle ij \rangle} \hat{n}_i \hat{n}_j + V_2 \sum_{\langle\langle ij \rangle\rangle} \hat{n}_i \hat{n}_j$$

NN interaction NNN interaction



(A) Fractional Mott insulator and fractional Thouless pump

phase diagram $\hat{H} - \mu \hat{N}$

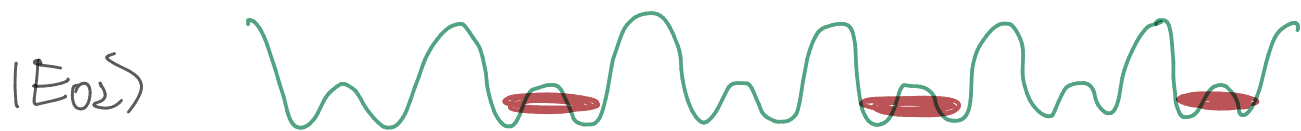
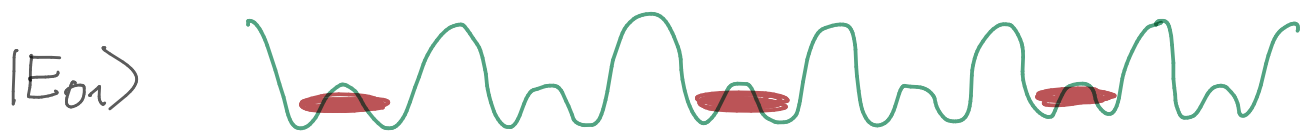


fractional MI

The MI phase at fractional filling is
degenerate

• atomic limit $t_2 \gg V_1, V_2 \gg t_1$

two density waves possible GS



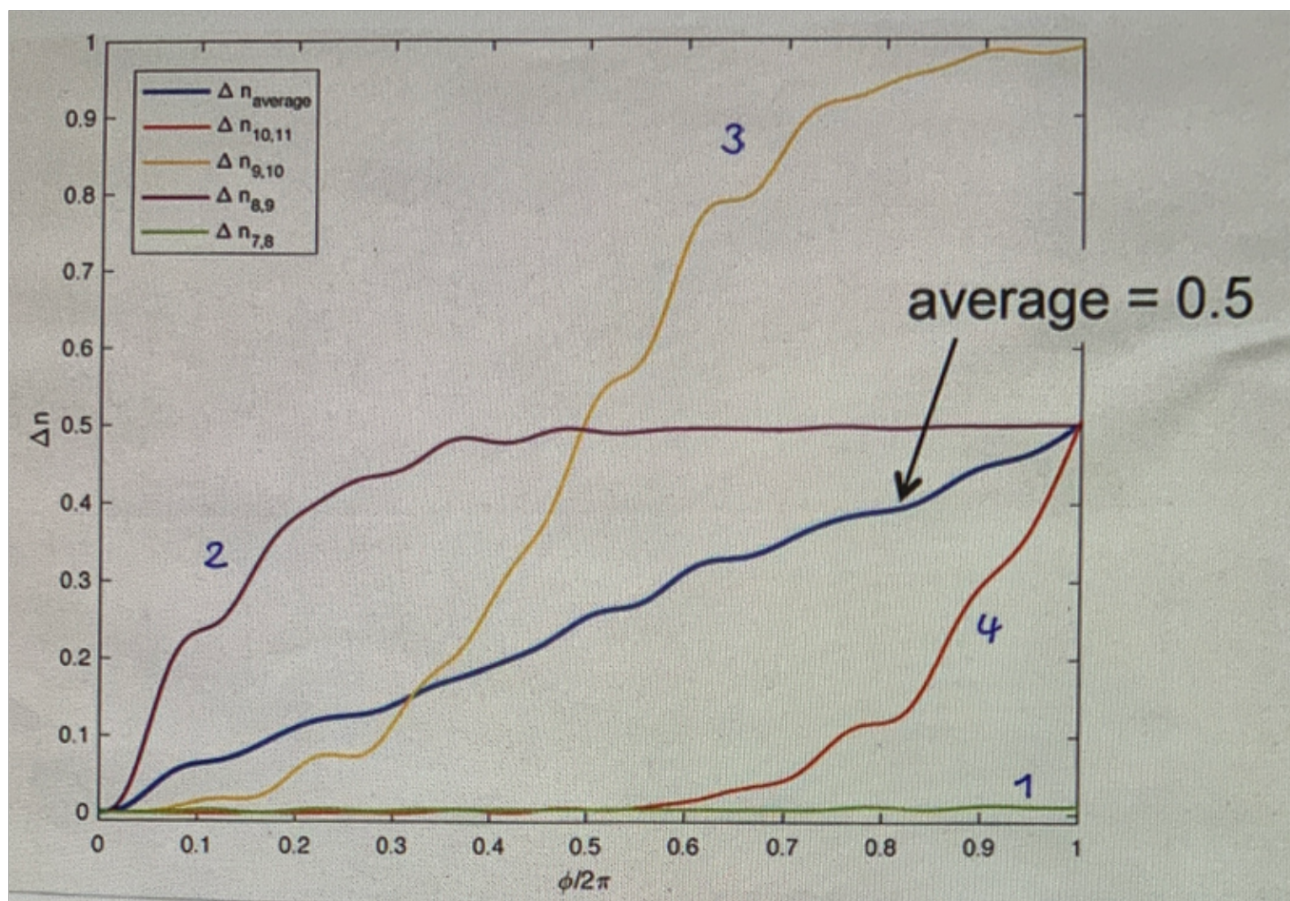
$$\langle E_{01} | E_{02} \rangle = 0$$

gapped many-body states with fractional filling
are degenerate and spontaneously break
a discrete symmetry (here lattice translation by
one unit cell) (Lieb-Schultz-Mattis)

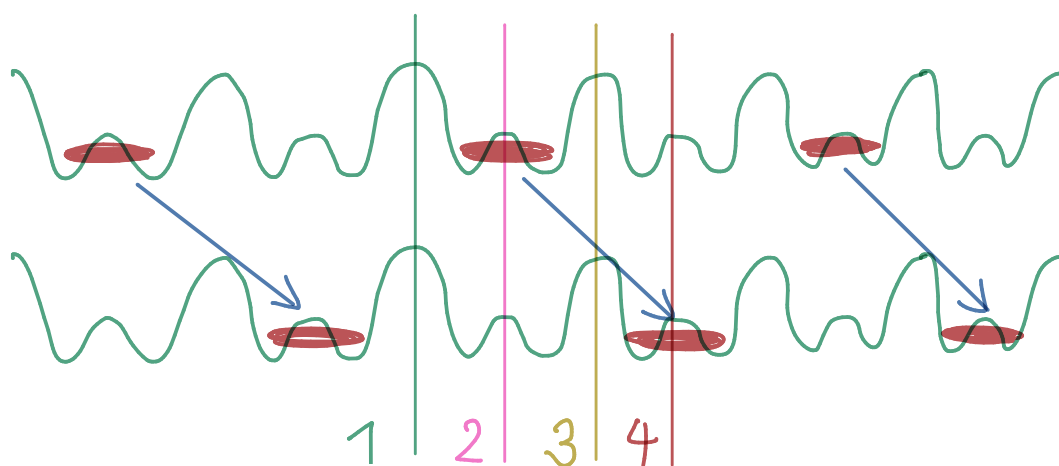
$\Rightarrow$ single cycle Thouless pump will in general not
return the bulk state into itself but

(478) $|E_{01}\rangle \longrightarrow e^{i\Delta\phi} |E_{02}\rangle$ possible

p - degree of degeneracy: p Thouless cycles
needed to return to initial state



transport in atomic limit



Niu, Thouless, Wu PRB 1985

consider p -fold loop

$\forall \mu=1,2,\dots,p$

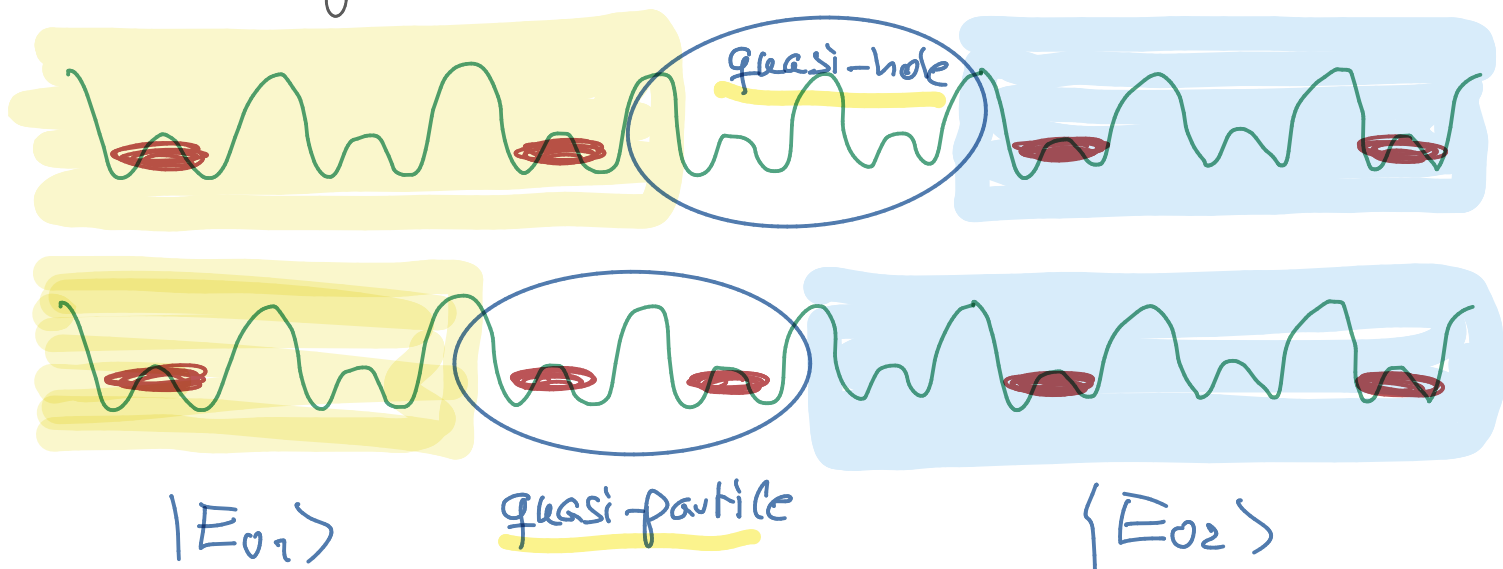
$$(479) \quad \nu = \frac{1}{2\pi} \int_0^{2\pi p} d\lambda \frac{\partial}{\partial \lambda} \int_{-\pi}^{\pi} d\beta \operatorname{Im} \langle E_{q\mu} | \frac{\partial}{\partial \beta} E_{q\mu} \rangle$$

fractional transport (in one cycle)

$$(480) \quad \Delta n = \frac{1}{p} \nu$$

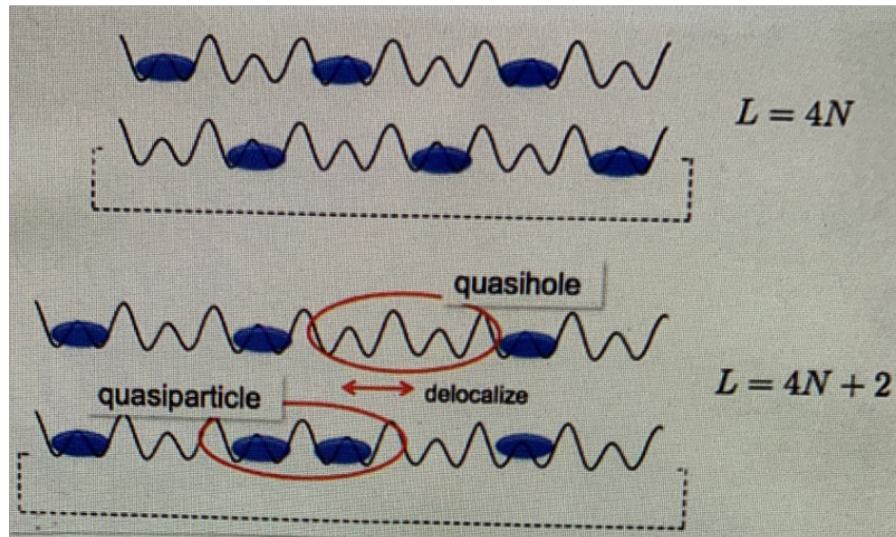
(B) topological excitations: domain walls

If there are degenerate ground states then low energy topological excitations exist at a spatial boundary between two different ground states

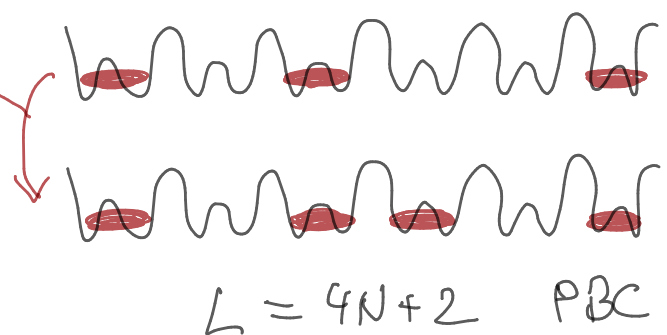
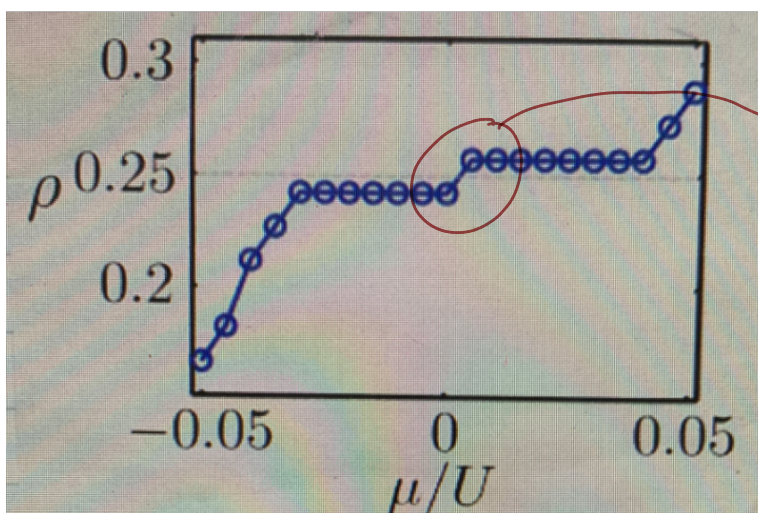


a quasi particle has an excitation energy $\sim \alpha V_1 + \beta V_2$

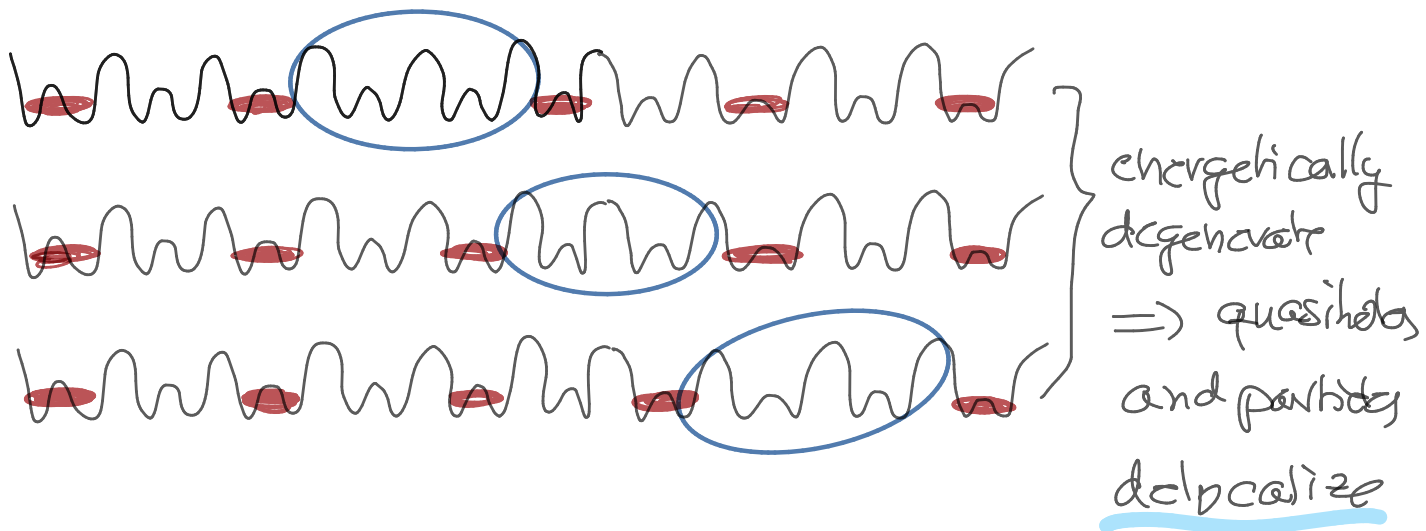
For a system with periodic boundary conditions the PBC may not be commensurate with the fractional filling and may enforce domain wall excitations



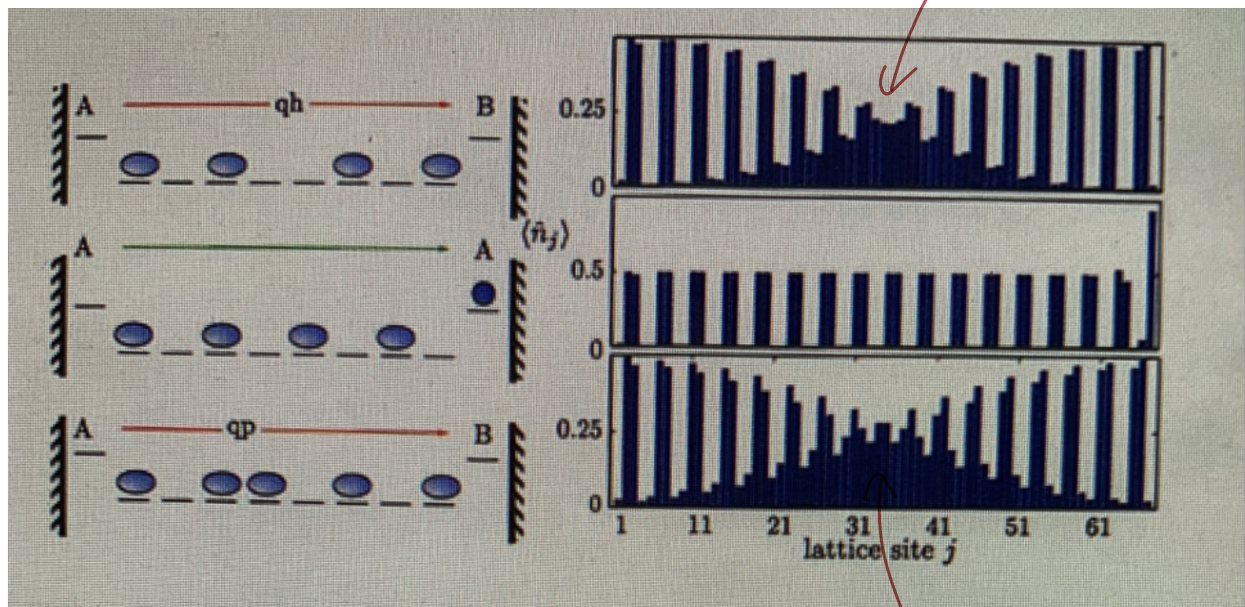
phase diagram in incommensurate case



domain walls are a 1D analogue of gh and qp in the FQHE



with open boundaries (DNRG)



10.4 Topological invariants for (quasi) degenerate systems

In the extended SCBHM we have seen that interaction-induced (topological) ground states