Topological insulators and the Quantum Hall effect

WS 2024/25 4SWS

Di 17:15 – 18:45 46/576
Fr 08:15 – 09:45 46/576

Literature

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0. Introduction

0.1 Phases of Matter and spontaneous symmetry breaking

Most phases of matter consisting of the same constituents can be distinguished by spontaneously broken symmetries

- example: Ferromagnet $\longleftrightarrow$ Paramagnet

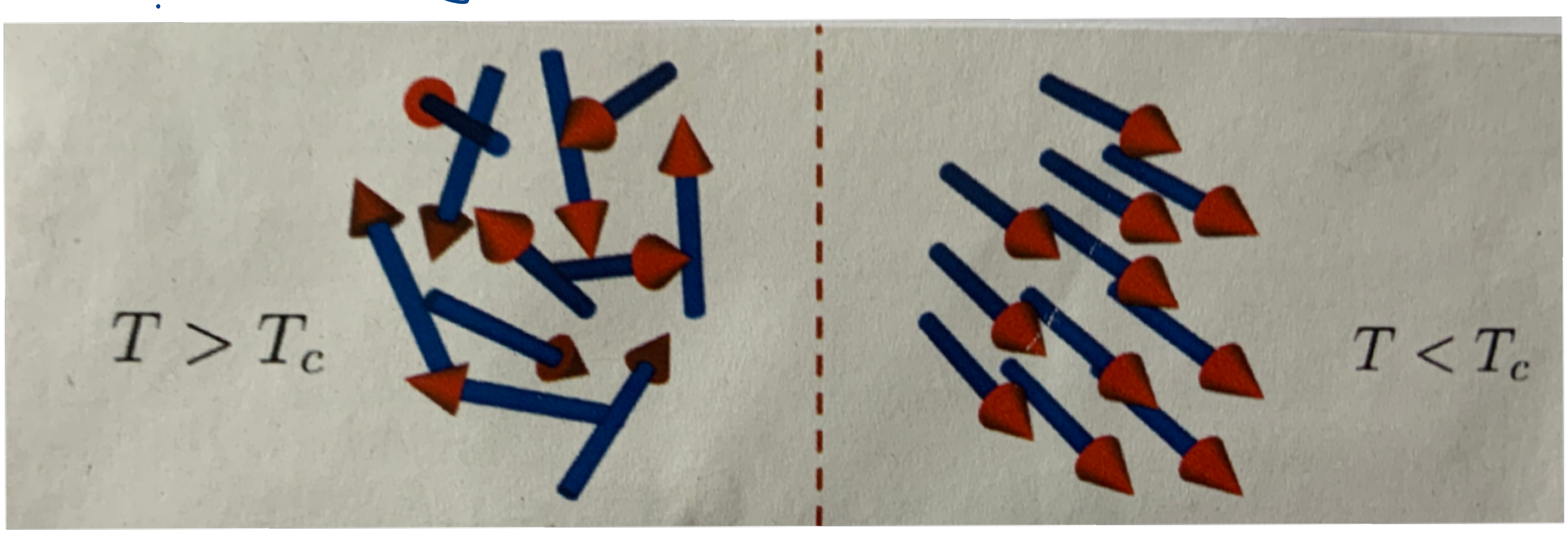
Heisenberg model $\quad H = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j \quad J > 0$

H has $O(3)$ symmetry; i.e.

$$D^{-1} H D = H \qquad D = e^{i\theta \vec{n} \cdot \vec{S}_{tot}}$$

still H describes two different phases

paramagnet $\qquad$ ferromagnet



$O(3)$ symmetric $\qquad$ spontaneously broken $O(3)$ symmetry

$$[\hat{\vec{S}}_{tot}, H] = 0$$

but

$$\langle \psi_0 | \vec{S}_{tot} | \psi_0 \rangle = 0 \qquad \langle \psi_0 | \vec{S}_{tot} | \psi_0 \rangle \neq 0$$

$$T > T_c \qquad\qquad T < T_c$$

Phase transitions of this type can be described by a Ginzburg-Landau theory:

- free energy $\quad F(\vec{M}) = U - TS$

$$F(\vec{M}) = F_0 + \alpha_1(T)\vec{M}^2 + \alpha_2(T)\vec{M}^4$$

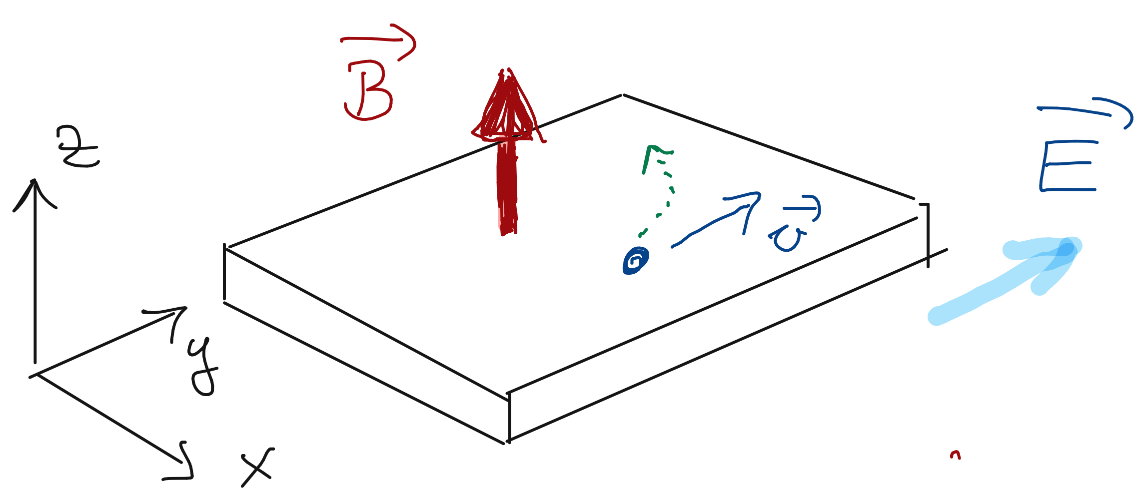
$$\alpha_1 > 0 \qquad \alpha_2 = \alpha(T - T_c)$$

$$F(\vec{M}) \overset{!}{=} \min \implies \begin{array}{ll} T > T_c & \vec{M} = 0 \\ T < T_c & |\vec{M}| = \sqrt{\frac{-\alpha_1}{\alpha_2}} \neq 0 \end{array}$$

0.2 Beyond symmetry breaking: Quantum Hall effect

(A) Classical Hall effect of 2D electron gas

free electrons in 2D metal / SC



Lorentz force $\vec{F} = -e\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right) = m\dot{\vec{v}}$

stationarity condition $\dot{\vec{v}} = 0$

$$\vec{B} = (0, 0, B) \qquad \vec{E} = (E_x, E_y, 0)$$

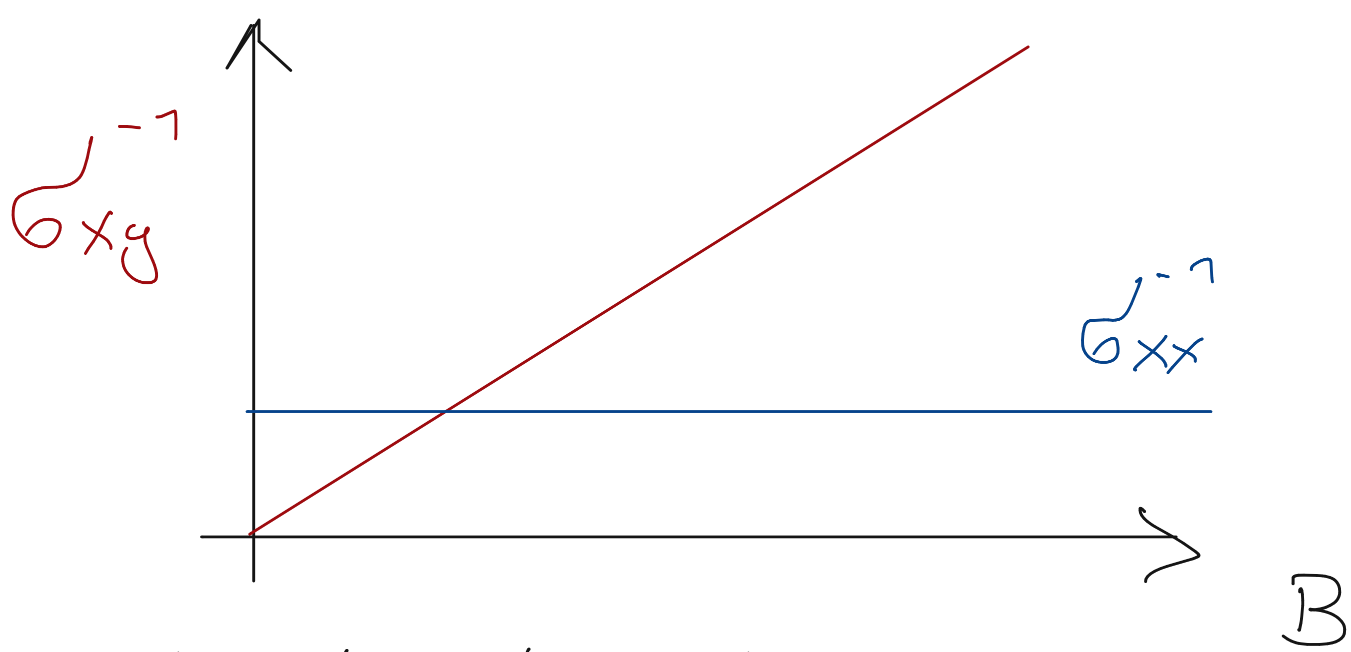
$$\vec{E} = -\frac{\vec{v}}{c} \times \vec{B}$$

current $\vec{j} = -e\rho\vec{v}$

$$j_x = -\frac{e\rho c}{B} E_y \qquad j_y = \frac{e\rho c}{B} E_x$$

Hall resistivity $\vec{j} = \underline{\underline{\sigma}}\,\vec{E}$

$$\sigma_{xy}^{-1} = \frac{E_x}{j_y} = \frac{B}{e\rho c}$$



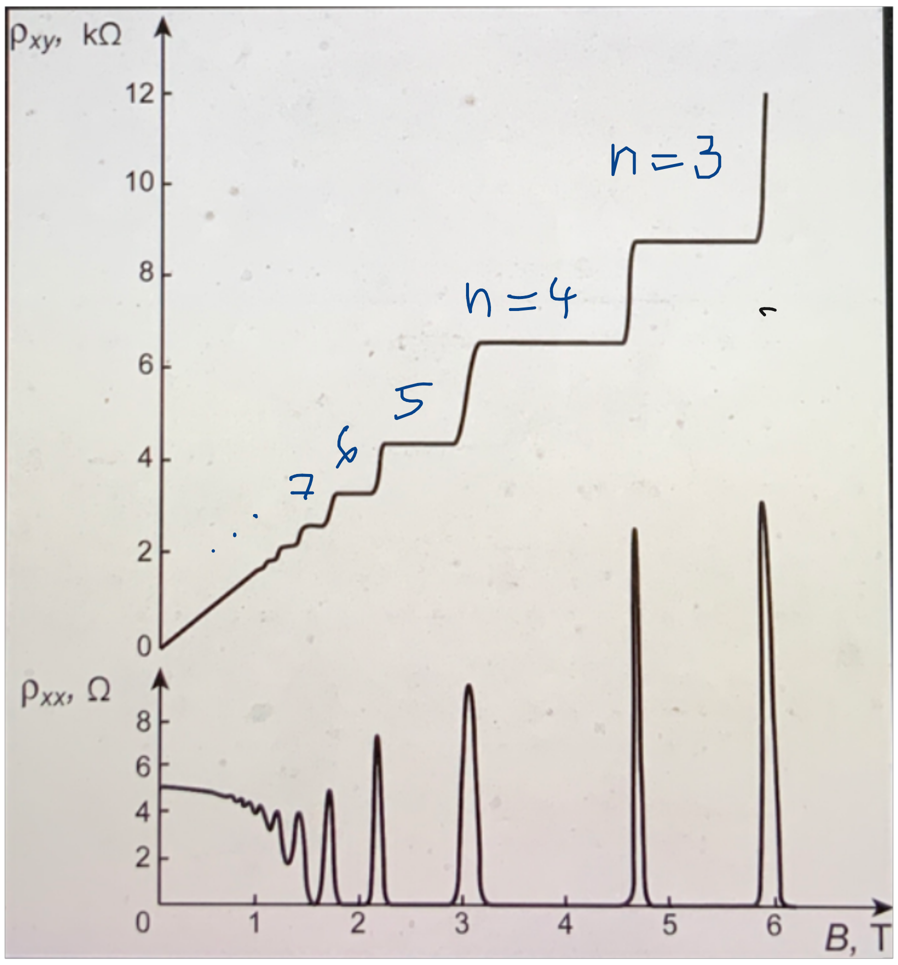
after adding a damping term $-m\frac{\vec{v}}{\tau}$

consistent with observations in classical regime, i.e. low B high temperature

$\rightarrow$ quantum regime?

(B) the discoveries of von Klitzing (1980) and Tsui, Störmer and Gossard (1982)

- Klaus v. Klitzing: resistivity in high magnetic field at low T



Wikipedia

integer quantum Hall effect

Nobel prize 1985

- plateaus at

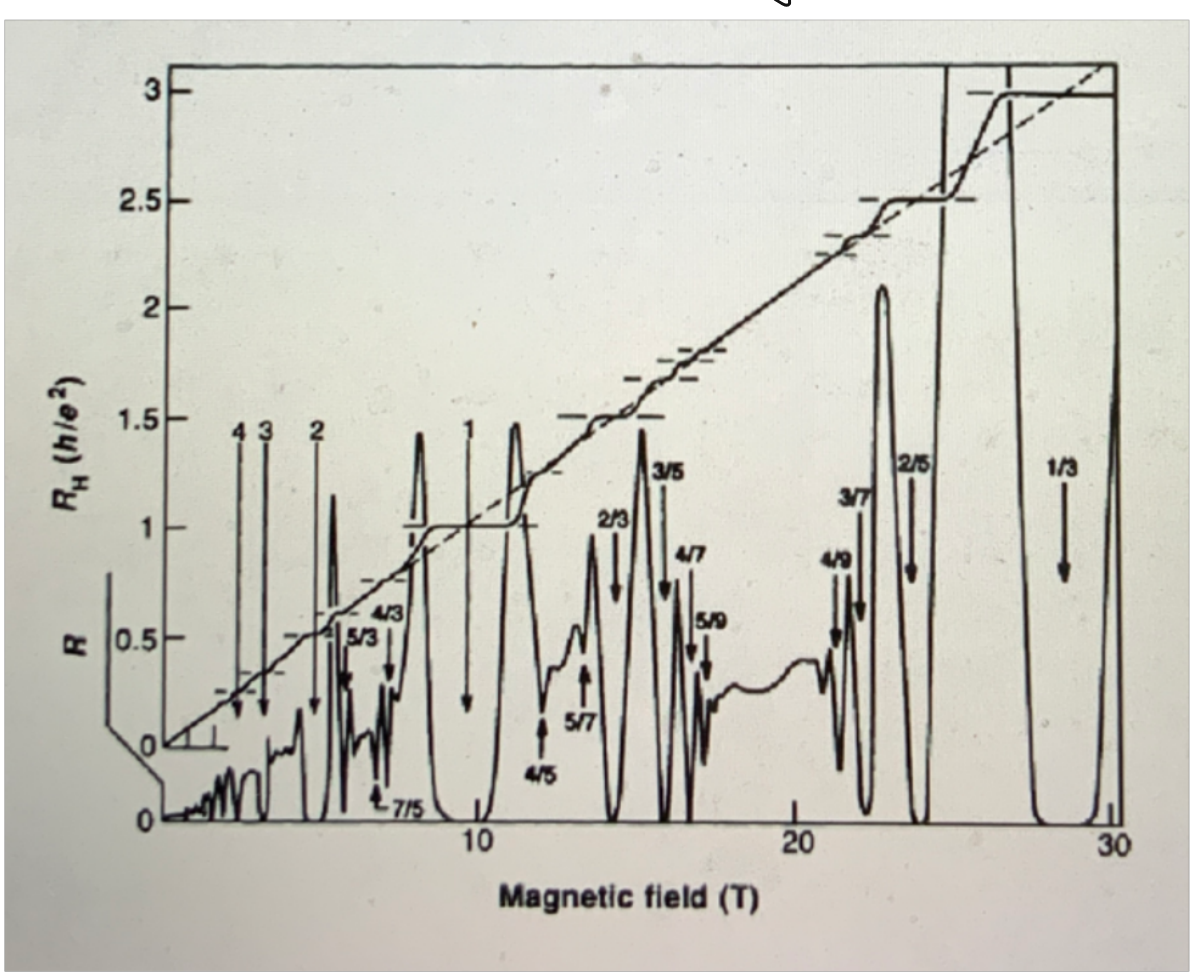
$$\boxed{\rho_{xy} = \frac{1}{n}\frac{h}{e^2}} \qquad n \in \mathbb{N}$$

$$h/e^2 = 25813.807\ \Omega$$

- and

$$\boxed{\rho_{xx} \longrightarrow 0}$$

- H. L. Störmer, D. C. Tsui, A. G. Gossard experiment
  R. Laughlin theory



Nobel prize
Störmer, Tsui, Laughlin 1998

- plateaus at fractional numbers

$$\boxed{\rho_{xy} = \frac{1}{f}\frac{h}{e^2} \qquad f = \frac{p}{q}}$$

fractional quantum Hall effect

phenomena robust against disorder
$\longrightarrow$ stable phases

<u>problem:</u> phase transitions but no spontaneously broken symmetries!?

$\Longrightarrow$ $\boxed{\text{beyond Landau Ginzburg}}$

- another system with different phases w/o spontaneously broken symmetries $\Longrightarrow$ spin liquids

But phases can be characterized by <u>discrete quantum numbers</u>; we will see that these are

$\boxed{\text{topological quantum numbers}}$

How does topology come into the game?
$\longrightarrow$ quantum states have a phase!

0.3 Berry phase

- Consider a Hamiltonian that depends smoothly on a parameter $\vec{\lambda}$

(1) $$H = H(\vec{\lambda})$$

define instantaneous eigenvalues / eigenstates

(2) $$H(\vec{\lambda})\,|n(\vec{\lambda})\rangle = E_n(\vec{\lambda})\,|n(\vec{\lambda})\rangle$$

Adiabatic theorem

> If $\vec{\lambda} = \vec{\lambda}(t)$ changes sufficiently slowly in time, a system prepared at time $t=0$ in an eigenstate $|n(t=0)\rangle$ will remain in that eigenstate at later times t

proof: (3) $$|\psi(t)\rangle = \sum_n c_n e^{i\alpha_n(t)} |n(t)\rangle \qquad \alpha_n = -\frac{1}{\hbar}\int_0^t d\tau\, E_n(\tau)$$

from SE $\frac{d}{dt}|\psi(t)\rangle = -\frac{i}{\hbar} H(t)|\psi(t)\rangle$ follows

(4) $$\sum_n \left(\dot{c}_n e^{i\alpha_n(t)} |n(t)\rangle + c_n e^{i\alpha_n(t)} |\dot{n}(t)\rangle \right) = 0$$

$$\Rightarrow \dot{c}_m(t) = -\sum_n c_n(t)\, e^{i(\alpha_n(t) - \alpha_m(t))} \langle m|\dot{n}\rangle$$

need $\langle m|\dot{n}\rangle = ?$ $\qquad \langle m|n\rangle = \delta_{mn}$

$\frac{d}{dt}\Big|$ $$H(t)|n(t)\rangle = E_n(t)|n(t)\rangle$$

$\langle m|\Big|$ $$\dot{H}|n\rangle + H|\dot{n}\rangle = \dot{E}_n|n\rangle + E_n|\dot{n}\rangle$$

$|m\rangle \neq |n\rangle$ $$\langle m|\dot{H}|n\rangle + E_m\langle m|\dot{n}\rangle = E_n\langle m|\dot{n}\rangle$$

(5) $$\Rightarrow \langle m|\dot{n}\rangle = \frac{\langle m|\dot{H}|n\rangle}{E_n - E_m}$$

i.e. $$\dot{c}_m = -\sum_{n\neq m} c_n(t)\, e^{i(\alpha_n(t)-\alpha_m(t))} \frac{\langle m|\dot{H}|n\rangle}{E_n - E_m}$$

now at $t=0$ system in state $|n(t=0)\rangle$, i.e.

$$c_n(0) = 1 \qquad c_{m\neq n}(0) \equiv 0$$

then

$$c_m(\delta t) = -i\hbar \frac{\langle m|\dot{H}|n\rangle}{(E_m-E_n)^2}\left(e^{i(E_m-E_n)\frac{\delta t}{\hbar}} - 1\right)$$

so $|c_m(\delta t)| \ll 1$ if

$$\boxed{\langle m|\dot{H}|n\rangle \ll \frac{|E_m-E_n|^2}{\hbar}} \quad m \neq n \tag{6}$$

adiabatic condition

The characteristic rate of change of the Hamiltonian must be small compared to energy separation from other states

$\Longrightarrow$ system with energy gap

Knowing that the system remains in the eigenstate is not sufficient since there is a phase ambiguity!

adiabatic limit $\quad c_n(0)=1 \Rightarrow |c_n(t)|=1$

Ansatz $\quad c_n(t) = e^{i\gamma(t)}$

$$\dot{c}_n = i\dot{\gamma}\, c_n \underset{\uparrow \text{ eq. (4)}}{=} -c_n \langle n|\dot{n}\rangle$$

$$\boxed{\gamma(t) = i\int_0^t d\tau\, \langle n|\frac{d}{d\tau}|n\rangle} \tag{7}$$

Berry phase

now

$$\dot{\gamma}(t) = i \langle n | \frac{\partial}{\partial \vec{\lambda}} | n(\vec{\lambda}) \rangle \, \dot{\vec{\lambda}}$$

(8) $$\gamma(t) = i \int d\vec{\lambda} \, \langle n(\vec{\lambda}) | \frac{\partial}{\partial \vec{\lambda}} | n(\vec{\lambda}) \rangle$$

Berry phase is not unique. It depends on gauge choice

$$|n(\vec{\lambda})\rangle \longrightarrow e^{i\phi(\vec{\lambda})} |n(\vec{\lambda})\rangle = |\tilde{n}(\vec{\lambda})\rangle$$

(9) $$\gamma' = \gamma - \int d\vec{\lambda} \cdot \vec{\nabla}_{\lambda} \phi$$

However if we choose a closed path i.e. $\vec{\lambda}(0) = \vec{\lambda}(T)$

(10) $$\boxed{\gamma = i \oint_C d\vec{\lambda} \, \langle n | \vec{\nabla}_{\lambda} | n \rangle}$$

is invariant under gauge transformations !

if a smooth gauge exists!

$$\boxed{\partial_j \equiv \frac{\partial}{\partial \lambda_j}}$$

Berry connection

(11) $$\boxed{A_j = i \langle n | \partial_j n \rangle = A_j^*}$$

(like a gauge potential in Edyn)

note $\partial_j \langle n | n \rangle = 0 \Rightarrow \langle \partial_j n | n \rangle + \langle n | \partial_j n \rangle = 0$

depends on gauge

(12) $$\vec{A}' = \vec{A} - \vec{\nabla}_{\lambda} \phi$$

Berry curvature

(12) $$\boxed{\begin{aligned}\Omega_{jk} &= i\left(\partial_j A_k - \partial_k A_j\right)\\ &= i\left(\langle\partial_j n|\partial_k n\rangle - \langle\partial_k n|\partial_j n\rangle\right)\end{aligned}}$$

(like field tensor in Edyn)

(13) $$\boxed{\Omega_i = \varepsilon_{ijk}\Omega_{jk} = i\,\varepsilon_{ijk}\left(\langle\partial_j n|\partial_k n\rangle - \langle\partial_k n|\partial_j n\rangle\right)}$$

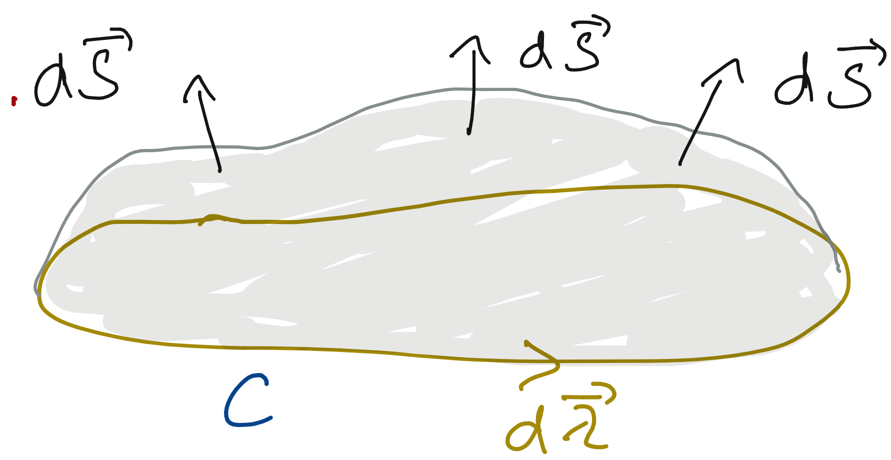
(like magnetic field in Edyn)

In a 3-dimensional parameter space $\vec{\lambda} \in \mathbb{R}^3$

(14) $$\vec{\Omega} = \vec{\nabla}_\lambda \times \vec{A}$$

and using Stokes theorem

(15) $$\boxed{\gamma = \int_C d\vec{\lambda}\cdot\vec{A} = \iint_{S(C)} d\vec{S}\cdot\vec{\Omega}}$$



note: S is not a closed surface!

There is an important general relation for $\lambda \in \mathbb{R}^3$

$$\vec{B} = \vec{B}_n = \mathrm{Im}\left(\vec{\nabla} \times \langle n|\vec{\nabla} n\rangle\right) = \dots$$

$$= \mathrm{Im} \sum_{m \neq n} \langle \vec{\nabla} n | m\rangle \times \langle m | \vec{\nabla} n\rangle$$

$$= -\mathrm{Im} \sum_{m \neq n} \frac{\langle n|\vec{\nabla} H|m\rangle \times \langle m|\vec{\nabla} H|n\rangle}{(E_m - E_n)^2}$$

$$\Rightarrow \quad \boxed{\sum_n \gamma_n = 0} \qquad (14)$$

sum of Berry phases over all eigenstates vanishes

- Generalization to degenerate eigenstates ?

So far we have assumed non-degenerate eigenstates. What happens if there are degeneracies (even only for few values of λ) ?

d-fold degeneracy $\{|n_1\rangle, \dots, |n_d\rangle\}$

Due to degeneracy (at some values of λ) no adiabatic following possible and transition between states possible $\Rightarrow$ consider whole subspace

$$(15)\quad \begin{bmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ c_d(t) \end{bmatrix} = \underline{\underline{U}}(t) \begin{bmatrix} c_1(0) \\ c_2(0) \\ \vdots \\ c_d(0) \end{bmatrix} = T e^{i\int_0^t d\tau\, \underline{\underline{A}}(\tau)} \begin{bmatrix} c_1(0) \\ c_2(0) \\ \vdots \\ c_d(0) \end{bmatrix}$$

i.e.

$$A_j \longrightarrow \underline{\underline{A}}_j$$ matrix in d-dimensional space

$$(16)\quad \boxed{\begin{gathered} \underline{\underline{A}}_j\Big|_{ke} = i\langle n_k | \partial_j n_e \rangle \\ \underline{\underline{A}}_j = \underline{\underline{A}}_j^{+} \end{gathered}}$$

matrix-valued Berry connection

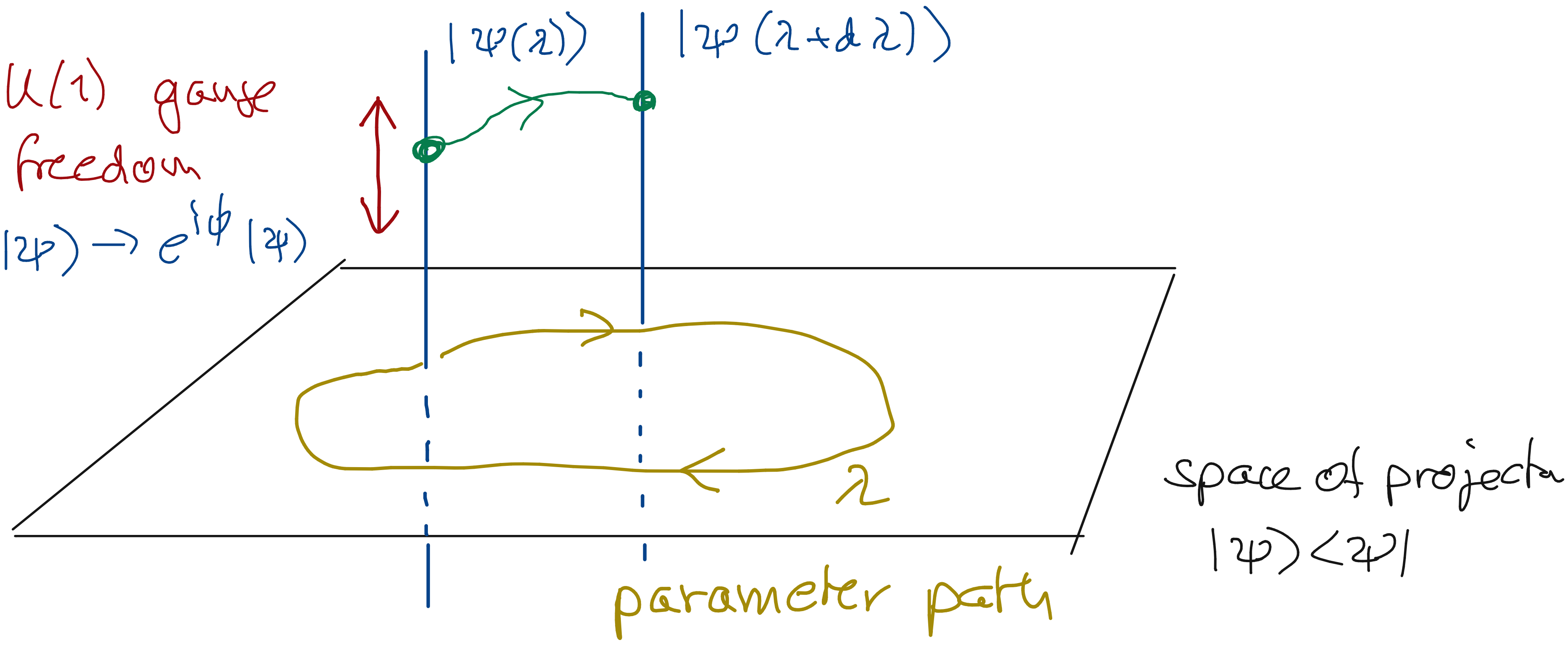
may be non-Abelian if e.g.

non-Abelian Berry connection

$$(17)\quad \left[\underline{\underline{A}}(\vec{\lambda}_1), \underline{\underline{A}}(\vec{\lambda}_2)\right] \neq 0$$

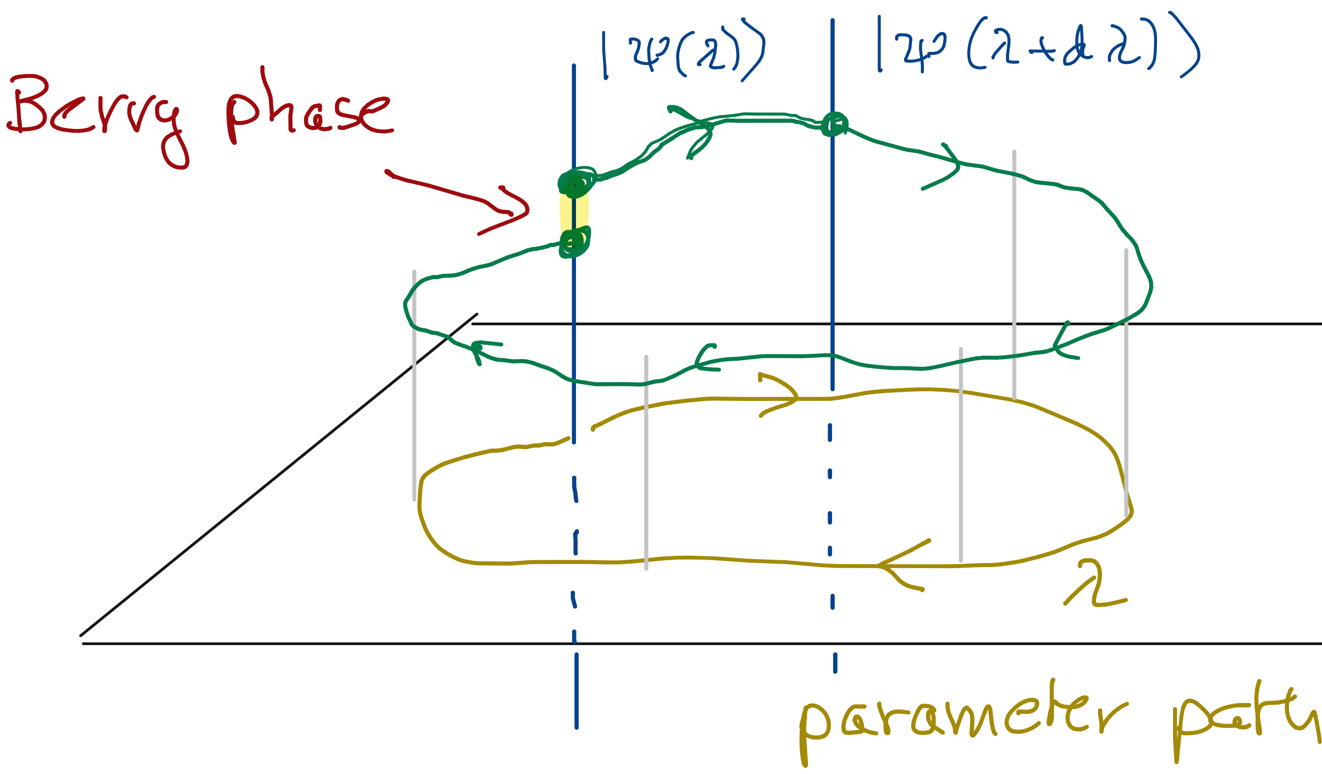
For (partially) degenerate states we have to consider (non-Abelian) Berry connections.

Berry phase and parallel transport



parallel transport of state $|\psi(\lambda)\rangle \to |\psi(\lambda+d\lambda)\rangle$

$$\left\| \, |\psi(\lambda)\rangle - |\psi(\lambda+d\lambda)\rangle \, \right\| \overset{!}{=} \min \tag{18}$$

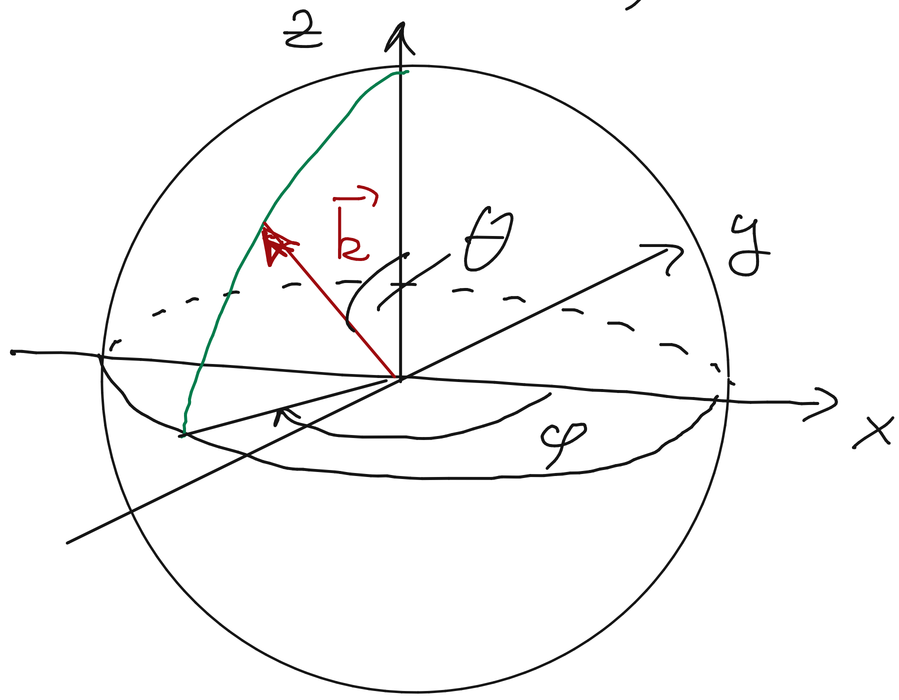


0.4 Absence of smooth gauge, Chern number

instructive example

$$H = \vec{h}(\vec{k}) \cdot \vec{\sigma} \qquad \vec{k} \in \mathbb{R}^3 \tag{19}$$

where $\vec{h}(\vec{k}) = \vec{e}_k = \begin{pmatrix} \sin\theta \cos\varphi \\ \sin\theta \sin\varphi \\ \cos\theta \end{pmatrix}$



eigenstates

$$E_- = -1 \qquad |E_-\rangle = \begin{pmatrix} \sin\frac{\theta}{2} e^{-i\varphi} \\ -\cos\frac{\theta}{2} \end{pmatrix}$$

(20)

$$E_+ = +1 \qquad |E_+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\varphi} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

- Berry connections in θ and φ direction

$$(21) \quad A_\theta^{(-)} = i\langle E_- | \partial_\theta E_-\rangle = 0 \qquad A_\varphi^{(-)} = i\langle E_- | \partial_\varphi E_-\rangle = \sin^2\theta/2$$

$\vec{A}$ defined everywhere except at south pole

since for $\theta = \pi$

$$|E_-\rangle = \begin{pmatrix} e^{-i\phi} \\ 0 \end{pmatrix} \qquad \phi \text{ not defined!}$$

- o.k. So lets choose a different gauge (phase of state)

$$|E'_\pm\rangle \equiv e^{i\phi}|E_\pm\rangle$$

i.e.

(22) $|E'_-\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} e^{-i\varphi} \end{pmatrix}$ $\qquad |E'_+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{-i\varphi} \end{pmatrix}$

now

(23) $A_\theta^{(-)\prime} = 0 \qquad A_\varphi^{(-)\prime} = -\cos^2\frac{\theta}{2}$

but now $\vec{A}'$ is undefined at north pole since for $\theta = 0$

$$|E'_-\rangle = \begin{pmatrix} 0 \\ -e^{i\varphi} \end{pmatrix} \qquad \phi \text{ not defined}$$

For this example we cannot find a gauge where the Berry connection is well defined in the whole parameter space!

$\Rightarrow$ $\nexists$ globally smooth gauge (i)

Now let us calculate the Berry curvature

$$\Omega_{\theta\varphi}^{(-)} = i\left(\partial_\theta A_\varphi^{(-)} - \partial_\varphi A_\theta^{(-)}\right)$$

in gauge (21) or (23) (Ω is gauge invariant)

(24) $\Omega_{\theta\varphi}^{(-)} = \Omega_{\theta\varphi}^{(-)\prime} = \frac{1}{2}\sin\theta$ $\cos\theta$

If we calculate the surface integral over the closed surface of the sphere

(25) $$\boxed{\frac{1}{2\pi}\int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \; \Omega^{(-)}_{\theta\varphi} = 1 = C}$$

we obtain an integer, called Chern number

$\Rightarrow$ The Chern number is non-zero (ii)

Summarizing (i) and (ii)

If on a closed surface one cannot define a globally smooth Berry connection the corresponding surface integral over the Berry connection, the Chern number is nonzero.

remark: The fact that the Chern number is an integer has an instructive interpretation using Gauß law ($\mathbb{R}^3$)

(26) $$\iint_{S(V)} d\vec{S} \cdot \vec{\Omega}^{(-)} = \iiint dV \operatorname{div} \vec{\Omega}^{(-)}$$

For our example $\vec{\Omega}_{\pm} = \mp \frac{1}{2} \frac{\vec{k}}{k^3}$

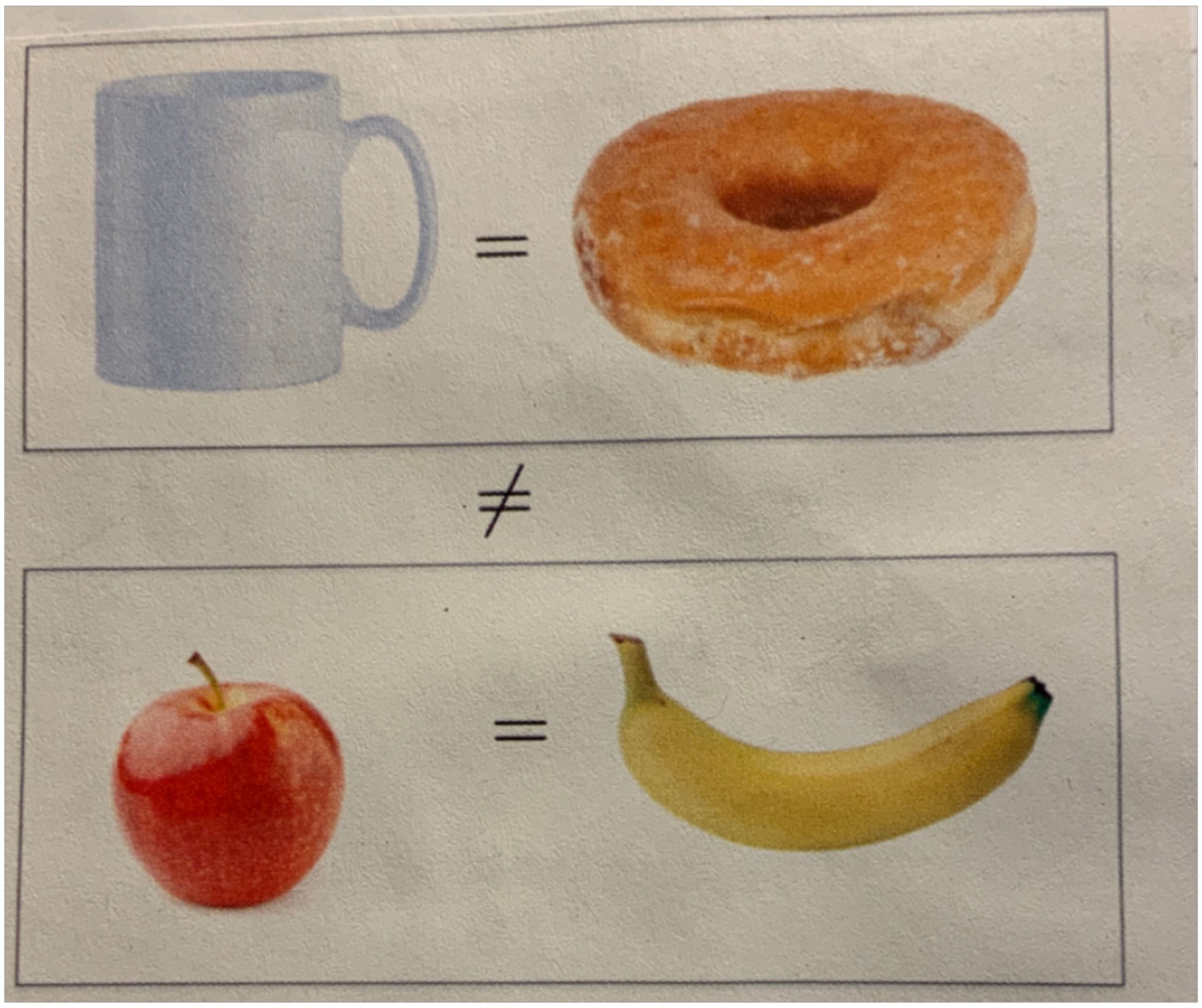
corresponds to a monopol at the origin

$$\operatorname{div} \vec{\Omega}_{\pm} = \pm 2\pi \delta^{(3)}(\vec{k})$$

$$(27) \quad C_{\pm} = \frac{1}{2\pi} \iint d\vec{S} \cdot \vec{\Omega}^{(\pm)} = \frac{1}{2\pi} \int dV \operatorname{div} \vec{\Omega}^{(\pm)} = \mp 1$$

0.5 Some elementary notions from topology

"A topologist is a mathematician which cannot distinguish a coffee cup from a donut but from a banana or an apple"



" topology is the mathematical study of the properties that are preserved through smooth deformations twisting and stretching of objects. Tearing, however is not allowed"

Eric W. Weisstein "topology"

objects that can smoothly be transformed into each other are called homotop

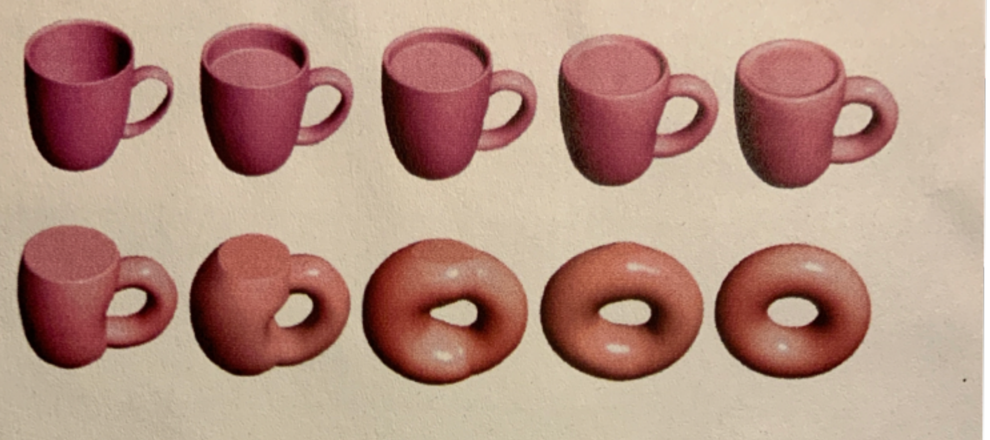
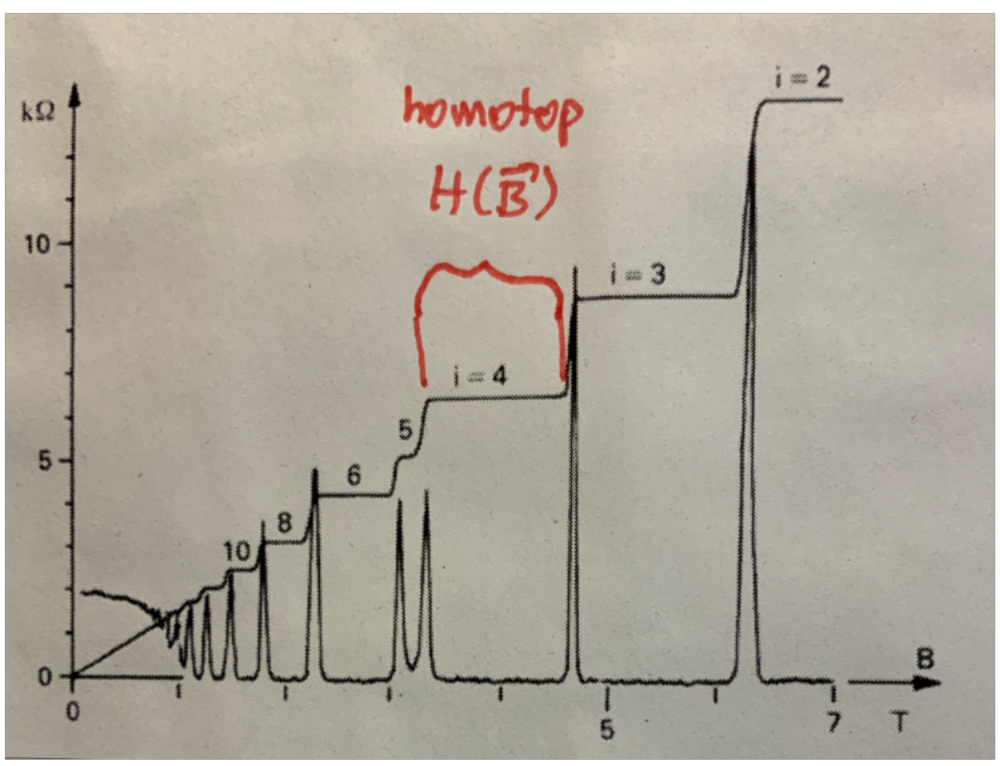
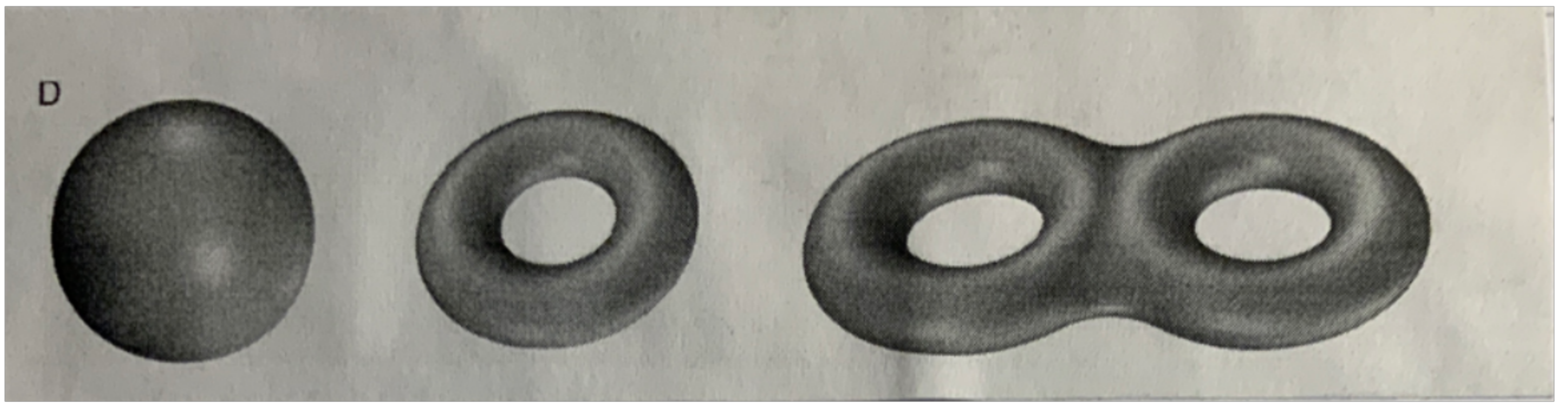


Figure: Homotopy between mug and torus.

We will see that all Hamiltonians of a quantum Hall system for magnetic fields corresponding to the same Hall plateau are homotop, i.e. belong to the same topological phase



topologically distinct surfaces in $\mathbb{R}^3$

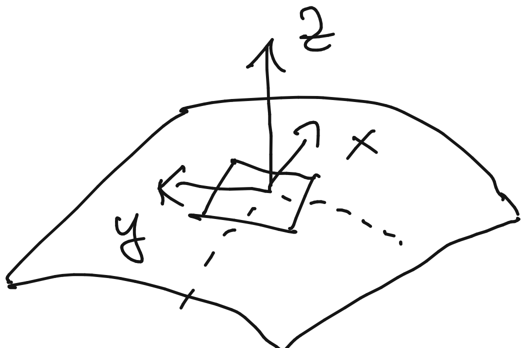


can be distinguished by integer numbers

topological invariants

<u>example of topological invariant</u>: Gauss-Bonnet theorem

2d compact surface embedded in $\mathbb{R}^3$



$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0 \quad \text{at } (0,0)$$

surface in neighborhood of $(0,0)$

$$z(x,y) \simeq \frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix}^T \underbrace{\begin{pmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial x \partial y} \\ \frac{\partial^2 z}{\partial x \partial y} & \frac{\partial^2 z}{\partial y^2} \end{pmatrix}}_{\text{Hesse matrix } H} \begin{pmatrix} x \\ y \end{pmatrix}$$

eigenvalues of H = inverse radii of curvature

$$\lambda_1, \lambda_2 = r_1^{-1}, r_2^{-1} \qquad \det(H) = (r_1 r_2)^{-1}$$

example: sphare $\lambda_1 = \lambda_2 = r^{-1}$

Gauß Bonnet

(28) $$\boxed{\int dA\ \lambda_1 \lambda_2 = 2\pi\chi = 2\pi(2-2g)}$$

$\left.\begin{array}{l}\chi : \text{ Euler characteristics} \\ g : \text{ genus}\end{array}\right\} \in \mathbb{Z}$

topological invariants

topological phase transitions

- lets modify the example from chapter 0.4

(29) $H = \vec{h}(\vec{k}) \cdot \vec{\sigma} \qquad \vec{h}(\vec{k}) = \vec{k}$

$$E_\pm = \pm|\vec{k}| \qquad |E_-\rangle = \begin{pmatrix} \sin\frac{\theta}{2} e^{-i\varphi} \\ -\cos\frac{\theta}{2} \end{pmatrix} \qquad |E_+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\varphi} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

$$C_\pm = \pm 1$$

- all Hamiltonians (29) are topologically equivalent for all $|\vec{k}|$

- now lets add a shift

(30) $$\vec{h}(\vec{k}) \longrightarrow \vec{h}'(\vec{k}) = \vec{k} + \vec{e}_x$$

then one finds :

(31) $$C_\pm(|\vec{b}|<1) = \pm 1$$

but $$C_\pm(|\vec{b}|>1) = 0$$

So something happens at $|\vec{b}|=1$!

Here

(32) $$E_+(|\vec{b}|=1) = E_-(|\vec{b}|=1)$$

i.e. there is a degeneracy

(31), (32) characterize what we call a topological phase transition