Please note: Exercises 23 and 24 are mandatory and have to be submitted to the postboxes in the 5th floor building 46.

Exercise 23. Wigner-Jordan Transformation
Consider a one-dimensional system of N spin-1/2 particles. This system can be described by the Pauli matrices $\sigma_j^x,\ \sigma_j^y,\ \sigma_j^z$ with $j = 1, 2, \ldots, N$. Show that the following transformation allows a mapping to fermions

$$\sigma_j^- = \exp\left(i\pi \sum_{l<j} c_l^\dagger c_l\right) c_j \qquad \sigma_j^+ = \exp\left(-i\pi \sum_{l<j} c_l^\dagger c_l\right) c_j^\dagger \tag{1}$$

respectively

$$c_j = \exp\left(-i\pi \sum_{l<j} \sigma_l^+ \sigma_l^-\right) \sigma_j^- \qquad c_j^\dagger = \exp\left(i\pi \sum_{l<j} \sigma_l^+ \sigma_l^-\right) \sigma_j^+ . \tag{2}$$

This means: Show, that spin and fermionic commutator relations are valid, respectively.

Exercise 24.
In the lecture it has been shown, that the magnetic susceptibility of an Ising ferromagnet in mean field is given by the self-consistency equation

$$\chi\left(\vec{r}_l - \vec{r}_{l'}\right) = \frac{\beta\delta_{l,l'} + \beta \sum_{l''} I_{l,l''} \chi\left(\vec{r}_{l''} - \vec{r}_{l'}\right)}{\cosh^2 \beta \bar{I}_l m}, \tag{3}$$

where $I_{l,l'} = I\left(|\vec{r}_l - \vec{r}_l'|\right)$ and $\bar{I}_l = \sum_{l'} I_{l,l'}$ have been defined.
Show, that for large distances $|\vec{r}_l - \vec{r}_{l'}|$

$$G_{l,l'} = k_\mathrm{B} T \chi\left(\vec{r}_l - \vec{r}_{l'}\right) \ \propto\ \frac{\exp\left(-|\vec{r}_l - \vec{r}_{l'}|/\xi\right)}{|\vec{r}_l - \vec{r}_{l'}|} \tag{4}$$

with correlation length ξ holds. To do so, first derive a self-consistency equation for the Fourier transformation $\tilde{\chi}(\vec{q})$ of $\chi(\vec{r})$. The behaviour for large $|\vec{r}|$ can be calculated by expanding $\tilde{\chi}(\vec{q})$ up to second order in $\vec{q}$ and by transforming the result back. Verify the result from the lecture for the correlation length ξ.