Please note: Exercises 12 and 13 are mandatory and have to be submitted to the postboxes in the 5. floor of building 46.

Exercise 12.

(a) A harmonic oscillator with small anharmonicities has the energy:

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right) - \gamma\hbar\omega\, n^2, \qquad n = 0,1,2,\ldots \tag{1}$$

where γ is a small parameter. Calculate the partition function Z by neglecting all terms which are non-linear in γ. Determine the free energy F in the same approximation.
(*Note*: $\sum_{n=0}^{\infty} n^2 \mathrm{e}^{-\alpha n} = \frac{\partial^2}{\partial\alpha^2}\sum_{n=0}^{\infty}\mathrm{e}^{-\alpha n}$)

(b) Expand the free energy of (a) for low temperatures and only keep terms of first order of $\exp(-\hbar\omega/k_B T)$. From this calculate the entropy S and the specific heat C_V and compare it to the results of a harmonic oscillator.

Exercise 13.

To describe ideal Fermi gases at low temperatures one has to solve integrals of the form

$$I = \int_0^\infty \mathrm{d}\varepsilon\; f(\varepsilon) n(\varepsilon), \tag{2}$$

where $n(\varepsilon)$ is the Fermi distribution of the energy ε. For low temperatures $n(\varepsilon)$ only slightly deviates from the step function $\Theta(\mu-\varepsilon)$. Therefore one can approximate:

$$\begin{aligned} I &= \int_0^\mu \mathrm{d}\varepsilon\; f(\varepsilon) + \int_0^\infty \mathrm{d}\varepsilon\; f(\varepsilon)\left[n(\varepsilon)-\Theta(\mu-\varepsilon)\right] && (3)\\ &\simeq \int_0^\mu \mathrm{d}\varepsilon\; f(\varepsilon) + \int_{-\infty}^\infty \mathrm{d}\varepsilon\; f(\varepsilon)\left[n(\varepsilon)-\Theta(\mu-\varepsilon)\right]. && (4)\end{aligned}$$

Expand $f(\varepsilon)$ at $\varepsilon=\mu$ and show that one obtains:

$$I = \int_0^\mu \mathrm{d}\varepsilon\; f(\varepsilon) + \frac{\pi^2}{6}(k_\mathrm{B}T)^2 f'(\mu) + \frac{7\pi^4}{360}(k_\mathrm{B}T)^4 f'''(\mu) + \ldots. \tag{5}$$

Use the expressions of the grand canonical potential and particle number derived in the lecture to prove that the following expansions hold true ($T_F = \varepsilon_F/k_\mathrm{B}$):

$$\begin{aligned} \mu &= \varepsilon_F\left[1-\frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2+\ldots\right], && (6)\\ p &= \frac{2}{5}\frac{N}{V}\varepsilon_F\left[1+\frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2+\ldots\right], && (7)\\ U &= \frac{3}{5}N\varepsilon_F\left[1+\frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2+\ldots\right]. && (8)\end{aligned}$$

What is the specific heat C_V of an ideal Fermi gas at constant volume?