4.5 Fault tolerant q-computing

(A) Goal and principle

Goal: Protection of quantum information during computation dynamics

remember: error correction idea

$|0\rangle \longrightarrow |000\rangle = |0\rangle_L$

$|1\rangle \longrightarrow |111\rangle = |1\rangle_L$

physical qubit error probability p → qubit code error probability $\mathcal{O}(p^2)$

- how about gate operations?

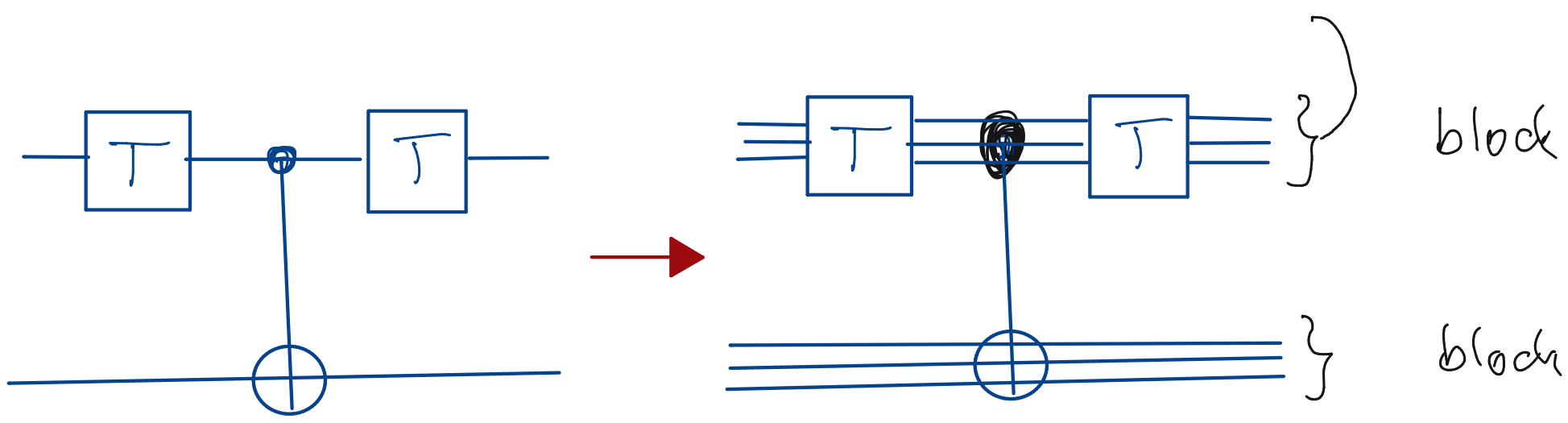
naive (and stupid) approach:

(i) decoding of logical qubits (ii) gate operation (iii) coding of results ?

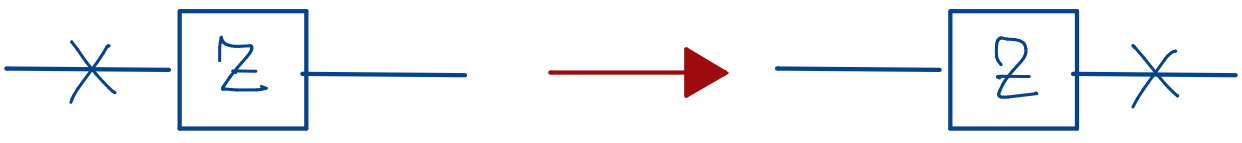
If during operation a single error on a physical qubit happens with probability p the whole gate operation has an error probability of p! ☹

(i) need circuits for computation to operate on encoded data.

i.e. for example

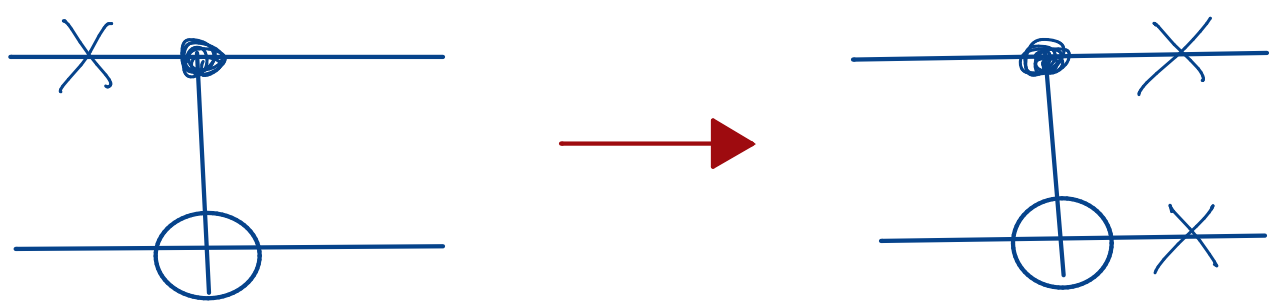


This is however not sufficient as quantum gates can cause errors to propagate and proliferate. E.g.: error X_1



X_1 error on input propagates to output

(230) $ZX|\psi\rangle = -XZ|\psi\rangle$



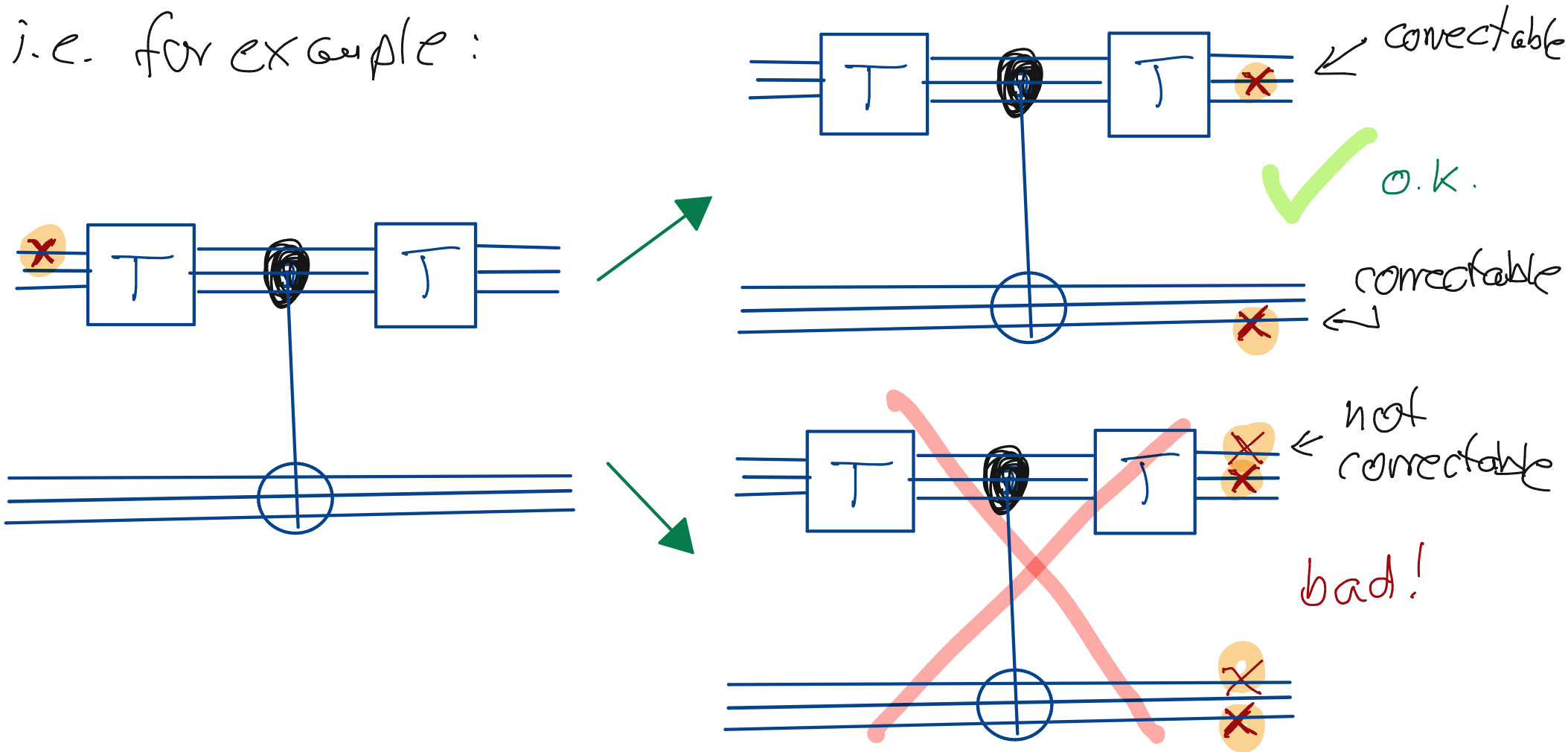
X_1 error on control input proliferates to X_1X_2 on both qubits

$U = CNOT$

$$UX_1|\psi\rangle = UX_1U^{-1}U|\psi\rangle = X_1X_2U|\psi\rangle \quad (231)$$

(ii) need to construct circuits such that error on one physical qubit of input leads to to an error in at most one physical qubit per output block

i.e. for example:



Finally the gate operation (measurement) itself may be faulty, e.g. a noisy CNOT gate

$$\mathcal{E} = \mathcal{E} \circ \mathcal{U}^{-1} \mathcal{U} \qquad \mathcal{U} = CNOT$$

$\equiv$ CNOT + subsequent error "$\mathcal{E} \circ \mathcal{U}^{-1}$"

$\mathcal{E} \circ \mathcal{U}^{-1}$ is an error such as $X_1 Z_2$ in the tensor space affecting both (logical, i.e. encoded) qubits.

Finally we need fault tolerant preparation of states and measurements which can be defined in an analogous way.

(B) Concatenating codes and threshold condition

Let us assume that the fault tolerant gate architecture discussed in section (A) can be realized with d being the number of operations needed in one gate operation. How many gates do we need in total if the algorithm altogether shall have an error probability

$$\leq \varepsilon \quad ?$$

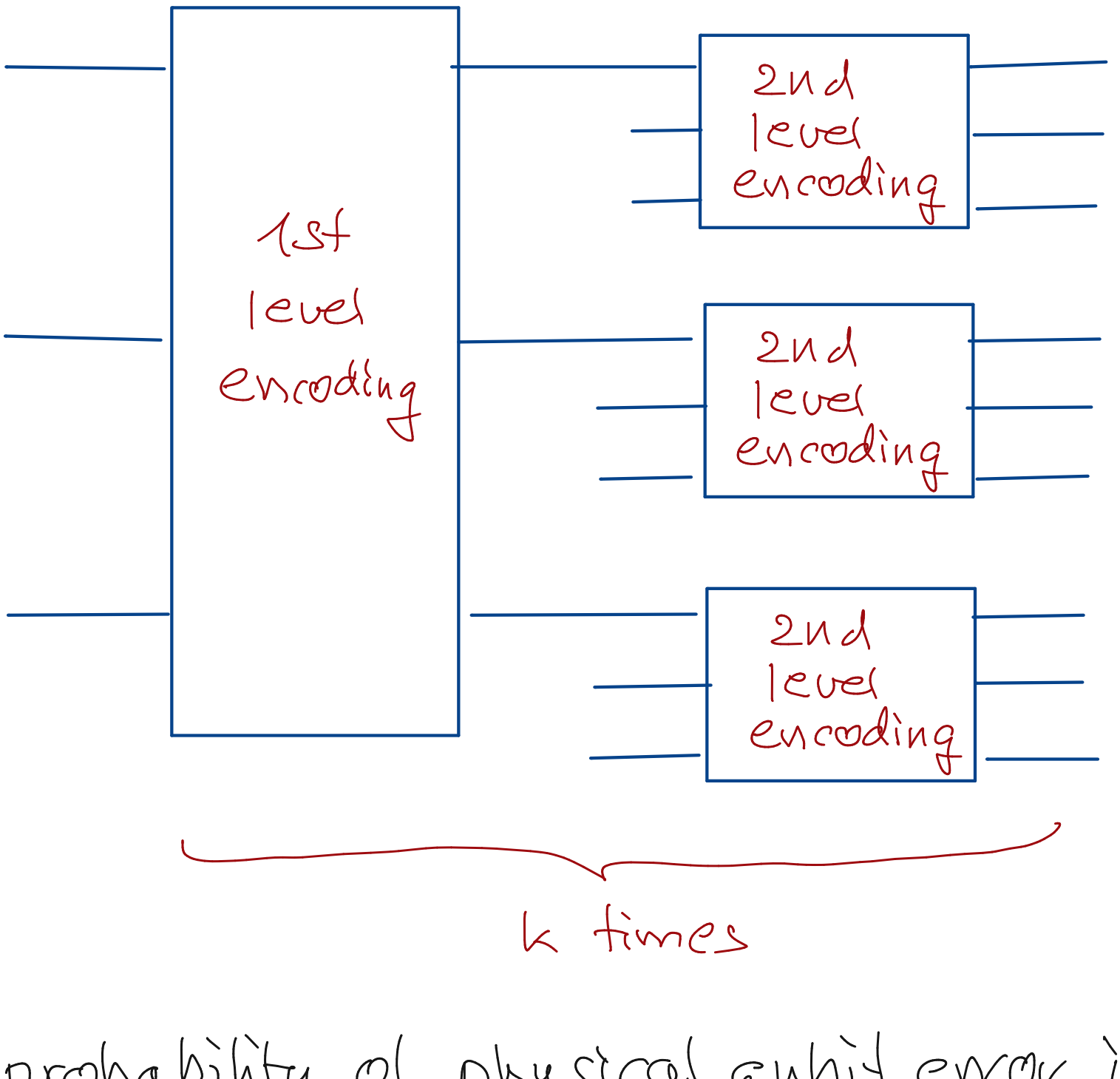
Let n be describe the size of the problem (e.g. # of input bits). Suppose are want to implement a circuit with

$$N(n) = \text{poly}(n) \tag{232}$$

gates. Then the allowed error probability per fault-tolerant gate must be less than

$$p_{\text{gate}} \leq \varepsilon / N(n) \tag{233}$$

This is demanding! However we can use a clever trick: concatinating codes for fault tolerant gates, e.g.



(i) probability of physical qubit error in one fault tolerant gate p

(ii) error probability of fault tolerant gate (1st level)

$$cp^2$$

(iii) error probability of fault tolerant gate (2nd level)

$$c(cp^2)^2 = \frac{(cp)^4}{c}$$

(iv) after k levels of concatinating this reduces to

(234) $$\frac{(cp)^{2^k}}{c}$$

(235) if $p < p_{th} = \frac{1}{c}$ exponential decrease!

From the requirement (233), i.e.

(236) $$\frac{(cp)^{2^k}}{c} < \frac{\varepsilon}{N(n)}$$

we can determine the minimum value of concatinating steps k. This gives the total number of elementary gate operations required

(237) $$d^k = \mathcal{O}\left(poly\left(\log \frac{N(n)}{\varepsilon}\right) N(n)\right)$$

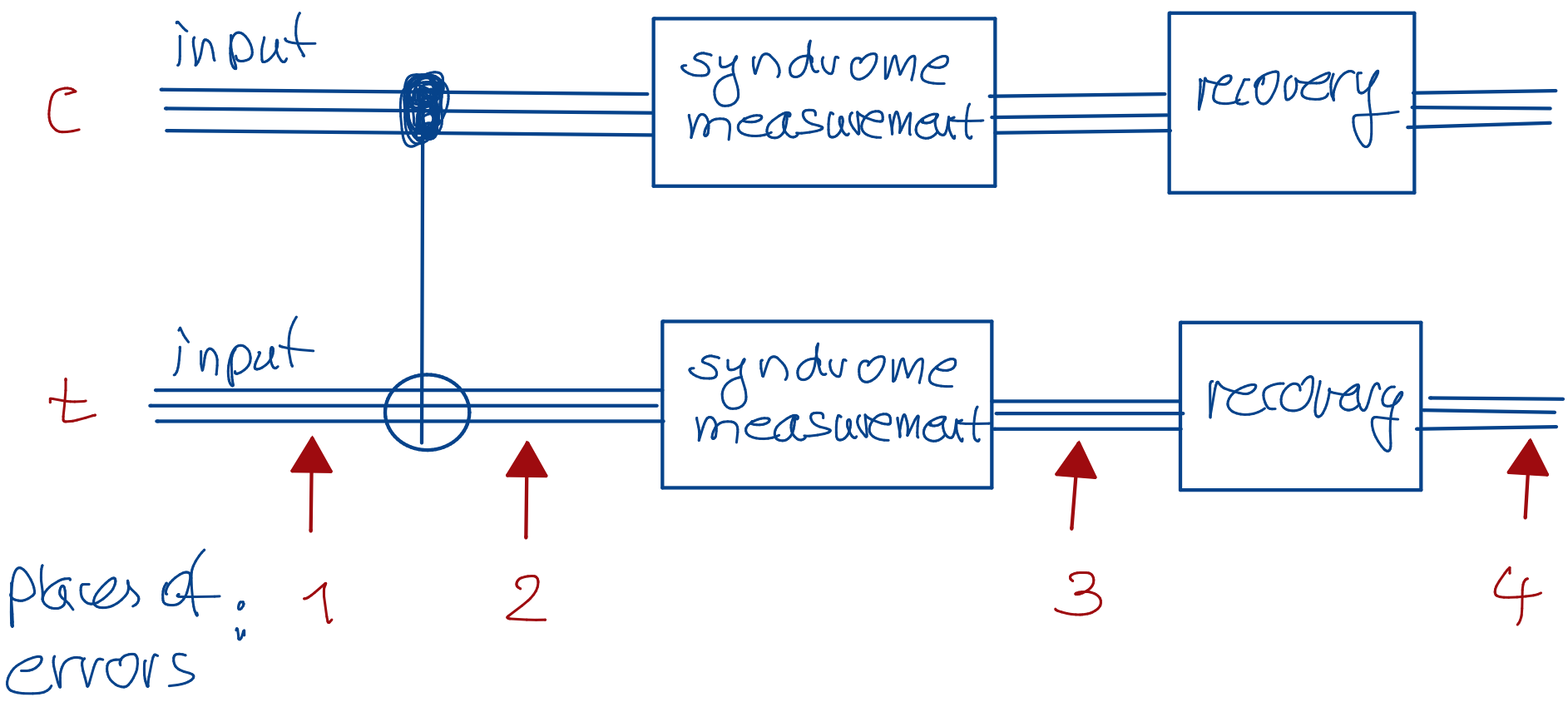
threshold theorem of quantum computation

A quantum circuit containing $N(n)$ logical gate operations may be simulated with error probability less than ε using

$$\mathcal{O}\left(poly\left(\log \frac{N(n)}{\varepsilon}\right) N(n)\right)$$

gates if the failure of each component happen with probability $p < p_{th} = 1/c$

- What is p_{th}? This depends very much on the code that is used. Let's discuss this for the 7 qubit Steane code and a fault tolerant CNOT



Two errors per encoded block cannot be corrected, so let's estimate the probability of such an event if the probability of a single error on a physical qubit is p.

(1) pre existing errors enter on both inputs (c and t) at step 1 and propagate via CNOT to one of the two outputs.

probability of pre existing error in one block

$$c_0 p$$

c_0: total number of places during syndrom measurement and recovery of previous

stage of circuit. For Steane code $C_0 \simeq 60+7 \simeq 70$

Probability of two errors in output

(238) $\leq C_0^2 p^2$ $\qquad C_0^2 \approx 5 \cdot 10^3$

(2) preexising error enters in step 1 and error in CNOT operation happens

$$\leq 2 \times C_0 p \times n p = C_1 p^2$$

n: number of places in the CNOT circuit where an error can happen. For Steane $n \simeq 7$

i.e. $C_1 = 2 \cdot 70 \cdot 7 \simeq 10^3$

(3) two failures during CNOT circuit

$$\leq \frac{1}{2} n(n-1) p^2 = C_2 p^2$$

For Steane $C_2 \simeq 20$

(4) failure during CNOT circuit and syndrome measurement (60 places)

$$\leq n \cdot 60 \cdot p^2 = C_3 p^2$$

For Steane $C_3 \simeq 10^2$

(5) Two failures occur during syndrome measurement

$$\leq C_4 p^2 \qquad \text{Steane } C_4 \approx 60^2 \approx 5 \cdot 10^3$$

(6) A failure during syndrome measurement and a failure in recovery

$$\leq C_5 p^2 \qquad \text{Steane } C_5 \approx 60 \times 7 \approx 500$$

(7) two failures during recovery

$$\leq C_6 p^2 \qquad \text{Steane } C_6 \approx 7^2 \approx 50$$

Total probability to have two or more errors in one of the two output blocks of the fault tolerant CNOT circuit

$$c p^2 \qquad c = C_0^2 + C_1 + C_2 + C_3 + C_4 + C_5 + C_6$$

For the Steane code the threshold probability for an error to occur to any physical qubit is

(238)

$$C \approx 10^4 \Rightarrow p_{th}^{\text{steane}} = 10^{-4}$$

(C) Fault-tolerant operations of Clifford gates

A universal set of elementary gates is e.g.

(239) $$\{H, S, CNOT, T\}$$

The first three are Clifford gates, the last one not. It turns out that the property (221) of the Clifford gates to map elements of the Pauli group back to elements of the Pauli group makes their fault tolerant construction relatively easy.

For the Steane code (7 qubit of CSS type) eqs. (213):

$$|0\rangle_L = \frac{1}{\sqrt{8}}\Big[|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle\Big]$$

$$|1\rangle_L = \frac{1}{\sqrt{8}}\Big[|1111111\rangle + |0101010\rangle + |0011001\rangle + |1001100\rangle + |1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle\Big]$$

it is easy to see that the logical Z and X operations read

(240) $$\overline{Z} = Z_1 Z_2 Z_3 Z_4 Z_5 Z_6 Z_7$$

(241) $$\overline{X} = X_1 X_2 X_3 X_4 X_5 X_6 X_7$$

as $\bar{Z}|0\rangle_L = |0\rangle_L \qquad \bar{Z}|1\rangle_L = -|1\rangle_L$

$\bar{X}|0\rangle_L = |1\rangle_L \qquad \bar{X}|1\rangle_L = |0\rangle_L$

Since

(242) $$HZ = HZH^+H = XH \qquad H = H^+$$

and

(243) $$HX = HXH^+H = ZH$$

it is immediately obvious that

(244) $$\boxed{\bar{H} = H_1 H_2 H_3 H_4 H_5 H_6 H_7}$$

This is fault tolerant because a failure in any component Hadamard will not propagate to other components. Property (244) is called transversality.

In a similar way we find that the logical phase gate $\bar{S}$ can be written as

(245) $$\boxed{\begin{aligned}\bar{S} &= \bar{Z}\cdot S_1 S_2 S_3 S_4 S_5 S_6 S_7 \\ &= Z_1 S_1 \cdot Z_2 S_2 \cdot Z_3 S_3 \cdots Z_7 S_7\end{aligned}}$$

and is also transversal and thus fault tolerant.

Remember $$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

so

(246) $$\bar{S}\,|0\rangle_L = \bar{Z}\,|0\rangle_L = |0\rangle_L$$

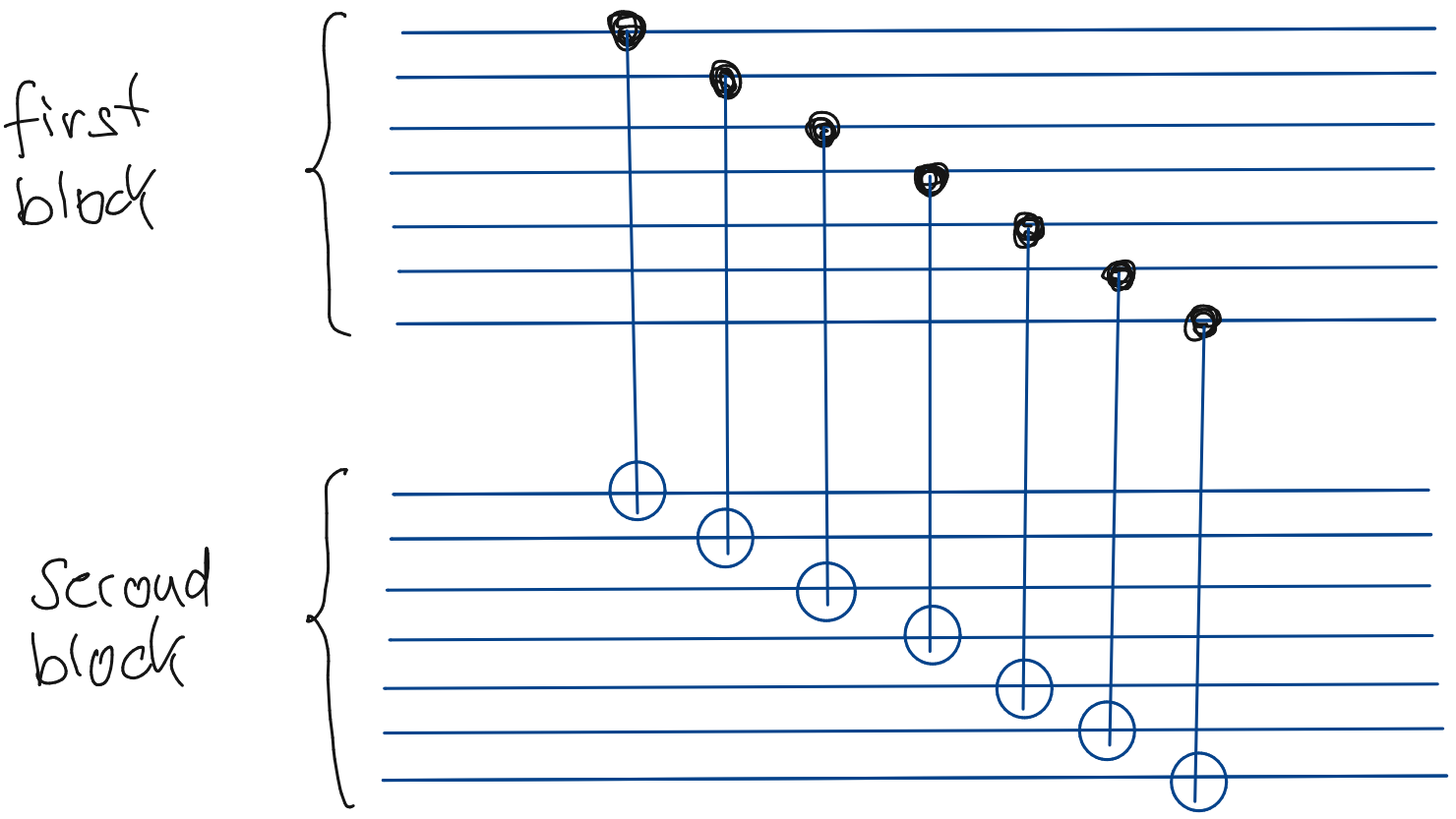
since always $(i)^4 = 1$. Furthermore

(247) $$\bar{S}\,|1\rangle_L = \bar{Z}\,(-i)\,|1\rangle_L = i\,|1\rangle_L$$

since either $i^3 = -i$ or $i^7 = -i$.

Also the logical CNOT gate is transversal

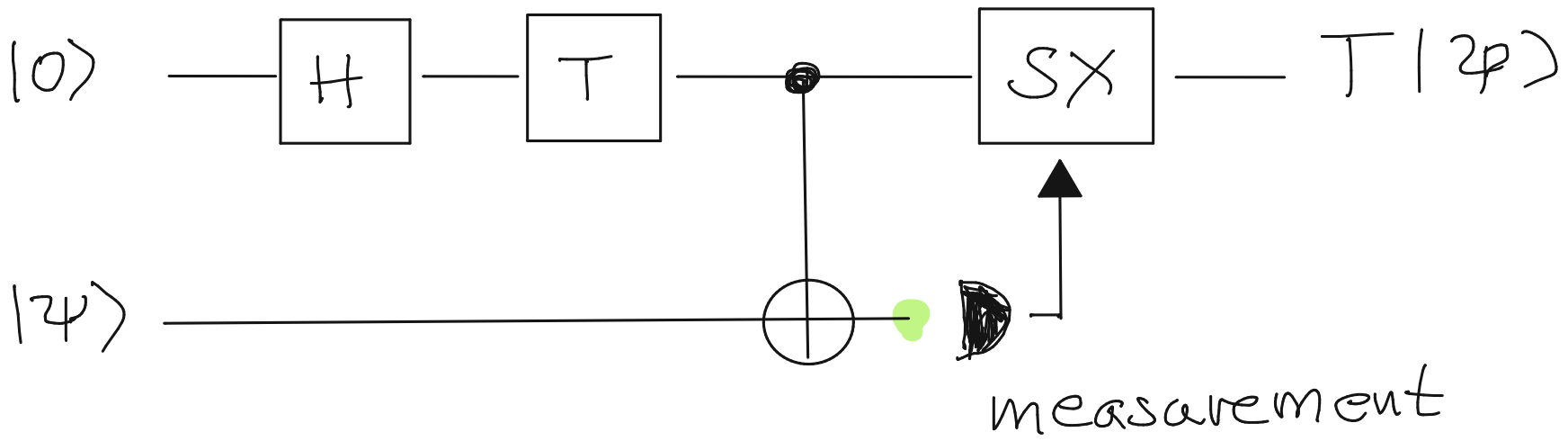
(248) $$\boxed{\overline{CNOT} = CNOT_{11}\, CNOT_{22} \cdots CNOT_{77}}$$



The fault tolerant implementation of the $\pi/8$ gate is a bit more involved but possible as we will see now.

(D) Fault-tolerant implementation of π/8 gate

In order to implement a T gate ($\pi/8$ gate) in a fault tolerant way we first convince ourselves that the T gate on an arbitrary state can be performed by a T gate on an ancilla state combined with a measurement



operation of H and T

(249) $|0\rangle \longrightarrow \frac{|0\rangle+|1\rangle}{\sqrt{2}} \longrightarrow |\theta\rangle = \frac{|0\rangle + e^{i\frac{\pi}{4}}|1\rangle}{\sqrt{2}}$

CNOT with $|\varphi\rangle = a|0\rangle + b|1\rangle$

(250)
$$|\theta\rangle|\varphi\rangle \xrightarrow[CNOT]{} \frac{1}{\sqrt{2}}\Big[|0\rangle(a|0\rangle+b|1\rangle) + e^{i\pi/4}|1\rangle(a|1\rangle+b|0\rangle)\Big]$$
$$= \frac{1}{\sqrt{2}}\Big[\left(a|0\rangle + be^{i\frac{\pi}{4}}|1\rangle\right)|0\rangle + \left(b|0\rangle + ae^{i\frac{\pi}{4}}|1\rangle\right)|1\rangle\Big]$$

Measurement of second qubit projects the first to

$$\text{(257)} \quad \begin{aligned} &\text{"0"} \longrightarrow a|0\rangle + b e^{i\frac{\pi}{4}}|1\rangle \\ &\text{"1"} \longrightarrow b|0\rangle + a e^{i\frac{\pi}{4}}|1\rangle \end{aligned}$$

In the case of a measurement outcome of "1" we have to apply

$$\text{(252)} \quad \begin{aligned} SX\left[b|0\rangle + a e^{i\frac{\pi}{4}}|1\rangle\right] &= i b|1\rangle + a e^{i\frac{\pi}{4}}|0\rangle \\ &= e^{i\frac{\pi}{4}}\left[a|0\rangle + e^{i\frac{\pi}{4}}|1\rangle\right] \\ &= e^{i\frac{\pi}{4}} T|\psi\rangle \qquad \square \end{aligned}$$

In order to translate this into a fault tolerant operation on encoded logical qubits we have to discuss how the measurement and how the preparation of the state $|\Theta\rangle$ can be performed in a fault tolerant way. The preparation can be done using again measurements whose fault tolerant implementation will be discussed in the next subsection.

$|\Theta\rangle$ is an eigenstate of

$$\text{(253)} \quad \hat{\Theta} = e^{-i\pi/4} SX$$

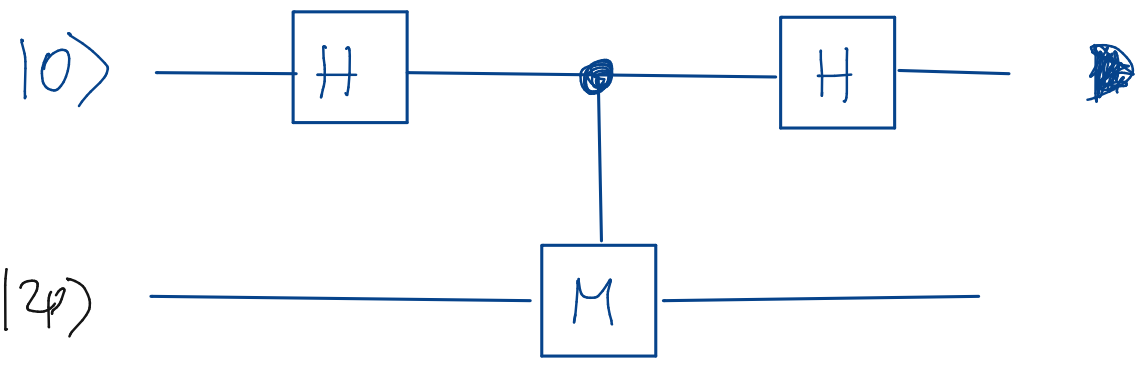
with eigenvalue $+1$. Note that $\hat{\Theta}^+ = \hat{\Theta}^{-1}$.

So measuring $\hat{\Theta}$ fault tolerantly with result +1 prepares the state $|\theta\rangle$. If we obtain the result −1 we just have to apply a Z.Sinp

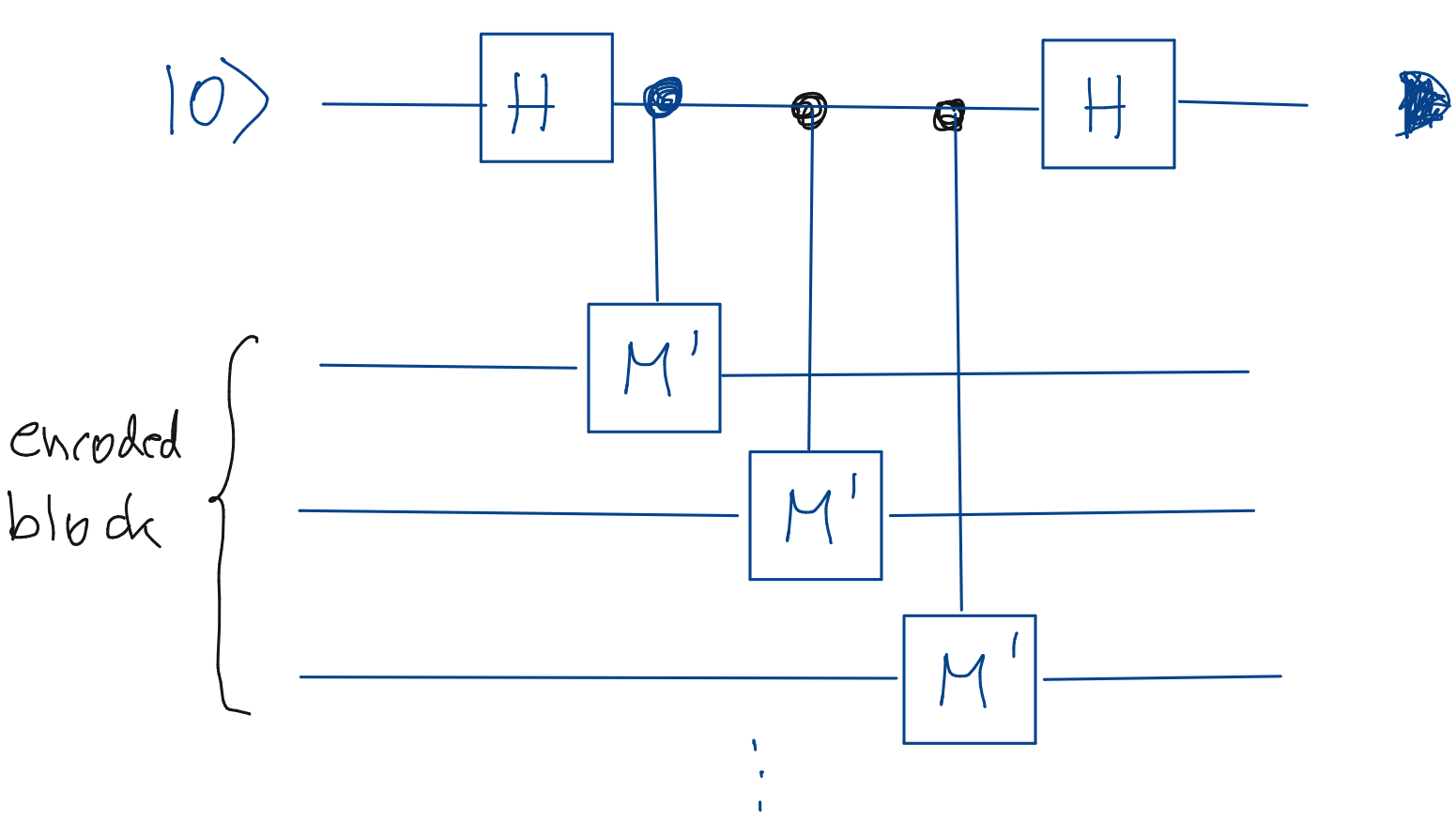
$$Z\, SXZ^{+} = -SX \tag{254}$$

(E) Fault tolerant measurements

We have already seen that the measurement of a single qubit operator $\hat{M}$ with eigenvalues ± 1 can be performed via the gate



This can easily be generalized to a transversal operator $\hat{M} = M'_1 M'_2 M'_3 \cdots M'_n$ on n qubits



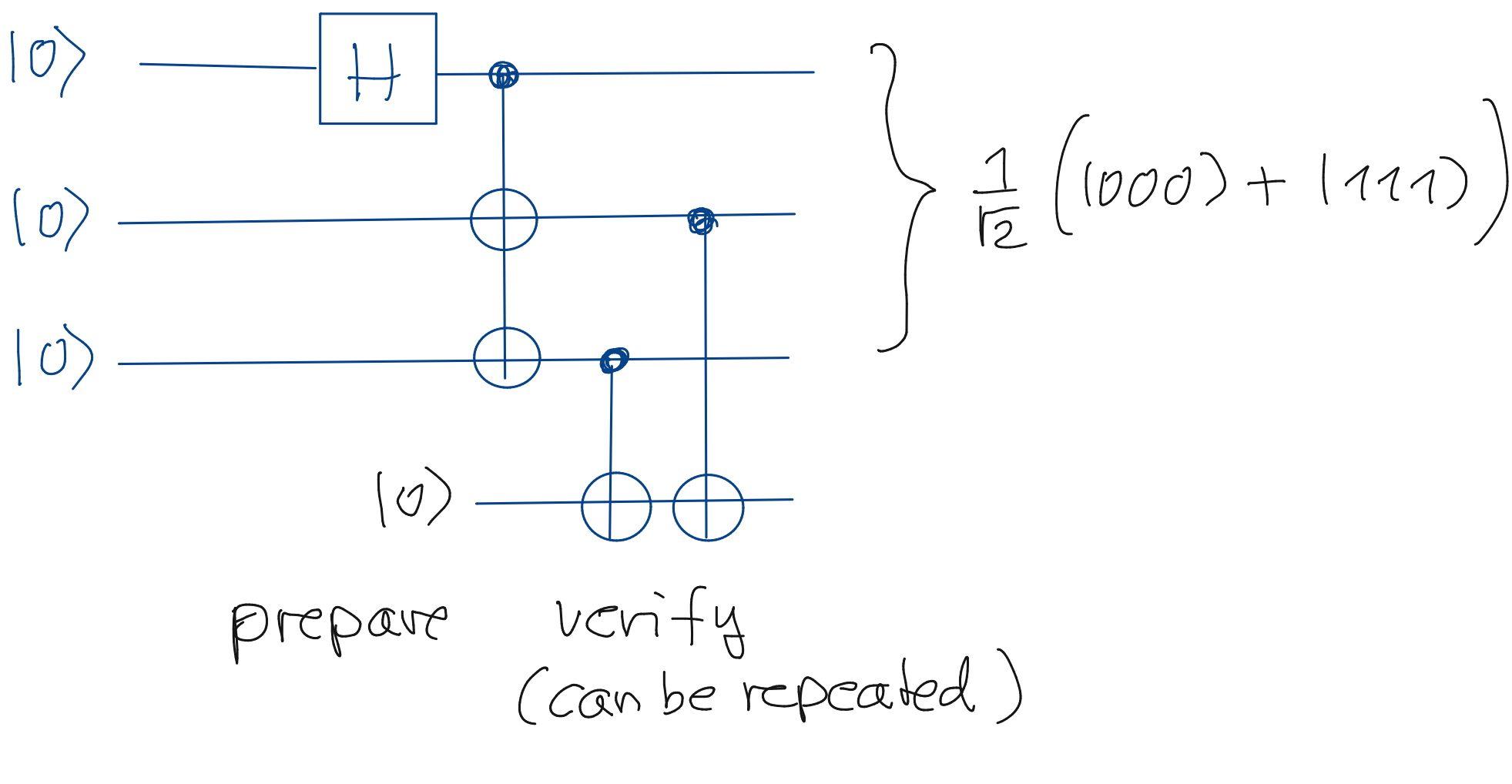
This is not yet fault tolerant since the first line operates on a single qubit. Note that the ancilla after the first Hadamard is

$$|0\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

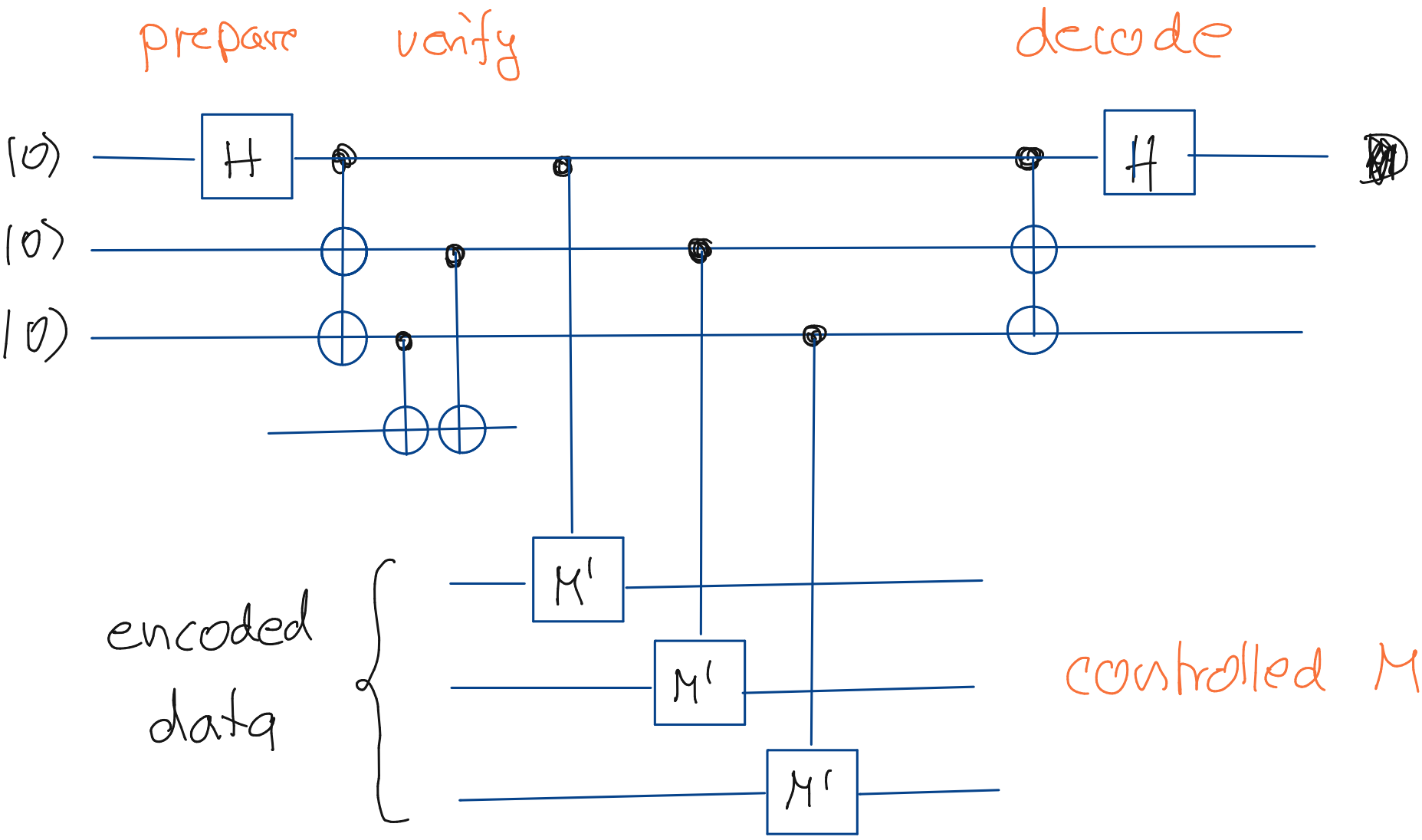
So we prepare the "encoded" state (cat state)

$$\frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right) \tag{255}$$

We note that the cat state is not in a code state and a circuit preparing this state is by itself not fault tolerant. Its preparation can however be amended with (several) verification steps



With this we can make the whole measurement circuit fault tolerant



V. Quantum communication

5.1 Classical information transfer, Shannon's theorems

We have already discussed that information is a measure of the a-priori ignorance about an event. A quantitative measure of information about a complete set of events A_i $i = 1, 2.., n$ which occur with