detection of error syndroms and correction

- **spin flip** e.g. in first block 1, 2, 3:

detect $\quad Z_1 Z_2 \quad$ and $\quad Z_2 Z_3$

Z_1Z_2	Z_2Z_3	
+	+	no error
−	+	1st qubit
−	−	2nd qubit
+	−	3rd qubit

analogously for other blocks

correct $\quad$ apply X_1 or X_2 or X_3

- **phase flip** e.g. on first block

$\Longrightarrow$ Changes sign on first qubit block

$$\big(|000\rangle + |111\rangle\big)\big(\;\cdots\;\big)\big(\;\cdots\;\big)$$

$$\updownarrow$$

$$\big(|000\rangle - |111\rangle\big)\big(\;\cdots\;\big)\big(\;\cdots\;\big)$$

detect

$X_1 X_2 X_3 X_4 X_5 X_6$ and $X_4 X_5 X_6 X_7 X_8 X_9$

$+$	$+$	no error
$-$	$+$	phase in 1st block
$-$	$-$	phase in 2nd block
$-$	$+$	phase in 3rd block

correct   apply $Z_1 Z_2 Z_3$ or $Z_4 Z_5 Z_6$ or $Z_7 Z_8 Z_9$

(D) Conditions for Error correction

The reason why classical error correction in digital information processing is so powerful is that classical bit errors are discrete!

In analog systems this is not the case!

$\Rightarrow$ How about quantum errors? They are not discrete but can be arbitrary superpositions of X, Z and $Y = XZ$

> The continuum of errors that can happen to a qubit can all be corrected by correcting only a discrete subset of errors.

consider single qubit e.g. in Shore code $(|0\rangle_L, |1\rangle_L)$

$$|\psi\rangle = \alpha|0\rangle_L + \beta|1\rangle_L$$

error

$$\mathcal{E}(|\psi\rangle\langle\psi|) = \sum_j \hat{E}_j |\psi\rangle\langle\psi| \hat{E}_j^\dagger \tag{203}$$

where $\hat{E}_j$ are Krauss operators of error

arbitrary error on first qubit of the code

$$\hat{E}_j = \varepsilon_{j0}\hat{I} + \varepsilon_{j1}\hat{X}_1 + \varepsilon_{j2}\hat{Z}_1 + \varepsilon_{j3}\hat{X}_1\hat{Z}_1 \tag{204}$$

so

$$\hat{E}_j|\psi\rangle = \varepsilon_{j0}|\psi\rangle + \varepsilon_{j1}\hat{X}_1|\psi\rangle + \varepsilon_{j2}\hat{Z}_1|\psi\rangle + \varepsilon_{j3}\hat{X}_1\hat{Z}_1|\psi\rangle$$

measureing error syndrom collapses this state to:

$$\begin{tikzcd}
 & |\psi\rangle \arrow[r] & |\psi\rangle \\
 & \hat{X}_1|\psi\rangle \arrow[r] & |\psi\rangle \\
\hat{E}_j|\psi\rangle \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] & \hat{Z}_1|\psi\rangle \arrow[r] & |\psi\rangle \\
 & \hat{X}_1\hat{Z}_1|\psi\rangle \arrow[r] & |\psi\rangle
\end{tikzcd}$$

syndrom detection

syndrom correction

Quantum error-correction condition

Let C be a quantum code and P the projector on the code (subspace). Suppose $\mathcal{E}$ is an quantum error with Krauss operators $\hat{E}_j$.
A necessary and sufficient condition for the existence of an error-correcting operation R on C which corrects $\mathcal{E}$ is

$$P \hat{E}_m \hat{E}_j^{+} P = \alpha_{mj} P \tag{205}$$

with some hermitian matrix $\underline{\underline{\alpha}}$. The set of errors $\mathcal{E}$ is called correctable set of errors

4.4 Stabilizer codes

We have seen in specific examples that error correction can be applied to quantum systems.
Can we systematically understand

- how to construct quantum codes?
- how to construct error syndrom detection?

• what are correctable errors?

An important class of quantum codes are the so called stabilizer codes, which we will now discuss.

(A) Stabilizer formalism

To discuss stabilizer codes it is useful to first introduce the stabilizer formalism. The latter can be illustrated with the following simple example

EPR state of two qubits

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle \right) \tag{206}$$

It is easy to see that $|\psi\rangle$ is an eigenstate of

$$\hat{X}_1 \hat{X}_2 \, |\psi\rangle = |\psi\rangle \quad \text{and} \quad \hat{Z}_1 \hat{Z}_2 \, |\psi\rangle = |\psi\rangle \tag{207}$$

We say $|\psi\rangle$ is stabilized by the operators $\hat{X}_1 \hat{X}_2$ and $\hat{Z}_1 \hat{Z}_2$. One can show that $|\psi\rangle$ is the only state which fulfills eq. (207).

It turns out that the discussion of many codes, their manipulation, description of errors and measurements can conveniently be discussed in terms of the stabilizers rather than the states.

stabilizer formalism $\hat{=}$ clever use of group theory

Pauli group of one qubit

$$G_1 = \{\pm\hat{I}, \pm i\hat{I}, \pm\hat{X}, \pm i\hat{X}, \pm\hat{Y}, \pm i\hat{Y}, \pm\hat{Z}, \pm i\hat{Z}\} \quad (208)$$

is a group under matrix multiplication

Pauli group of n qubits G_n: all tensor products

Def: Consider a subgroup S of G_n. Let V_S be the set of n-qubit states that are stabilized by the elements of S. Then S is called stabilizer of the space V_S

- linear combination of two elements of V_S is in V_S

<u>example:</u> $n = 3$

(209) $S = \{\hat{I}, \hat{Z}_1\hat{Z}_2, \hat{Z}_2\hat{Z}_3, \hat{Z}_1\hat{Z}_3\}$

$\hat{I}$: all 3 qubit states

Z_1Z_2 : $|000\rangle, |001\rangle, |110\rangle, |111\rangle$

Z_2Z_3 : $|000\rangle, |100\rangle, |011\rangle, |111\rangle$

Z_1Z_3 : $|000\rangle, |010\rangle, |101\rangle, |111\rangle$

(210) V_S : $|000\rangle, |111\rangle$

here looking at elements Z_1Z_2 and Z_2Z_3 is sufficient $\Rightarrow$

<u>Def:</u> A subset of elements $\{g_1, g_2, \dots g_\ell\}$ of a group G is called generator of G if all elements of G can be written as products of the g_j

notation

(211) $G = \langle g_1, \dots, g_\ell \rangle$

in our example:

(212) $S = \langle Z_1Z_2, Z_2Z_3 \rangle$

proof: $Z_1Z_2 \cdot Z_2Z_3 = Z_1Z_3$

$(Z_1Z_2)^2 = \hat{I}$ $\square$

Def: The elements of a generator are called independent if removing one generator makes the stabilizer smaller

- how to see if $S = \langle g_1, g_2, \ldots, g_\ell \rangle$ is an independent set of generators?

check matrix

g_j : n-qubit operator e.g. $\hat{X}_1 \hat{Z}_2 \hat{I}_3 \hat{X}_4 \hat{Y}_5 \cdots$
consisting of $\hat{X}, \hat{Y}, \hat{Z}, \hat{I}$

$$\ell \left\{ \left(\begin{array}{c|c} \text{---}\; g_1 \;\text{---} & \text{---}\; g_1 \;\text{---} \\ \text{---}\; g_2 \;\text{---} & \text{---}\; g_2 \;\text{---} \\ \vdots & \vdots \\ \text{---}\; g_\ell \;\text{---} & \text{---}\; g_\ell \;\text{---} \\ \end{array} \right) \right. \qquad \begin{array}{c|c} \underbrace{\qquad\qquad\quad}_{n} & \underbrace{\qquad\qquad\quad}_{n} \end{array} \tag{213}$$

$$\left(\begin{array}{c|c} 0010001\cdots & 0100100\cdots \\ 1001001\cdots & 1000100\cdots \\ \uparrow & \uparrow \\ \text{"1" at position of } \boxed{X} & \text{"1" at position of } \boxed{Z} \end{array} \right)$$

- for a $\hat{Y}$ at som position j put a 1 in the left (X) side and a 1 in the right (Z) side

The elements of a generator of a stabilizer $S = \langle g_1, g_2, \dots, g_l \rangle$ are independent if the $2n$ dimensional row vectors of the check matrix are linear independent and S does not contain $-I$

Lets denote the $2n$-dimensional row vectors $r(g)$

Two elements g_1 and g_2 commute if

$$r(g_1) \Lambda r(g_2)^T = 0$$

where

$$\Lambda = \begin{pmatrix} 0 & \mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix}$$

- Why condition of $-I$ not being an element of S ?

Note that not every subgroup S of the Pauli group is a useful stabilizer, e.g. $G_1 = \{\pm I, \pm X\}$ has only the trivial vector as space V_S since

$$-I\,|\psi\rangle = |\psi\rangle \quad \Rightarrow \quad |\psi\rangle = 0$$

condition for non-trivial stabilizer $S \subset G_n$

(i) all elements commute

(ii) $-I$ is not an element of S

to emphasize the compactness of the stabilizer formalism lets consider the **7-qubit Steane code**

$$|0\rangle_L = \frac{1}{\sqrt{8}}\Big[|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle$$
$$+ |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle\Big] \tag{213}$$
$$|1\rangle_L = \frac{1}{\sqrt{8}}\Big[|1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle$$
$$+ |1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle\Big]$$

stabilizers of Steane code

(214)

g_1	1	1	1	X	X	X	X
g_2	1	X	X	1	1	X	X
g_3	X	1	X	1	X	1	X
g_4	1	1	1	Z	Z	Z	Z
g_5	1	Z	Z	1	1	Z	Z
g_6	Z	1	Z	1	Z	1	Z

- What is the dimension of the Hilbert space V_S of code states stabilized by S?

Let $S = \langle g_1, g_2, \ldots, g_{n-k} \rangle$ be a stabilizer for n qubits, such that $-I$ is not element. If all $n-k$ generators are independent and commute than the vector space of code states V_S is 2^k dimensional

(B) Unitary operations in stabilizer formalism

We will now see that all unitary operations i.e. gate operations but also errors due to interaction with environments can conveniently be described in the stabilizer formalism

let $|\psi\rangle \in V_S$ and $g \in S$, i.e.

$$g|\psi\rangle = |\psi\rangle$$

$$\Rightarrow \quad U|\psi\rangle = Ug|\psi\rangle = UgU^{-1}U|\psi\rangle$$

$\Rightarrow$ vector space UV_S is stabilized by the group of all $\{UgU^{-1}\} = USU^{-1}$

i.e. to compute effect of U it is sufficient to know its action on the independent generators

$$U g_1 U^{-1} \quad U g_2 U^{-1} \quad \dots \quad U g_e U^{-1}$$

- for particular unitaries U this takes a very simple form:

$\boxed{U = H}$ single qubit Hadamard $\quad H^{+} = H^{-1}$

(215)
$$H X H^{+} = Z \qquad H Z H^{+} = X$$
$$\Rightarrow \quad H Y H^{+} = -Y$$

$\boxed{U = X, Y, Z}$

(216) $$X X X^{+} = X \qquad X Z X^{+} = -Z$$

(217) $$Z X Z^{+} = -X \qquad Z Z Z^{+} = Z$$

$\boxed{U = S}$ phase gate $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

(218) $$S X S^{+} = Y \qquad S Z S^{+} = Z$$

This allows for a very compact description of states that are stabilized (in the stabilizer space), e.g.

$$S = \langle Z_1, Z_2, \dots, Z_n \rangle \quad \longrightarrow \quad |0\rangle^{\otimes n}$$

$$S = \langle X_1, X_2, \dots X_n \rangle = \langle H Z_1 H^\dagger, H Z_2 H^\dagger, \dots \rangle$$

$$\longrightarrow \frac{1}{\sqrt{2^n}} \left(|0\rangle + |1\rangle \right)^{\otimes n}$$ has 2^n coefficients

O.k., but may be not so impressive since stabilized states are not entangled!

- but consider two qubit operation CNOT

$$\boxed{U = CNOT}$$

(219)
$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

one finds:

(220)
$$U X_1 U^\dagger = X_1 X_2$$
$$U Z_1 U^\dagger = Z_1$$
$$U X_2 U^\dagger = X_2$$
$$U Z_2 U^\dagger = Z_1 Z_2$$

closed!

$\Rightarrow$ Gottesman – Knill theorem