II. Entangled states

2.1. The thought experiment by Einstein, Podolsky and Rosen (1935) EPR

In 1935 EPR put forward an argument that quantum theory is either incomplete or that the principle of locality in the strict sense must be abandoned:

Consider a pair of particles A and B with position and momentum operators (in 1D for simplicity) $\hat{X}_A, \hat{X}_B$ and $\hat{P}_A, \hat{P}_B$. Since

(39) $$[\hat{X}_A + \hat{X}_B, \hat{P}_A - \hat{P}_B] = 0$$

they can be in a state which is an eigenstate of both observables $\hat{X}_A + \hat{X}_B$ and $\hat{P}_A - \hat{P}_B$

(40) $$|\Psi_{AB}\rangle = |x_A = -x_B, p_A = p_B\rangle$$

measuring the position x_A of A immediately fixes the position x_B of B; but measuring the momentum p_A of A also fixes p_B of B.

$\Rightarrow$ Either x_B and p_B are actually given at the same time or quantum mechanics is non local

Bohm's variant of EPR (1951)

2 spin $1/2$ particle in total $S=0$ state

$$A \longleftarrow \underset{S=0}{\circ\circ} \longrightarrow B$$

$$|EPR\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle_A |\downarrow_z\rangle_B - |\downarrow_z\rangle_A |\uparrow_z\rangle_B\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B\right)$$

(47)

$$= \frac{1}{\sqrt{2}}\left[\frac{1}{2}\left(|0\rangle - |1\rangle\right)\left(|0\rangle + |1\rangle\right) - \frac{1}{2}\underbrace{\left(|0\rangle + |1\rangle\right)}_{\uparrow x}\underbrace{\left(|0\rangle - |1\rangle\right)}_{\downarrow x}\right]$$

entangled state

$$= \frac{1}{\sqrt{2}}\left(|\uparrow_x\rangle_A |\downarrow_x\rangle_B - |\downarrow_x\rangle_A |\uparrow_x\rangle_B\right)$$

$$= \frac{1}{\sqrt{2}}\left(|\uparrow_y\rangle_A |\downarrow_y\rangle_B - |\downarrow_y\rangle_A |\uparrow_y\rangle_B\right)$$

measurement of any spin direction $(\sigma_x, \sigma_y, \sigma_z)$ of A immediately gives an eigenstate of $(\sigma_x, \sigma_y, \sigma_z)$ of B

if A and B are space-like separated

$\Rightarrow$ EPR paradoxon

- since $[\sigma_\mu^B, \sigma_\nu^B] \neq 0$ for $\mu \neq \nu$

the spin in μ and ν direction of B cannot be determined precisely at the same time

but! we can choose to measure any spin direction for particle A and thus fix any spin direction of B

$\Rightarrow$ either there are hidden variables in quantum mechanics

OR

quantum mechanics is non-local

2.2. Bell inequalities

If local hidden variables exist then correlation measurements in non-orthogonal basis states have to fulfill certain inequalities

Bell inequalities (1964)

spin projections $\vec{a}, \vec{b}$ with $|\vec{a}| = |\vec{b}| = 1$

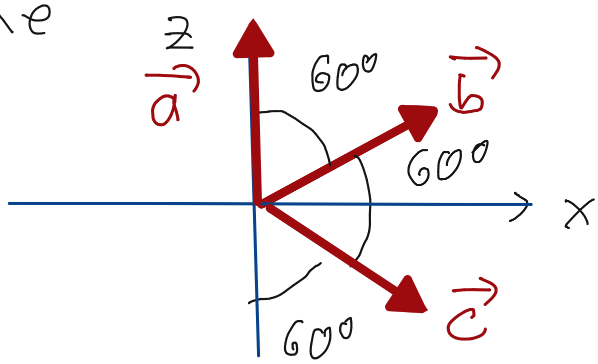
$$E(\vec{a}, \vec{b}) = \langle \underbrace{(\vec{\sigma}_A \cdot \vec{a})}_{\text{spin A}} \underbrace{(\vec{\sigma}_B \cdot \vec{b})}_{\text{spin B}} \rangle \tag{42}$$

Bell:

(43) $$\boxed{\left| E(\vec a,\vec b) - E(\vec a,\vec c)\right| \le 1 + E(\vec b,\vec c)}$$

in hidden variable (classical) theory

Consider EPR state (41) and 3 directions in one plane



EPR state is isotropic so choose $\vec a \overset{!}{=} \vec e_z$

(44) $\vec\sigma\cdot\vec a = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$ $\qquad \vec\sigma\cdot\vec e = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi\end{pmatrix}$

(45) $$E(\vec a,\vec b) = \langle EPR| \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}_A \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi\end{pmatrix}_B |EPR\rangle$$

with $|EPR\rangle = \frac{1}{\sqrt2}\left(\begin{pmatrix}1\\0\end{pmatrix}_A\begin{pmatrix}0\\1\end{pmatrix}_B - \begin{pmatrix}0\\1\end{pmatrix}_A\begin{pmatrix}1\\0\end{pmatrix}_B\right) \Rightarrow$

$$E(\vec a,\vec e) = \frac12\left[(1\,0)_A(0\,1)_B - (0\,1)_A(1\,0)_B\right] \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}_A \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi\end{pmatrix}_B \left[\underset{A}{\begin{pmatrix}1\\0\end{pmatrix}}\underset{B}{\begin{pmatrix}0\\1\end{pmatrix}} - \underset{A}{\begin{pmatrix}0\\1\end{pmatrix}}\underset{B}{\begin{pmatrix}1\\0\end{pmatrix}}\right]$$

(46)

$$= \frac12\left\{\underbrace{\begin{pmatrix}1\\0\end{pmatrix}^T\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}}_{A}\cdot\underbrace{\begin{pmatrix}0\\1\end{pmatrix}^T\begin{pmatrix} c & s \\ s & -c\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}}_{B} + \begin{pmatrix}0\\1\end{pmatrix}^T\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\underbrace{\begin{pmatrix}1\\0\end{pmatrix}^T\begin{pmatrix} c & s \\ s & -c\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}}_{B}\right\}$$

$\Rightarrow 0 \qquad\qquad \Rightarrow 0$

(47) so $E(\vec{a},\vec{e}) = \frac{1}{2}\left(1\cdot(-\cos\varphi) + (-1)\cdot\cos\varphi\right) = -\cos\varphi$

$\varphi \measuredangle(\vec{a},\vec{e})$

$\Rightarrow \quad E(\vec{a},\vec{b}) = -\cos(60°) = -\frac{1}{2} = E(\vec{b},\vec{c})$

rot. invariance

$E(\vec{a},\vec{c}) = -\cos(120°) = \frac{1}{2}$

$\Rightarrow \quad |E(\vec{a},\vec{b}) - E(\vec{a},\vec{c})| = 1 \quad > \quad 1 + E(\vec{b},\vec{c}) = \frac{1}{2}$

$\boxed{|EPR\rangle \text{ leads to violation of Bell inequalities}}$

experiments (Clauser 1969, Aspect 1982) have proven violation of Bell inequalities

$\Rightarrow$ no hidden variables !

Nobel prize 2022

Clauser-Horne-Shimony-Holt (CHSH) variant

(49) $$\boxed{|E(a,b) - E(a,b') + E(a',b) + E(a',b')| \leq 2}$$

quantum upper bound gives $2\sqrt{2}$

$\boxed{\text{Only entangled states can lead to violation of Bell inequalities}}$

Problem: finite detection efficiency

- important maximally entangled (max. violation of Bell inequalities) two-partite states

"Bell states"

$$|\phi_+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \quad |\psi_+\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$$

$$|\phi_-\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) \quad |\psi_-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

2.3. Multi-partite entanglement

The difference between classical correlations and entanglement becomes even more apparent when considering multi-partite states

Greenberger, Horn, Zeilinger (GHZ) (1989)

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(\underset{A}{|\uparrow_z\rangle}\underset{B}{|\uparrow_z\rangle}\underset{C}{|\uparrow_z\rangle} - \underset{A}{|\downarrow_z\rangle}\underset{B}{|\downarrow_z\rangle}\underset{C}{|\downarrow_z\rangle}\right) \tag{50}$$

One can show that the GHZ-state is an eigenstate of

$$(51)\qquad \begin{matrix} \hat\sigma_y^{A}\,\hat\sigma_y^{B}\,\hat\sigma_x^{C} \\ \hat\sigma_y^{A}\,\hat\sigma_x^{B}\,\hat\sigma_y^{C} \\ \hat\sigma_x^{A}\,\hat\sigma_y^{B}\,\hat\sigma_y^{C} \end{matrix}$$

with eigenvalue $+1$

and

$$(52)\qquad \hat\sigma_x^{A}\,\hat\sigma_x^{B}\,\hat\sigma_x^{C}$$

with eigenvalue -1

for classical correlated states with random values

$$S_{x,y} = \pm 1$$

if

$$(53)\qquad \begin{aligned} S_y^{A} S_y^{B} S_x^{C} &\overset{!}{=} +1 \\ S_y^{A} S_x^{B} S_y^{C} &\overset{!}{=} +1 \\ S_x^{A} S_y^{B} S_y^{C} &\overset{!}{=} +1 \end{aligned}$$

then multiplying the left hand sides gives

$$(54)\qquad \underbrace{(S_y^{A})^2}_{=1} S_x^{A}\, \underbrace{(S_y^{B})^2}_{=1} S_x^{B}\, \underbrace{(S_y^{C})^2}_{=1} S^{C}_x = S_x^{A} S_x^{B} S_x^{C} = +1 \ \lightning$$

in contrast to quantum case eq. (52)

$$\boxed{\text{entanglement is a genuine quantum property}}$$

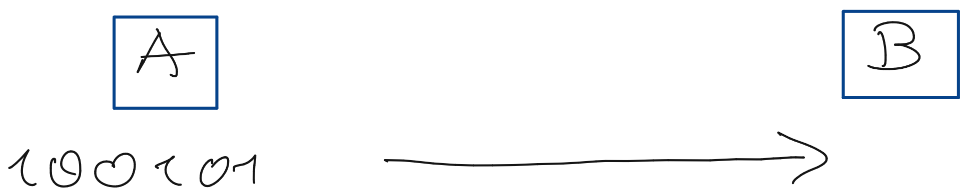
qualitatively different types of entangled states for multi-partite systems

$$|GHZ\rangle \sim |000\rangle - |111\rangle \tag{55}$$
$$|W\rangle \sim |100\rangle + |010\rangle + |001\rangle$$

2.4 Dense coding: entanglement as ressource

Bennett & Wiesner 1992:

- Alice wants to send classical bits to Bob



If the sequence of bits of A are random (i.e. cannot be compressed) Alice needs to send one bit per bit of her sequence

- Assume Alice and Bob share a priori a pair of particles in Bell state $|\phi_+\rangle$. Alice can do some operation with her qubit and send it to Bob.

- Bob can then by a local measurement of the two quits (his one and the one sent by Alice) distinguish 4 orthogonal Bell states by local measurements. Thus Alice can encode 2 bits of information per qubit sent!

(56)

$$|\phi_+\rangle \sim |00\rangle + |11\rangle \quad \begin{cases} I_A \rightarrow |00\rangle + |11\rangle \\ \sigma_z^A \rightarrow |00\rangle - |11\rangle \\ \sigma_x^A \rightarrow |10\rangle + |01\rangle \\ \sigma_x^A \sigma_z^A \rightarrow |10\rangle - |01\rangle \end{cases}$$

4 alternatives

2.5 Quantum teleportation

Is there a way to transfer an unknown qubit state from A to B without having to send the (fragile) spin 1/2 particle?

Remember: Measurement and cloning do not work!

Bennett, Brasard, Crepeau, Josza, Peres & Wooters 1993 (PRL 70, 1895(1993))

use entangled states!

assume Alice and Bob possess a pair of qubits in a Bell state (drop normalization)

(57) $$|\phi_+\rangle = |0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B$$

and Alice has an unknown (to her) qubit

(58) $$|\psi_A\rangle = \alpha|0\rangle_A + \beta|1\rangle_A$$

The total state then reads

$$|\psi_A\rangle|\phi_+\rangle = \left(\alpha|0\rangle_A + \beta|1\rangle_A\right)\left(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right)$$

(59) $$= \underbrace{\begin{aligned} &\left(|0\rangle|0\rangle + |1\rangle|1\rangle\right)_A \left(\alpha|0\rangle_B + \beta|1\rangle_B\right) && (1)\\ +&\left(|0\rangle|0\rangle - |1\rangle|1\rangle\right)_A \left(\alpha|0\rangle_B - \beta|1\rangle_B\right) && (2)\\ +&\left(|1\rangle|0\rangle + |0\rangle|1\rangle\right)_A \left(\alpha|1\rangle_B + \beta|0\rangle_B\right) && (3)\\ +&\left(|1\rangle|0\rangle - |0\rangle|1\rangle\right)_A \left(-\alpha|1\rangle_B + \beta|0\rangle_B\right) && (4)\end{aligned}}_{\text{4 orthogonal Bell state}}$$

$\rightarrow$ can be discriminated by Bell state measurement

measurement outcome

(1) $\Rightarrow$ Bob has correct state already

(2) $\Rightarrow$ Bob applies $\hat{\sigma}_z$

(3) $\Rightarrow$ Bob applies $\hat{\sigma}_x$

(4) $\Rightarrow$ Bob applies $\hat{\sigma}_z$ and then $\hat{\sigma}_x$

$\Rightarrow$ State is transfered without measureing α, β or sending the qubit from $A \to B$

collapse of state is instantaneous
$\Rightarrow$ information transfer faster than light?

No! Without knowing the state Alice measured Bob has no information

$$\rho_B = \mathrm{Tr}_A\{\rho_0\} = \frac{\mathbb{1}}{2} \tag{60}$$

$$= \frac{1}{4} \sum_{i=1}^{4} |\phi_B^i\rangle\langle\phi_B^i|$$

where $|\phi_B^1\rangle = \frac{1}{\sqrt{2}}(\alpha|0\rangle + \beta|1\rangle)$

$|\phi_B^2\rangle = \frac{1}{\sqrt{2}}(\alpha|0\rangle - \beta|1\rangle)$ etc.

$\boxed{\text{teleportation: entangled state as ressource + LOCC}}$

2.6 Criterion and measure of entanglement

(A) pure bi-partite states

$|\psi\rangle$ is entangled if it cannot be written as

$$|\psi\rangle = |\psi\rangle_A |\psi'\rangle_B$$

i.e. necessary and sufficient condition of entanglement

(61) Schmidt number $S > 1$

for separable states

(62) $\rho_A = \mathrm{Tr}_B\{|\psi\rangle\langle\psi|\} = |\psi_A\rangle\langle\psi_A|$ i.e. $\rho_A^2 = \rho_A$

$\rho_B = \mathrm{Tr}_A\{|\psi\rangle\langle\psi|\} = |\psi'_B\rangle\langle\psi'_B|$ i.e. $\rho_B^2 = \rho_B$

i.e. reduced density matrix is a pure state

entangl. criterion

$|\psi\rangle$ separable
$\Updownarrow$
Schmidt number $S = 1$
$\Updownarrow$
reduced density operator = pure

quantitative measure?

should have properties

(i) $E \geq 0$ (ii) $E = 0$ for $\varsigma = 1$

(iii) E extensive (in thermod. sense) (iv) $E = \max$ if all coefficients of Schmidt decomposition equal

(63)
$$\boxed{E(|\psi\rangle\langle\psi|) = -\mathrm{Tr}_B\{\varsigma_B \ln \varsigma_B\} = -\mathrm{Tr}_A\{\varsigma_A \ln \varsigma_A\}}$$

von Neumann entropy

remark: Schmidt decomposition $M = \dim(\mathcal{H}_A) \leq \dim(\mathcal{H}_B)$

$$|\psi\rangle = \sum_{j=1}^{M} \sqrt{\lambda_j}\, |j_A\rangle |j_B\rangle \qquad \{|j_{A,B}\rangle\} \text{ ONB}$$

$$\varsigma_B = \sum_{j=1}^{M} \lambda_j\, |j_B\rangle\langle j_B|$$

$$\varsigma_A = \sum_{j=1}^{M} \lambda_j\, |j_A\rangle\langle j_A|$$

$$\Rightarrow E = -\mathrm{Tr}_B\left\{\sum_{j=1}^{M} \lambda_j\, |j_B\rangle\langle j_B| \ln\left(\sum_{k=1}^{M} \lambda_k\, |k_B\rangle\langle k_B|\right)\right\}$$

(64) $$E = -\sum_{j=1}^{M} \lambda_j \ln \lambda_j \qquad \text{fulfills (i)–(iii)}$$

(65) $$E_{max} = \ln M \qquad \forall\ \lambda_j = \frac{1}{M} \qquad \text{fulfills (v)}$$

A separable state $|\psi\rangle = |\psi_A\rangle|\psi_B'\rangle$ cannot be transferred to an entangled state by LOCC operations.

(B) Mixed bi-partite states

A pure bi-partite state that is separabel has the density matrix

(66) $$\rho = |\psi_A\rangle\langle\psi_A| \circ |\psi_B'\rangle\langle\psi_B'|$$

We will call ρ still separabel if it is a convex sum of separable pure states with probability p_j

(67) $$\rho = \sum_{j=1}^{K} p_j \left(|\psi_A^j\rangle|\psi_B'^j\rangle\right)\left(\langle\psi_A^j|\langle\psi_B'^j|\right)$$ separabel

otherwise ρ is entangled. Interpretation: ρ is ensemble of separabel pure states we just don't know in which of the separabel states the system actually is. Also here holds

A separabel state ρ cannot be turned into an entangled one by LOCC operations.

A necessary condition for entanglement is given by Peres' theorem PRL 77, 1413 (1996)

(68) $\boxed{\text{if } \rho \text{ separable } \Rightarrow \rho^{T_A} \geq 0 \text{ and } \rho^{T_B} = (\rho^{T_A})^T \geq 0}$

ρ^{T_A} is partial transpose with respect to subsystem A

proof: ρ separabel $\rho = \sum_{j=1}^{k} p_j \, |j_A\rangle\langle j_A| \cdot |j'_B\rangle\langle j'_B|$

$$\rho^{T_A} = \sum_{j=1}^{k} p_j \left(|j_A\rangle\langle j_A|\right)^T |j'_B\rangle\langle j'_B|$$

using $A^+ = (A^T)^*$ and $\left(|j_A\rangle\langle j_A|\right)^+ = |j_A\rangle\langle j_A|$

$$\Rightarrow \rho^{T_A} = \sum_{j=1}^{k} p_j \, |j_A^*\rangle\langle j_A^*| \cdot |j'_B\rangle\langle j'_B| \geq 0 \quad \square$$

Horodecki theorem

In $\mathbb{C}^2 \otimes \mathbb{C}^2$ or $\mathbb{C}^2 \otimes \mathbb{C}^3$

ρ separabel $\Leftrightarrow \rho^{T_A} \geq 0$

- a general, practical necessary and sufficient condition for mixed state entanglement is not yet found

quantitative measures of mixed-state entanglement

von Neumann entropy no entanglement measure other measures exist but are not easily calculable

- entanglement of formation
- entanglement of destillation
- relative entanglement entropy
- logarithmic negativity

- for mixed states of two qubits

Concurrence

(69) $$\boxed{C(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)}$$

where $\lambda_1, \lambda_2 \ldots$ are eigenvalues of decreasing order of

(70) $$R = \left(\sqrt{\rho}\, \tilde{\rho} \sqrt{\rho}\right)^{1/2}$$

with

(71) $$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\, \rho^* \, (\sigma_y \otimes \sigma_y)$$

and ρ is in the computational basis of σ_z eigen-states.