Note: Please insert your assignment into the mailboxes 5th floor, building 46. The problems marked by a ☗ have to be submitted.

☗ **Problem 30.** *Virial theorem*

Given the Hamilton operator in one dimension

$$\hat{H} = \hat{H}_{\text{kin}} + \hat{H}_{\text{pot}} = \frac{\hat{p}^2}{2m} + V(\hat{x}). \tag{1}$$

(a) Show that in position space the commutator of $\hat{H}$ and $\hat{x}\hat{p}$ reads:

$$\frac{1}{i\hbar}\left[\hat{H}, \hat{x}\hat{p}\right] = \hat{x}\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right) - \frac{1}{m}\hat{p}^2. \tag{2}$$

(b) Let φ_E be an eigenstate of $\hat{H}$ with Energy E. Show from equation (2) that

$$\left\langle \frac{\hat{p}^2}{2m} \right\rangle_E = \frac{1}{2}\left\langle \hat{x}\frac{\mathrm{d}}{\mathrm{d}x}V(x)\right\rangle_E, \tag{3}$$

where $\langle \bullet \rangle_E$ denotes the expectation value in the state φ_E.

(c) Show that in the special case

$$V(\lambda x) = \lambda^n V(x), \tag{4}$$

i.e. if $V(x)$ is a homogeneous function of order n, holds:

$$\left\langle \hat{H}_{\text{kin}} \right\rangle_E = \frac{n}{2}\left\langle \hat{H}_{\text{pot}} \right\rangle_E \qquad \text{(Virial theorem)}. \tag{5}$$

Hint: Calculate the first derivative (4) with respect to λ.

☗ **Problem 31.** *Angular momentum; ladder operators*

In Cartesian coordinates holds:

$$\hat{\vec{L}}^2 = \frac{1}{2}\left(\hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+\right) + \hat{L}_z^2, \tag{6}$$

where $\hat{L}_\pm = \hat{L}_x \pm \mathrm{i}\hat{L}_y$ are the ladder operators of the angular momentum.

(a) Making use of the representation of the angular momentum operators $\hat{L}_x, \hat{L}_y, \hat{L}_z$ in spherical coordinates, detrmine the representation of the ladder operators in the spherical coordinates.

(b) Derive with the help of (6) the representation of $\hat{\vec{L}}^2$ in spherical coordinates and show that in three dimensions the Laplace operator in spherical coordinates can be expressed through $\hat{\vec{L}}^2$ by:

$$\vec{\nabla}^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\hat{\vec{L}}^2}{\hbar^2 r^2} \tag{7}$$

Problem 32. *Spin 1*

(a) Show that the spin matrices

$$\hat{s}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{s}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \hat{s}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

obey the commutation relations $[\hat{s}_i, \hat{s}_j] = i\hbar\epsilon_{ijk}\hat{s}_k$ and $[\hat{s}_i, \hat{\vec{s}}^{\,2}] = 0$. What is $\hat{\vec{s}}^{\,2}$ and determine the eigenvalues of the operator.

(b) The eigenstates (eigenspinors) of $\hat{s}_z$ are

$$\Psi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \Psi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Which linear combinations of $\Psi_\pm$ are eigenspinors of $\hat{s}_x$ and $\hat{s}_y$?

Problem 33. *Spin 2*

The general spin-components of a spin-1/2 system is $\hat{s}_{\vec{n}} := 1/2\vec{n}\hat{\vec{\sigma}}$, where $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\phi)$ is the generalized unity vector. Determine the eigenvalues of $\hat{s}_{\vec{n}}$ and a complete set of orthonormal eigenstates. Determine the expectation value $\langle\hat{\sigma}_z\rangle$ in these eigenstates.