**Problem 30. Virial theorem**

Given the Hamilton operator in one dimension

\[
\hat{H} = \hat{H}_{\text{kin}} + \hat{H}_{\text{pot}} = \frac{\hat{p}^2}{2m} + V(\hat{x}).
\]

(a) Show that in position space the commutator of \( \hat{H} \) and \( \hat{x}\hat{p} \) reads:

\[
\frac{1}{i\hbar} [\hat{H}, \hat{x}\hat{p}] = \hat{x}\left(\frac{dV}{dx}\right) - \frac{1}{m}\hat{p}^2.
\]

(b) Let \( \varphi_E \) be an eigenstate of \( \hat{H} \) with Energy \( E \). Show from equation (2) that

\[
\langle \hat{p}^2 / 2m \rangle_E = \frac{1}{2} \langle \hat{x} \frac{d}{dx} V(x) \rangle_E,
\]

where \( \langle \bullet \rangle_E \) denotes the expectation value in the state \( \varphi_E \).

(c) Show that in the special case

\[
V(\lambda x) = \lambda^n V(x),
\]

i.e. if \( V(x) \) is a homogeneous function of order \( n \), holds:

\[
\langle \hat{H}_{\text{kin}} \rangle_E = \frac{n}{2} \langle \hat{H}_{\text{pot}} \rangle_E \quad \text{(Virial theorem)}.
\]

Hint: Calculate the first derivative (4) with respect to \( \lambda \).

**Problem 31. Angular momentum; ladder operators**

In Cartesian coordinates holds:

\[
\hat{\mathbf{L}}^2 = \frac{1}{2} \left( \hat{L}_+ \hat{L}_- + \hat{L}_- \hat{L}_+ \right) + \hat{L}_z^2,
\]

where \( \hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y \) are the ladder operators of the angular momentum.

(a) Making use of the representation of the angular momentum operators \( \hat{L}_x, \hat{L}_y, \hat{L}_z \) in spherical coordinates, determine the representation of the ladder operators in the spherical coordinates.

(b) Derive with the help of (6) the representation of \( \hat{\mathbf{L}}^2 \) in spherical coordinates and show that in three dimensions the Laplace operator in spherical coordinates can be expressed through \( \hat{\mathbf{L}}^2 \) by:

\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{\hat{\mathbf{L}}^2}{\hbar^2 r^2}
\]
Problem 32. Spin 1

(a) Show that the spin matrices
\[ \hat{s}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{s}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \hat{s}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

obey the commutation relations \([\hat{s}_i, \hat{s}_j] = i\hbar\epsilon_{ijk}\hat{s}_k\) and \([\hat{s}_i, \hat{s}^2] = 0\). What is \(\hat{s}^2\) and determine the eigenvalues of the operator.

(b) The eigenstates (eigenspinors) of \(\hat{s}_z\) are
\[ \Psi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \Psi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

Which linear combinations of \(\Psi_{\pm}\) are eigenspinors of \(\hat{s}_x\) and \(\hat{s}_y\)?

Problem 33. Spin 2

The general spin-components of a spin-1/2 system is \(\hat{s}_{\vec{n}} := \frac{1}{2}\vec{n} \cdot \hat{\vec{\sigma}}\), where \(\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\phi)\) is the generalized unity vector. Determine the eigenvalues of \(\hat{s}_{\vec{n}}\) and a complete set of orthonormal eigenstates. Determine the expectation value \(\langle \hat{\sigma}_z \rangle\) in these eigenstates.